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Solving optimal control problems by using hermite polynomials

Ayat Ollah Yari*
Department of Applied Mathematics,
Faculty of Mathematical Sciences,
Payame Noor University,
P. O. BOX 19395-3697, Tehran, Iran.
E-mail: a_yary@yahoo.com

Mirkamal Mirnia
Department of Applied Mathematics,
Faculty of Mathematical Sciences,
University of Tabriz, Tabriz, Iran.
E-mail: mirnia-kam@tabrizu.ac.ir

Abstract In this paper, one numerical method is presented for numerical approximation of linear constrained optimal control problems with quadratic performance index. The method with variable coefficients is based on Hermite polynomials. The properties of Hermite polynomials with the operational matrices of derivative are used to reduce optimal control problems to the solution of linear algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords. Optimal control; Hermite polynomials ; Best approximating; Operational matrix of derivative.
2010 Mathematics Subject Classification. 34Ho5 ;34K35 ;65Kxx ;90Cxx ;33C45.

1. Introduction

One of the widely used methods to solve optimal control problems is the direct method. There are a large number of research papers that employ this method to solve optimal control problems (see for example [2–5, 8, 9, 14–17, 19, 27–34, 37–40] Razzaghi, et. al. used direct method for variational problems by using hybrid of block-pulse and Bernoulli polynomials [38]. Optimal control of switched systems based on Bezier control points presented in [19]. A new approach using linear combination property of intervals and discretization is proposed to solve a class of nonlinear optimal control problems, containing a nonlinear system and linear functional [43, 44]. Time varying quadratic optimal control problem was solved by using Bezier control points [18]. Hybrid functions approach for nonlinear constrained optimal control problems presented by Mashayekhi et. al. [34]. The optimal control problem of a linear distributed parameter system is studied via shifted Legendre polynomials (SLPs) in [24]. An accurate method is proposed to solve problems such as identification, analysis and optimal control using the Bernstein orthonormal polynomials operational matrix of integration [42]. In [23] Jaddu and Shimemura proposed a method to solve

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∗ Corresponding author.

the linear-quadratic and the nonlinear optimal control problems by using Chebyshev polynomials to parameterize some of the state variables, then the remaining state variables and the control variables are determined by the state equations. Also Razzaghi and Elnagar [39] proposed a method to solve the unconstrained linear-quadratic optimal control problem with equal number of state and control variables. Their approach is based on using the shifted Legendre polynomials to parameterize the derivative of each of the state variables. The approach proposed in [34] is based on approximating the state variables and control variables with hybrid functions. In [28] operational matrices with respect to Hermite polynomials and their applications is presented for solving linear dfferential equations with variable coeffcients. In [47] investigation of optimal control problems and solving them using Bezier polynomials is presented.

In this paper, we present a computational method for solving linear constrained quadratic optimal control problems by using Hermite polynomials. The method is based on approximating the state variables and the control variables with Hermite polynomials. Our method consists of reducing the optimal control problem into a set of linear algebraic equations by initial expanding the state rate $x(t)$ the control $u(t)$ as a Hermite polynomial with unknown coefficients. In order to approximate the integral and differential parts of the problem and the performance index, differentiation D_ϕ is given.

The paper is organized as follows: In Section 2 we describe the basic formulation of the Hermite functions required for our subsequent development. Section 3 is devoted to the formulation of optimal control problems. Section 4 summarizes the application of this method to the optimal control problems, and in Section 5, we report our numerical finding and demonstrate the accuracy of the proposed method.

2. Hermite polynomials and their properties

Hermite polynomials are a classical orthogonal polynomial sequence that arise in probability. They are named after Charles Hermite (1864). The explicit expression of Hermite polynomials of degree n is defined by [28]:

$$H_n(t) = n! \sum_{i=0}^{[\frac{n}{2}]} \frac{(-1)^i (2t)^{n-2i}}{i!(n-2i)!}, \tag{2.1}$$

where t is real number ($t \in R$), and Rodrigues formula is the following

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}(e^{-t^2}). \tag{2.2}$$

Eqs. (2.1) and (2.2) are solutions for the following equation

$$x'' - 2tx' + 2nx = 0. \tag{2.3}$$

Namely $x(t) = H_n(t)$. The first few Hermite polynomials are
$H_0(t) = 1$, $H_1(t) = 2t$, $H_2(t) = 4t^2 - 2$, $H_3(t) = 8t^3 - 12t$

2.1. **Some properties of Hermite polynomials.** These polynomials are satisfed in the following three terms recurrence formula

$$H_{i+1}(t) = 2tH_i(t) - 2iH_{i-1}(t). \tag{2.4}$$

An important property of the Hermite polynomials is the following derivative relation[28]

$$H_i'(t) = 2iH_{i-1}(t), \tag{2.5}$$

where $i = 0, \ldots, n$ and $H_i'(t)$ is derivation of Hermite polynomials of degree i.
Further, $H_i(t)$ are orthogonal in $L_w^2(\Lambda)$, weher $\Lambda = (-\infty, +\infty)$ with respect to the weight function $w(t) = e^{-t^2}$ and satisfy in the following relation

$$\int_{-\infty}^{+\infty} H_i(t)H_j(t)w(t)dt = 2^i i! \sqrt{\pi}\delta_{i,j}, \tag{2.6}$$

where $\delta_{i,j}$ is kronecker delta function. Some properties for Hermite polynomials are
$H_i(-t) = (-1)^n H_i(t)$, $H_{2i}(0) = (-1)^i \frac{(2i)!}{i!}$, $H_{2i+1}(0) = 0$,
$H'_{2i}(0) = 0$, $H'_{2i+1}(0) = (-1)^i \frac{(2i+2)!}{(i+1)!}$.

2.2. **The operational matrix of the Hermite polynomials.** A function $x(t) \in L_w^2(\Lambda)$, can be expressed in terms of Hermite polynomials as

$$x(t) = \sum_{-\infty}^{+\infty} a_i H_i(t), \tag{2.7}$$

where the coeffcients a_i are given by

$$a_i = \frac{1}{2^i i! \sqrt{\pi}} \int_{-\infty}^{+\infty} H_i(t)x(t)w(t)dt. \tag{2.8}$$

In practice, only the first $n+1$ term of the Hermite polynomials are considered. Then we have:

$$x_n(t) = \sum_{i=0}^{n} a_i H_i(t) = A\Phi_n(t), \tag{2.9}$$

where Hermite coefficients vector A and Hermite vector $\Phi(t)$ are given by

$$A = [a_0, \ldots, a_n],$$

$$\Phi_n(t) = [H_0(t), \ldots, H_n(t)]^T, \tag{2.10}$$

where T denotes transposition.
The operational matrix of derivative: The differentiation of vector $\Phi_n(t)$ can be expressed as

$$\Phi_n'(t) = D_\phi \Phi_n(t), \tag{2.11}$$

where D_ϕ is the $(n+1)(n+1)$ operational matrix of derivative for the Hermite polynomials given as follows:

$$D_\phi = (d_{i,j}) = \begin{cases} 2i, & j = i-1, \\ 0, & otherwise \end{cases} \tag{2.12}$$

2.3. **Approximations by Hermite polynomials.** Now in this section, we present some useful theorems which show the approximations of functions by Hermite polynomials. For this purpose, let us define
$S_n = span\{H_0(t), H_1(t), \ldots, H_n(t)\}$. Any polynomial $h(t)$ of degree m can be expanded in terms of $H_i(t), i = 0 \ldots n$ as follows

$$h(t) = \sum_{i=0}^{n} c_i H_i(t). \tag{2.13}$$

Also the $L^2(\Lambda)$-orthogonal projection $p_n : L^2(\Lambda) \to S_n$ is a mapping in a way that for any $y(t) \in L^2(\Lambda)$, we have:

$$\langle p_n(y) - y, \phi \rangle = 0, \quad \forall \phi \in S_n.$$

Due to the orthogonality, we can write

$$p_n(y) = \sum_{i=0}^{n-1} c_i H_i(t), \tag{2.14}$$

where c_i are constants in the following form

$$c_i = \frac{1}{\gamma_i} \langle y(t), H_i(t) \rangle_{L_w^2},$$

where $\gamma_i = 2^i i! \sqrt{\pi}$. In the literature of spectral methods, $p_n(y)$ is named as Hermite expansion of $y(t)$ and approximates $y(t)$ on $(-\infty, +\infty)$. Also estimating the distance between $y(t)$ and it's Hermite expansion as measured in the weighed norm $\| \, . \, \|_w$ is an important problem in numerical analysis. The following theorem provide the basic approximation results for Hermite expansion.

Theorem 1. we have

$$\| \tfrac{d^l}{dt^l}(p_n(y) - y) \|_{w(t)} \le n^{(l-m)/2} \| \tfrac{d^m}{dt^m} y(t) \|_{w(t)},$$

$$0 \le l \le m, \qquad \forall y \in B^m(\Lambda),$$

where

$$B^m(\Lambda) = \{\forall y \in L_w^2 : \tfrac{d^l}{dt^l} y \in L_w^2(\Lambda), 0 \le l \le m\}.$$

Proof: see [17].

3. Problem Statement

Consider the following class of nonlinear systems with inequality constraints,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3.1}$$

$$x(a) = x^0,\ x(b) = x^1, \tag{3.2}$$

where $A = (a_{i,j})_{n\times n}$ and $B = (b_{i,j})_{m\times m}$ are constant matrices and $x(t)$ and $u(t)$ are $n \times 1$ and $m \times 1$ state and control vectors respectively. The purpose is to find the optimal control $u(t)$ and the corresponding state trajectory $x(t)$, $t \in (a, b)$ satisfying Eqs. (3.1) and (3.2) while minimizing (or maximizing) the quadratic performance index

$$Z = \frac{1}{2}x^T(b)Gx(b) + \frac{1}{2}\int_a^b (x^T(t)Q(t)x(t) + u^T(t)R(t)u(t))dt, \tag{3.3}$$

where a and b are constant, also $G(t) = (g_{i,j}(t))_{n\times n}$, $Q(t) = (q_{i,j}(t))_{n\times n}$ are symmetric positive semi-definite matrices and $R(t) = (r_{i,j}(t))_{m\times m}$ is a symmetric positive definite matrix.

4. The proposed method

Let

$$x_i(t) \simeq X^i\Phi_n(t), \tag{4.1}$$

$$u_j(t) \simeq U^j\Phi_n(t), \tag{4.2}$$

where $X^i, i = 1,\dots,n$, and $U^j, j = 1,\dots,m$ are $1\times(n+1)$ state and control coefficient vectors respectively. Then using (2.4) we get

$$\dot{x}_i(t) \simeq X^i[D_\phi\Phi_n(t)]. \tag{4.3}$$

Using Eqs. (4.1) and (4.2) we have

$$x(t) \simeq X\Phi_n(t) = [\sum_{j=0}^{n} X_j^1 H_j(t),\dots,\sum_{j=0}^{n} X_j^n H_j(t)], \tag{4.4}$$

$$u(t) \simeq U\Phi_n(t) = [\sum_{j=0}^{n} U_j^1 H_j(t),\dots,\sum_{j=0}^{n} U_j^m H_j(t)], \tag{4.5}$$

where $X = (X_i^k)_{n\times(n+1)}$ and $U = (U_j^r)_{m\times(n+1)}$ are state and control coefficient matrices respectively. The boundary conditions in Eq. (3.2) can be rewritten as

$$x(a) = x^0 = d^0\otimes E\Phi_n(t), \tag{4.6}$$

$$x(b) = x^1 = d^1\otimes E\Phi_n(t), \tag{4.7}$$

where d^0 and d^1 are $n \times 1$ constant vectors, $E = [1, 0\dots, 0]$ is $1 \times (n + 1)$ constant vector, and the symbol $'\otimes'$ denotes Kronecker product [30]. If $x(a)$ or $x(b)$ is unknown in Eq. (3.2), then we put

$$x(a) \simeq X\Phi_n^T(a) = \sum_{j=0}^{n} X_j^1 H_j(a), \dots, \sum_{j=0}^{n} X_j^n H_j(a), \tag{4.8}$$

$$x(b) \simeq X\Phi_n^T(b) = \sum_{j=0}^{n} X_j^1 H_j(b), \dots, \sum_{j=0}^{n} X_j^m H_j(b). \tag{4.9}$$

4.1. **Performance Index Approximation.** By substituting Eqs. (4.4), (4.5) and (4.7) in Eq. (3.3) we get

$$min(max) Z = \frac{1}{2} x^1 G(b) {x^1}^T + \frac{1}{2} X[\int_a^b \Phi_n(t) Q(t) \Phi_n^T(t) dt] X^T + \frac{1}{2} U \int_a^b \Phi_n(t) R(t) \Phi_n(t)^T dt] U^T. \tag{4.10}$$

For problems with time-varying performance index, $Q(t)$ and $R(t)$ are functions of time. Let

$$P_x = \int_a^b \Phi_n(t) Q(t) \Phi_n^T(t) dt, \text{ and } P_u = \int_a^b \Phi_n(t) R(t) \Phi_n^T(t) dt. \tag{4.11}$$

Eq. (4.11) can be evaluated by numerical integration techniques. By substituting Eqs. (4.9) and (4.11) in Eq. (4.10) we get

$$Z[X, U] = \frac{1}{2} X(\hat{P} + P_x) X^T + \frac{1}{2} U P_u U^T, \tag{4.12}$$

where

$$\hat{P} = \Phi_n(b) G(b) \Phi_n^T(b).$$

The boundary conditions in Eq. (3.2) can be expressed as

$$q_k^0 = x_k(a) - x_k^0 \quad , \quad k = 1, \dots, n, \tag{4.13}$$

$$q_k^1 = x_k(b) - x_k^1 \quad , \quad k = 1, \dots, n. \tag{4.14}$$

We now find the extremum of Eq. (4.12) subject to Eqs.(4.13) and (4.14) using the Lagrange multiplier technique. Let

$$Z[X, U, \lambda^0, \lambda^1] = Z[X, U] + \lambda^0 Q^0 + \lambda^1 Q^1. \tag{4.15}$$

where $Q^0 = (q_k^0), k = 1, \dots, n$ and $Q^1 = (q_k^1), k = 1, \dots, n$ are $(n \times 1)$ constant vectors. The necessary condition for the extremum of (4.15) is

$$\nabla Z[X, U, \lambda^0, \lambda^1] = 0. \tag{4.16}$$

5. Illustrative Examples

This section is devoted to numerical examples. We implemented the proposed method in last section with MALAB (2012) in personal computer. To illustrate our technique, we present four numerical examples, and make a comparison with some of the results in the literature.

Example 1 This example is adapted from [18] and also studied by using least square method based on Bezier control points which minimizes

$$Z = \frac{1}{2}\int_0^1 u^2(t)dt, \tag{5.1}$$

subject to

$$\dot{x}_1 = x_2 + u, \tag{5.2}$$

$$x_2 - u = 0, \tag{5.3}$$

with the boundery conditions

$$x_1(0) = 1, \quad x_1(\frac{1}{2}) = x_1(1) = 0. \tag{5.4}$$

Here we solve this problem with Hermite polynomials by choosing $n = 3$. Let

$$x_1(t) = X^1\Phi_3(t), \tag{5.5}$$

$$x_2(t) = X^2\Phi_3(t), \tag{5.6}$$

$$u(t) = U\Phi_3(t), \tag{5.7}$$

where

$$X^1 = [X_0^1, X_1^1, X_2^1, X_3^1],$$
$$X^2 = [X_0^2, X_1^2, X_2^2, X_3^2],$$

and

$$U = [U_0, U_1, U_2, U_3].$$

Using Eqs. (2.4), (5.5) and (5.6) we get

$$\dot{x}_1(t) = X^1[D_\phi\Phi_3(t)], \tag{5.8}$$

$$\dot{x}_2(t) = X^2[D_\phi\Phi_3(t)], \tag{5.9}$$

where D_ϕ is the operational matrix of derivative given in Eq. (2.5). By substituting Eqs. (5.5)-(5.9) in Eqs. (5.2) and (5.3) we obtain

$$[X^1D_\phi - X^2 - U]\Phi_3(t) = 0, \tag{5.10}$$

$$[X^2 - U]\Phi_3(t) = 0. \tag{5.11}$$

Let

$$Z_{\chi[t_0,t_1]} = \frac{1}{2}\int_{t_0}^{t_1} u^2(t)dt, \tag{5.12}$$

Using Eq. (5.7) in Eq. (5.1) we have

$$\begin{aligned} Z_{\chi[t_0,t_1]} &= \frac{1}{2}\int_{t_0}^{t_1} u^2(t)dt = \frac{1}{2}\int_{t_0}^{t_1} (U\Phi_3(t))(U\Phi_3(t))^T dt \\ &= \frac{1}{2}\int_{t_0}^{t_1} (U\Phi_3(t))(\Phi_3^T(t)U^T)dt = \frac{1}{2}U\left(\int_{t_0}^{t_1} \Phi_3(t)\Phi_3^T(t)dt\right)U^T \\ &= \frac{1}{2}UV_{\chi[t_0,t_1]}U^T, \end{aligned} \tag{5.13}$$

where $V_{\chi[t_0,t_1]} = \int_{t_0}^{t_1} \Phi_3(t)\Phi_3^T(t)dt$ is of order (4×4) constant matrix. Using the Lagrange multiplier technique to find the extremum of (5.13) subject to the conditions (5.4) ,(5.10) and (5.11), we have

$$Z[U,\lambda_1,\lambda_2,\lambda_3] = \begin{cases} Z[U,\lambda_1,\lambda_2] = Z[U] + \lambda_1 Q_1 + \lambda_2 Q_2, & t \in [0,\frac{1}{2}], \\ Z[U,\lambda_2,\lambda_3] = Z[U] + \lambda_2 Q_2 + \lambda_3 Q_3, & t \in [\frac{1}{2},1], \end{cases} \tag{5.14}$$

where $Q_1 = X^1\Phi_3(0) - 1$,$Q_2 = X^1\Phi_3(\frac{1}{2})$ and $Q_3 = X^1\Phi_3(1)$.
The necessary conditions are

$$\nabla Z[U,\lambda_1,\lambda_2,\lambda_3] = 0. \tag{5.15}$$

The exact solutions of the problem are:

$$x_1(t) = \begin{cases} -2t+1, & t \in [0,\frac{1}{2}], \\ 0, & t \in [\frac{1}{2},1], \end{cases}$$

$$x_2(t) = \begin{cases} -1, & t \in [0,\frac{1}{2}], \\ 0, & t \in [\frac{1}{2},1], \end{cases}$$

$$u(t) = \begin{cases} -1, & t \in [0,\frac{1}{2}], \\ 0, & t \in [\frac{1}{2},1]. \end{cases}$$

We obtain the state and control solutions as following:

TABLE 1. Results for Example 1. and t$\in [0, \frac{1}{2}]$

New Method	Method[47]	Method[18]
$X^1 : [1,-1,0,0]$	$[8, \frac{16}{3}, \frac{8}{3}, 0]$	$[8, \frac{16.02400}{3}, \frac{7.97601}{3}, 0]$
$X^2 : [-1,0,0,0]$	$[-8,-8,-8,-8]$	$[-7.97601, \frac{-24.07199}{3}, \frac{-24.07199}{3}, -7.96601]$
$U : [-1,0,0,0]$	$[-8,-8,-8,-8]$	$[-7.976003, \frac{-24.07198}{3}, \frac{-24.07198}{3}, -7.97601]$
Z : 0.25	0.25	0.2500003224

for $t \in [0, \frac{1}{2}]$ we obtain

$$x_1(t) = X^1\Phi_3(t) = H_0(t) - H_1(t) + 0H_2(t) + 0H_3(t) = -2t + 1.$$

$$x_2(t) = u(t) = X^2\Phi_3(t) = -H_0(t) + 0H_1(t) + 0H_2(t) + 0H_3(t) = -1.$$

for $t \in [\frac{1}{2}, 1]$
$X^1 = X^2 = U = [0,0,0,0]$, then we have $x_1(t) = x_2(t) = u(t) = 0$,

TABLE 2. The approximate values of Z, for Example 2.

n	Presented Method	error
3	2.792372881355932	$7.1290e-004$
4	2.791662024685567	$2.0516e-006$
5	2.791660082922831	$1.0761e-007$
6	2.791659975445970	$1.3590e-010$
7	2.791659975313821	$3.7578e-012$
8	2.791659975310065	$2.0228e-015$

which is the exact solution.

Example 2 This example is adapted from [40] and also studied by using Bezier parameterization for optimal control by differential evolution

$$\min Z = \int_0^1 (3x^2(t) + u^2(t))dt \tag{5.16}$$

subject to

$$\dot{x}(t) = x(t) + u(t),$$
$$x(0) = 1.$$

Let $n = 3$ then we have

$$x(t) = X\Phi_3(t), \quad u(t) = U\Phi_3(t). \tag{5.17}$$

We obtain the state and control solutions which are presented in Table 2, and the analytical solutions are [40]:

$$x(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} + \frac{1}{3e^{-4}+1}e^{-2t},$$
$$u(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} - \frac{3}{3e^{-4}+1}e^{-2t}. \tag{5.18}$$

The optimal values of objective functional is:

$$Z^* = 2.791659975310063.$$

Figure 1 and Figure 2 show the errors for state and control functions for example 2.

Example 3[40] Find the minimum of the functional

$$Z = \int_0^1 (x_2(t) + u^2(t))dt \tag{5.19}$$

subject to

$$\dot{x}_1(t) = x_2(t),\ x_1(0) = 0,\ x_1(1) = 1 \tag{5.20}$$
$$\dot{x}_2(t) = u(t),\ x_2(0) = 0. \tag{5.21}$$

The exact solutions are:

$$x_1(t) = \frac{3}{2}t^2 - \frac{1}{2}t^3, \quad x_2(t) = 3t - \frac{3}{2}t^2, \quad u(t) = 3 - 3t.$$

FIGURE 1. Plots of errors for state (left) and control (right) functions for n=3

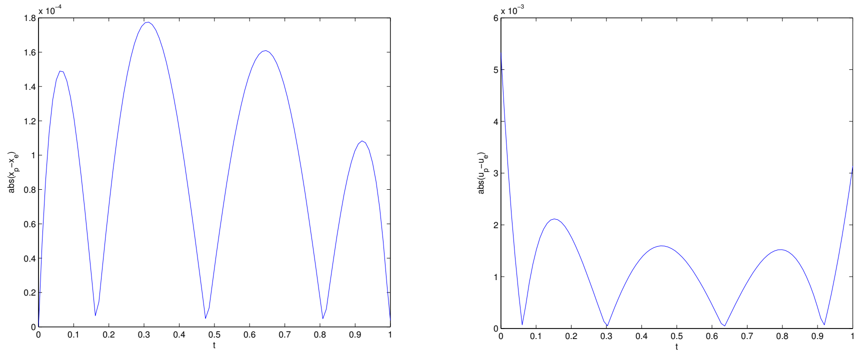
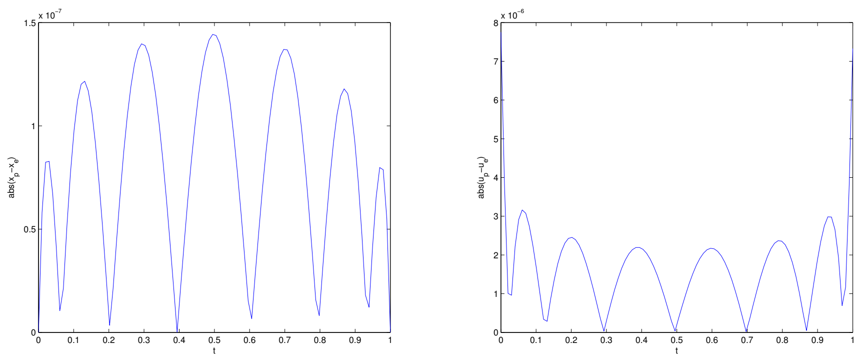


FIGURE 2. Plots of errors for state (left) and control (right) functions for n=7



For $n = 3$ we get the exact solution as:

$$X^1 = [\frac{3}{4}, -\frac{3}{8}, \frac{3}{8}, -\frac{1}{16}]$$
$$\to x_1(t) = \frac{3}{4}H_0(t) - \frac{3}{8}H_1(t) + \frac{3}{8}H_2(t) - \frac{1}{16}H_3(t) = \frac{3}{2}t^2 - \frac{1}{2}t^3.$$
$$X^2 = [-\frac{3}{4}, \frac{3}{2}, -\frac{3}{8}, 0]$$
$$\to x_2(t) = -\frac{3}{4}H_0(t) + \frac{3}{2}H_1(t) - \frac{3}{8}H_2(t) = 3t - \frac{3}{2}t^2.$$
$$U = [3, -\frac{3}{2}, 0, 0] \to u(t) = 3H_0(t) - \frac{3}{2}H_1(t) = 3 - 3t.$$
$$Z = 4.$$

TABLE 3. The approximate values of Z, for Example 4.

n	Presented Method	error
3	0.7616037565468665	$9.6006e-06$
4	0.7615941626680414	$6.7123e-09$
5	0.7615941563301999	$7.7435e-011$
6	0.7615941559557914	$2.6534e-014$
7	0.7615941559557651	$2.2204e-016$

Example 4 Find the extremum of the functional [12]

$$Z = \int_0^1 (x^2(t) + u^2(t))dt, \tag{5.22}$$

where

$$\dot{x}(t) = u(t), \tag{5.23}$$

with the conditions

$$x(0) = 1, \;\; x(1) \;\; unspecified, \tag{5.24}$$

By using Euler equation and free payoff term the exact solutions as following:

$$x(t) = -\frac{\sinh(1-t)}{\cosh 1},$$
$$u(t) = \frac{\cosh(1-t)}{\cosh 1},$$
$$Z = 0.761594155955765.$$

Euler equation and free payoff term respectively are:

$$\text{Euler equation:} \qquad \frac{\partial F}{\partial x} - \frac{d}{dt}(\frac{\partial F}{\partial \dot{x}}) = 0,$$
$$\text{free payoff term:} \qquad \frac{\partial F}{\partial \dot{x}}(1) = 0,$$

by using Eq. (5.23) $F(x, u)$ as following:

$$F(x, u) = F(x, \dot{x}) = x^2 + \dot{x}^2.$$

With free payoff term the other condition is obtained as following:

$$\dot{x}(1) = 0. \tag{5.25}$$

only be used free payoff term in the solving problemes with presented method and isnt nesessary Euler equation. Figure 3 and figure 4 show plots of errors for state and control functions for example 4. Table 4 shows the optimal values of objective functional.

FIGURE 3. Plots of errors for state (left) and control (right) functions for n=3

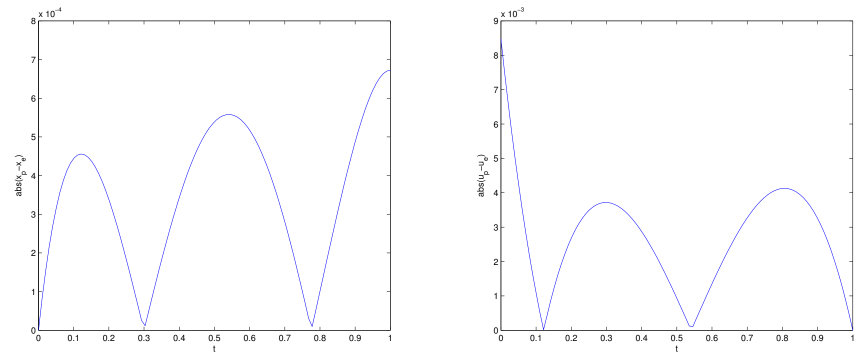
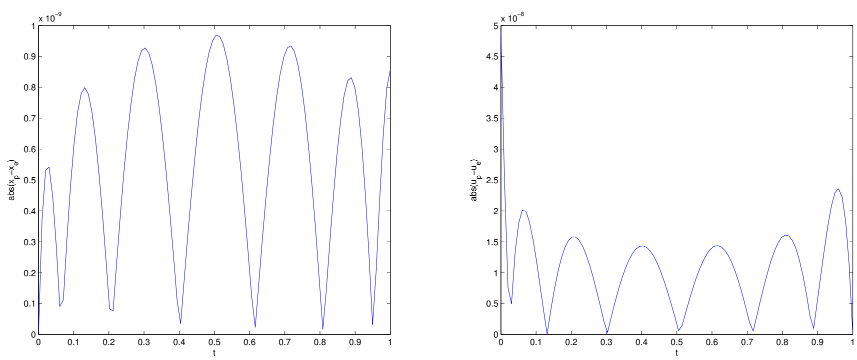


FIGURE 4. Plots of errors for state (left) and control (right) functions for n=7



Example 5 Consider the performance measure [29]

$$Z = \frac{1}{2}\int_0^2 (\dot{x_1}(t) + \dot{x_2}(t))^2 dt \tag{5.26}$$

where

$$\dot{x_1}(t) = x_2(t), \ \dot{x_2}(t) = -x_2(t) + u(t), \tag{5.27}$$

with the conditions

$$x_1(0) = x_2(0) = 0, \tag{5.28}$$

$$x_1(2) + 5x_2(2) = 15. \tag{5.29}$$

FIGURE 5. Plots of state and control functions for n=5

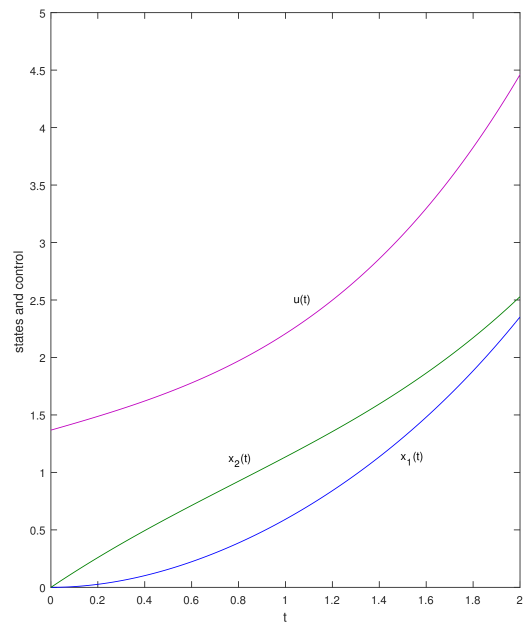
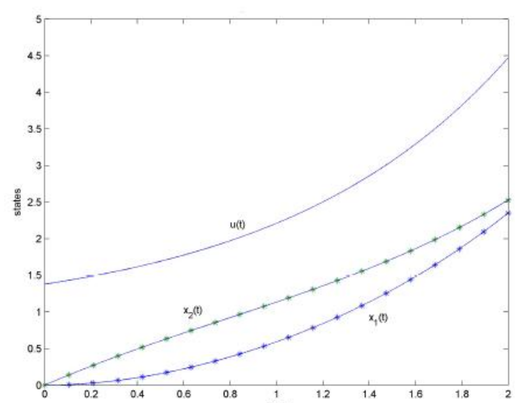


FIGURE 6. Plots of exact solutions for state and control functions with[29]



6. Conclusion

In this paper, we presented a numerical scheme for solving quadratic optimal control problems with liner constrained. The Hemite polynomials was employed. Also several test problems were used to see the applicability and efficiency of the method. The obtained results show that the new approach can solve the problem effectively.

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