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Application of spline to approximate the solution of singularly perturbed boundary-value problems

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Abstract We develop a class of new methods based on modification of polynomial spline function for the numerical solution of singularly perturbed boundary-value problems. The modified spline contains the exponential terms and named by tension spline, which is infinity smooth. Tension spline contain parameter, by choosing arbitrary values of such parameters the various classes of spline can be obtained. The proposed methods are accurate for solution of linear and non-linear singularly perturbed boundary-value problems. Boundary formulas are developed to associate with spline methods. These methods are converging. The analysis of convergence is shown to yield up to $O(h^8)$ approximation to the solution of singularly perturbed boundary-value problems. Comparison are carried out, numerical examples are given to showing the efficiency of our methods.

Keywords. Singularly perturbed boundary-value problems, Tension spline, Boundary formula, Convergence analysis.

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1. Introduction

Consider the following second-order singularly perturbed boundary value problem:

$$-\epsilon u^{''}(x) = f(x,u), \quad a < x < b, \quad a, b, x \in R \tag{1.1}$$

with the following boundary conditions

$$u(a) = \lambda_1, \quad u(b) = \lambda_2 \tag{1.2}$$

whereλ_1, λ_2 are finite real constants and $u(x)$ and $f(x,u)$ are continuous functions defined in the interval $[a,b]$ and ϵ is a parameter such that $0 < \epsilon < 1$. Singularly perturbed problems occurs naturally in various fields of science and engineering, for example, combustion, nuclear engineering, control theory, elasticity, fluid mechanics, fluid dynamics, quantum mechanics, optimal control, chemical reactor theory, hydrodynamics, convection-diffusion process, geophysics, etc. Secondly, the occurrence of sharp boundary-layers as ϵ, the coefficient of highest derivative, approaches zero creates difficulty for most standard numerical schemes. Many researcher try to over come such difficulty, recently.

A collocation method with tension splines, sextic spline, cubic spline, quintic spline, spline approximation method, trigonometric quintic B-splines, Asymptotic initial-value method and a three-point difference scheme, in solving singularly perturbed boundary-value problems has been of considerable concern and is well covered in papers see [1, 2, 6, 8, 9, 10, 11, 16, 17, 18, 19, 25, 27]. Mohanty et al. [12] introduced convergent spline in tension methods for singularly perturbed two-point singular boundary value problems. Mohanty et al. [13]- [15] used spline in compression method, a class of variable mesh spline in compression methods and a family of non-uniform mesh tension spline methods for the solution of singularly perturbed two point singular boundary value problems. EL-Danaf et al. [3]- [5] applied finite difference method, cubic B-spline and Non-polynomial spaline for solving the GRLW equation. B-spline method for the solution of KDV equation has been developed by Raslan et al. [20]- [22]. Sine-cosine functions method for the solution of CGRLW equation has been developed by Raslan et al. [23]. Also Raslan et al. [24] used modified extended tanh method for the solving the space-time fractional RLW and MRLW equations.

The paper is organized in four sections. We use the consistency relation of tension spline for approximate the solution of singularly perturbed boundary-value problems. Section 2 is devoted to the description of the method and development of boundary conditions and also we obtain the method of order 16th. The new approach for convergence analysis is discussed in section 3. Finally, in section 4, numerical evidences are included to show the practical applicability of our method. The obtained computation results are compared with the other existing methods.

2. Formulation of spline and numerical methods

To develop the tension spline approximation for the singularly perturbed boundary-value problem Eqs. (1.1) and (1.2), First of all the interval $[a,b]$ is divided into n equal

Table 1: Maximum absolute errors of example 1 (n=16)

ϵ	Our method	method in [19]	method in [17]	method in [16]
1×10^{-4}	0	1.776×10^{-15}	6.5×10^{-5}	2.6×10^{-2}
1×10^{-5}	0	1.776×10^{-15}	3.6×10^{-5}	2.4×10^{-2}
1×10^{-6}	0	1.776×10^{-15}	3.3×10^{-5}	1.7×10^{-2}
1×10^{-7}	0	1.776×10^{-15}	2.6×10^{-5}	6.9×10^{-3}
1×10^{-8}	0	1.776×10^{-15}	2.0×10^{-5}	2.3×10^{-3}
1×10^{-9}	0	1.776×10^{-15}	2.0×10^{-5}	7.6×10^{-4}
1×10^{-10}	0	3.552×10^{-15}	1.1×10^{-5}	2.4×10^{-4}

Table 2: Maximum absolute errors of example 1 (n=32)

ϵ	Our method	method in [19]	method in [17]	method in [16]
1×10^{-4}	0	1.776×10^{-15}	5.9×10^{-5}	6.5×10^{-3}
1×10^{-5}	0	1.776×10^{-15}	2.1×10^{-5}	6.4×10^{-3}
1×10^{-6}	0	1.776×10^{-15}	3.5×10^{-5}	5.6×10^{-3}
1×10^{-7}	0	1.776×10^{-15}	3.9×10^{-5}	3.1×10^{-3}
1×10^{-8}	0	1.776×10^{-15}	2.1×10^{-5}	1.2×10^{-3}
1×10^{-9}	0	1.776×10^{-15}	2.1×10^{-5}	3.8×10^{-4}
1×10^{-10}	0	3.552×10^{-15}	1.4×10^{-5}	1.3×10^{-4}

subintervals using the grid:

$$x_0 = a, \quad x_i = a + ih, \quad h = \frac{b-a}{n}, \quad i = 0, 1, ..., n, \quad x_n = b.$$

We define the following tension spline S_i on each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$,

$$S_i(x) = \sum_{j=0}^{7} a_{ij}(x - x_i)^j + b_i(e^{k(x-x_i)} + e^{-k(x-x_i)}) + c_i(e^{k(x-x_i)} - e^{-k(x-x_i)}), \tag{2.1}$$

where k is arbitrary parameter, $a_{ij}(j = 0, 1, .., 6, 7), b_i$ and c_i are real finite coefficients which have to be determine so that, the spline is defined in terms of its 1^{th}, 2^{th}, 3^{th} and 8^{th} derivatives and we denote these values at knots as:

$$S_i(x_l) = u_l, S_i^{'}(x_l) = m_l, S_i^{''}(x_l) = v_l, S_i^{(3)}(x_l) = z_l, S_i^{(8)}(x_l) = p_l$$

for

$$i = 0, 1, 2, ..., n-1, l = i, i+1. \tag{2.2}$$

Assuming $u(x)$ to be the function which has be interpolated by $S_i(x)$, and u_i be an approximation to $u(x_i)$, using the continuity conditions of fourth, fifth, sixth and seventh ($S_{i-1}^{(\mu)}(x_i) = S_i^{(\mu)}(x_i)$ where $\mu = 4, 5, 6$ and 7), and also by eliminating of m_i, z_i and p_i, we obtain the consistency relation for second derivative of tension spline(2.1):

$$\begin{gathered} \alpha_1 v_{i-4} + \alpha_2 v_{i-3} + \alpha_3 v_{i-2} + \alpha_4 v_{i-1} + \alpha_5 v_i + \alpha_4 v_{i+1} + \alpha_3 v_{i+2} + \\ \alpha_2 v_{i+3} + \alpha_1 v_{i+4} = \tfrac{-1}{h^2}(\beta_1 u_{i-4} + \beta_2 u_{i-3} + \beta_3 u_{i-2} + \beta_4 u_{i-1} + \beta_5 u_i + \\ \beta_4 u_{i+1} + \beta_3 u_{i+2} + \beta_2 u_{i+3} + \beta_1 u_{i+4}), \quad i = 4, 5, ..., n-4, \end{gathered} \tag{2.3}$$

Table 3: Maximum absolute errors of example 2 (Our sixteen-order method)

n	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$	$\epsilon = \frac{1}{256}$	$\epsilon = \frac{1}{512}$
16	1.72×10^{-11}	1.47×10^{-11}	1.03×10^{-10}	1.21×10^{-8}	7.40×10^{-7}	2.37×10^{-5}
32	1.52×10^{-17}	2.20×10^{-17}	2.14×10^{-15}	4.90×10^{-13}	8.38×10^{-11}	9.55×10^{-9}
64	2.66×10^{-22}	3.15×10^{-22}	2.14×10^{-20}	7.24×10^{-18}	2.10×10^{-15}	4.89×10^{-13}
128	1.28×10^{-27}	1.56×10^{-27}	1.33×10^{-25}	5.52×10^{-23}	2.12×10^{-20}	7.24×10^{-18}
256	5.21×10^{-33}	6.54×10^{-33}	6.52×10^{-31}	2.99×10^{-28}	1.32×10^{-25}	5.52×10^{-23}
512	2.02×10^{-38}	2.58×10^{-38}	2.81×10^{-36}	1.35×10^{-33}	6.47×10^{-31}	2.98×10^{-28}
1024	7.78×10^{-44}	1.00×10^{-43}	1.14×10^{-41}	5.66×10^{-39}	2.79×10^{-36}	1.35×10^{-33}

Table 4: Maximum absolute errors of example 2 (Our twelve-order method)

n	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$	$\epsilon = \frac{1}{256}$	$\epsilon = \frac{1}{512}$
16	1.72×10^{-11}	8.36×10^{-11}	1.04×10^{-10}	1.21×10^{-8}	7.43×10^{-7}	2.38×10^{-5}
32	9.89×10^{-17}	5.75×10^{-17}	2.23×10^{-15}	4.95×10^{-13}	8.43×10^{-11}	9.59×10^{-9}
64	2.39×10^{-20}	1.39×10^{-20}	1.63×10^{-19}	1.11×10^{-17}	2.19×10^{-15}	4.95×10^{-13}
128	5.77×10^{-24}	3.36×10^{-24}	4.24×10^{-23}	2.54×10^{-21}	1.58×10^{-19}	1.11×10^{-17}
256	1.40×10^{-27}	8.19×10^{-28}	1.06×10^{-26}	6.53×10^{-25}	4.09×10^{-23}	2.54×10^{-21}
512	3.42×10^{-31}	1.99×10^{-31}	2.62×10^{-30}	1.62×10^{-28}	1.03×10^{-26}	6.53×10^{-25}
1024	8.36×10^{-35}	4.87×10^{-35}	6.42×10^{-34}	3.97×10^{-32}	2.53×10^{-30}	1.62×10^{-28}

Table 5: Maximum absolute errors of example 2 (Our eighth-order method)

n	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$	$\epsilon = \frac{1}{256}$	$\epsilon = \frac{1}{512}$
16	2.62×10^{-10}	1.50×10^{-10}	2.48×10^{-10}	6.00×10^{-9}	6.81×10^{-7}	2.39×10^{-5}
32	8.85×10^{-13}	4.86×10^{-13}	1.62×10^{-12}	2.16×10^{-11}	2.46×10^{-10}	4.89×10^{-9}
64	3.24×10^{-15}	1.75×10^{-15}	8.35×10^{-15}	1.13×10^{-13}	1.54×10^{-12}	2.14×10^{-11}
128	1.24×10^{-17}	6.67×10^{-18}	3.61×10^{-17}	5.26×10^{-16}	7.84×10^{-15}	1.12×10^{-13}
256	4.83×10^{-20}	2.58×10^{-20}	1.45×10^{-19}	2.17×10^{-18}	3.39×10^{-17}	5.24×10^{-16}
512	1.88×10^{-22}	1.00×10^{-22}	5.73×10^{-22}	8.62×10^{-21}	1.36×10^{-19}	2.16×10^{-18}
1024	7.36×10^{-25}	3.94×10^{-25}	2.24×10^{-24}	3.38×10^{-23}	5.38×10^{-22}	8.59×10^{-21}

Table 6: Maximum absolute errors of example 2 (Our fourth-order method)

n	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$	$\epsilon = \frac{1}{256}$	$\epsilon = \frac{1}{512}$
16	1.28×10^{-4}	6.99×10^{-5}	2.13×10^{-5}	5.08×10^{-5}	8.53×10^{-5}	1.88×10^{-4}
32	7.35×10^{-6}	3.74×10^{-6}	4.79×10^{-6}	1.33×10^{-5}	3.14×10^{-5}	5.76×10^{-5}
64	4.32×10^{-7}	2.14×10^{-7}	4.72×10^{-7}	1.49×10^{-6}	4.65×10^{-6}	1.29×10^{-5}
128	2.65×10^{-8}	1.30×10^{-8}	3.39×10^{-8}	1.16×10^{-7}	4.18×10^{-7}	1.44×10^{-6}
256	1.64×10^{-9}	8.08×10^{-10}	2.20×10^{-9}	7.81×10^{-9}	2.97×10^{-8}	1.13×10^{-7}
512	1.02×10^{-10}	5.04×10^{-11}	1.39×10^{-10}	4.98×10^{-10}	1.93×10^{-9}	7.59×10^{-9}
1024	6.43×10^{-12}	3.15×10^{-12}	8.74×10^{-12}	3.13×10^{-11}	1.21×10^{-10}	4.84×10^{-10}

Table 7: Maximum absolute errors of example 2 (Eighth-order method in [18])

n	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$	$\epsilon = \frac{1}{256}$	$\epsilon = \frac{1}{512}$
16	6.91×10^{-9}	8.48×10^{-9}	5.86×10^{-8}	8.35×10^{-7}	2.49×10^{-5}	1.84×10^{-4}
32	2.04×10^{-11}	1.57×10^{-10}	1.36×10^{-9}	2.84×10^{-9}	6.24×10^{-7}	4.61×10^{-6}
64	6.86×10^{-14}	3.71×10^{-14}	1.92×10^{-13}	4.97×10^{-12}	3.90×10^{-9}	2.88×10^{-8}
128	2.51×10^{-15}	1.44×10^{-15}	3.44×10^{-15}	1.36×10^{-14}	9.76×10^{-11}	7.21×10^{-10}
256	3.07×10^{-16}	3.08×10^{-16}	1.27×10^{-15}	1.06×10^{-14}	2.48×10^{-12}	1.80×10^{-11}

Table 8: Maximum absolute errors of example 2 (Fourth-order method in [2])

n	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$
16	1.57×10^{-5}	8.27×10^{-6}	1.84×10^{-5}	1.03×10^{-4}
32	8.79×10^{-7}	4.41×10^{-7}	8.67×10^{-7}	2.61×10^{-6}
64	5.32×10^{-8}	2.62×10^{-8}	6.65×10^{-8}	2.23×10^{-7}
128	3.30×10^{-9}	1.62×10^{-9}	4.39×10^{-9}	1.54×10^{-8}
256	2.05×10^{-10}	1.00×10^{-10}	2.74×10^{-10}	9.44×10^{-10}

Table 9: Maximum absolute errors of example 2 (Fourth-order method in [19])

n	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$
16	4.07×10^{-5}	2.00×10^{-5}	5.45×10^{-5}	1.83×10^{-4}
32	2.53×10^{-6}	1.24×10^{-6}	3.42×10^{-6}	1.22×10^{-5}
64	1.58×10^{-7}	7.74×10^{-8}	2.14×10^{-7}	7.68×10^{-7}
128	9.87×10^{-9}	4.83×10^{-9}	1.34×10^{-8}	4.81×10^{-8}
256	6.17×10^{-10}	3.02×10^{-10}	8.39×10^{-10}	3.01×10^{-9}

Table 10: Maximum absolute errors of example 2 (method in [25])

n	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$
16	8.06×10^{-3}	7.11×10^{-3}	6.58×10^{-3}	6.36×10^{-3}
32	2.02×10^{-3}	1.79×10^{-3}	1.66×10^{-3}	1.61×10^{-3}
64	5.08×10^{-4}	4.48×10^{-4}	4.15×10^{-4}	4.03×10^{-3}
128	1.27×10^{-4}	1.12×10^{-4}	1.04×10^{-4}	1.01×10^{-4}
256	3.17×10^{-5}	2.80×10^{-5}	2.60×10^{-5}	2.52×10^{-5}

where
$\alpha_1 = \frac{5}{\gamma_1}(-7! + \theta(7! + 840\theta^2 + 42\theta^4 + \theta^6)csch(\theta))$,
$\alpha_2 = \frac{-5}{\gamma_2}(\theta(7! + 840\theta^2 + 42\theta^4 + \theta^6)cosh(\theta) - 12(-1260\theta + 42\theta^5 + 5\theta^7 + 1680Sinh(\theta)))$,
$\alpha_3 = \frac{-5}{\gamma_3}(84\theta(-120+10\theta^2-\theta^4)-149\theta^7+6\theta(-1260+42\theta^4+5\theta^6)cosh(\theta)+17640sinh(\theta))$,
$\alpha_4 = \frac{-5}{\gamma_2}(\gamma_7\ cosh(\theta) - 4(-163800\theta + 1680\theta^3 - 294\theta^5 + 317\theta^7 + 35280sinh(\theta)))$,
$\alpha_5 = \frac{-5}{\gamma_2}(-\gamma_7 + 16\theta(-6300 + 840\theta^2 - 210\theta^4 + 151\theta^6)cosh(\theta) + 176400sinh(\theta))$,
$\beta_1 = \frac{-1}{\gamma_4}(5\theta^2 csch(\theta)(120\theta + 20\theta^3 + \theta^5 - 120sinh(\theta)))$,
$\beta_2 = \frac{1}{\gamma_5}(5\theta^2(\theta(120 + 20\theta^2 + \theta^4)cosh(\theta) - 12(-30\theta + \theta^5 + 40Sinh(\theta))))$,
$\beta_3 = \frac{1}{\gamma_6}(10\theta^2(-\theta(120 - 10\theta^2 + \theta^4) + 3\theta(-30 + \theta^4)cosh(\theta) + 210sinh(\theta)))$,
$\beta_4 = \frac{1}{\gamma_5}(5\theta^2(15\theta(120 - 12\theta^2 + \theta^4)cosh(\theta) + 4(390\theta - 40\theta^3 + 7\theta^5 - 840sinh(\theta))))$,
$\beta_5 = \frac{-1}{\gamma_5}(25\theta^2(360\theta - 36\theta^3 + 3\theta^5 + 16\theta(30 - 4\theta^2 + \theta^4)cosh(\theta) - 840sinh(\theta)))$,
$\gamma_1 = 1008(60(-42 + \theta^2) + \theta(2520 + 360\theta^2 + 11\theta^4)csch(\theta))$,
$\gamma_2 = 504(\theta(2520 + 360\theta^2 + 11\theta^4) + 60(-42 + \theta^2)sinh(\theta))$,
$\gamma_3 = 126(\theta(2520 + 360\theta^2 + 11\theta^4) + 60(-42 + \theta^2)sinh(\theta))$,
$\gamma_4 = 24(60(-42 + \theta^2) + \theta(2520 + 360\theta^2 + 11\theta^4)csch(\theta))$,
$\gamma_5 = 12(\theta(2520 + 360\theta^2 + 11\theta^4) + 60(-42 + \theta^2)sinh(\theta))$,
$\gamma_6 = 3(\theta(2520 + 360\theta^2 + 11\theta^4) + 60(-42 + \theta^2)sinh(\theta))$,
$\gamma_7 = 3\theta(210(120 - 12\theta^2 + \theta^4) + 397\theta^6)$,
and $\theta = kh$.

The above relation is useful to solve singularly perturbed boundary-value problems (1.1)-(1.2), but if $k \to 0, (\theta = kh)\theta \to 0$
then
$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to -(\frac{1}{72}, \frac{251}{36}, \frac{1826}{9}, \frac{44117}{36}, \frac{78095}{36})$ and
$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) \to (1, 118, 952, 154, -2450)$,
and the relations defined by (2.3) reduce into nonic degree spline function which is special case of exponential spline. We assume that

$$-\epsilon u_i^{''} = f(x_i, u_i) = f_i \equiv f(x_i, u(x_i)), \tag{2.4}$$

Table 11: Maximum absolute errors of example 3 (Our sixteen-order method)

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	2.11×10^{-15}	4.89×10^{-16}	8.38×10^{-11}	9.55×10^{-9}	6.40×10^{-7}	2.21×10^{-5}
32	2.12×10^{-20}	7.24×10^{-18}	2.10×10^{-15}	4.89×10^{-13}	8.38×10^{-11}	9.55×10^{-9}
64	1.32×10^{-25}	5.52×10^{-23}	2.12×10^{-20}	7.24×10^{-18}	2.10×10^{-15}	4.89×10^{-13}
128	6.47×10^{-31}	2.98×10^{-28}	1.32×10^{-25}	5.52×10^{-23}	2.12×10^{-20}	7.24×10^{-18}
256	2.79×10^{-36}	1.35×10^{-33}	6.47×10^{-31}	2.98×10^{-28}	1.32×10^{-25}	5.52×10^{-23}
512	1.13×10^{-41}	5.66×10^{-39}	2.79×10^{-36}	1.35×10^{-33}	6.47×10^{-31}	2.98×10^{-28}
1024	4.46×10^{-47}	2.25×10^{-44}	1.13×10^{-41}	5.66×10^{-39}	2.79×10^{-36}	1.35×10^{-33}

Table 12: Maximum absolute errors of example 3 (Our twelve-order method)

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	2.20×10^{-15}	4.95×10^{-13}	8.43×10^{-11}	9.59×10^{-9}	6.42×10^{-7}	2.21×10^{-5}
32	1.56×10^{-19}	1.11×10^{-17}	2.19×10^{-15}	4.95×10^{-13}	8.43×10^{-11}	9.59×10^{-9}
64	4.05×10^{-23}	2.54×10^{-21}	1.58×10^{-19}	1.11×10^{-17}	2.19×10^{-15}	4.95×10^{-13}
128	1.02×10^{-26}	6.52×10^{-25}	4.09×10^{-23}	2.54×10^{-21}	1.58×10^{-19}	1.11×10^{-17}
256	2.51×10^{-30}	1.61×10^{-28}	1.03×10^{-26}	6.53×10^{-25}	4.09×10^{-23}	2.54×10^{-21}
512	6.16×10^{-34}	3.97×10^{-32}	2.53×10^{-30}	1.62×10^{-28}	1.03×10^{-26}	6.53×10^{-25}
1024	1.50×10^{-37}	9.70×10^{-36}	6.21×10^{-34}	3.97×10^{-32}	2.53×10^{-30}	1.62×10^{-28}

Table 13: Maximum absolute errors of example 3 (Our eighth-order method)

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	1.50×10^{-12}	2.15×10^{-11}	2.46×10^{-10}	4.89×10^{-9}	5.95×10^{-7}	2.23×10^{-5}
32	7.75×10^{-15}	1.12×10^{-13}	1.54×10^{-12}	2.14×10^{-11}	2.46×10^{-10}	4.89×10^{-9}
64	3.36×10^{-17}	5.23×10^{-16}	7.84×10^{-15}	1.12×10^{-13}	1.54×10^{-12}	2.14×10^{-11}
128	1.35×10^{-19}	2.16×10^{-18}	3.39×10^{-17}	5.24×10^{-16}	7.84×10^{-15}	1.12×10^{-13}
256	5.34×10^{-22}	8.59×10^{-21}	1.36×10^{-19}	2.16×10^{-18}	3.39×10^{-17}	5.24×10^{-16}
512	2.09×10^{-24}	3.37×10^{-33}	5.38×10^{-22}	8.59×10^{-21}	1.36×10^{-19}	2.16×10^{-18}
1024	8.17×10^{-27}	1.31×10^{-25}	2.10×10^{-24}	3.37×10^{-23}	5.38×10^{-22}	8.59×10^{-21}

Table 14: Maximum absolute errors of example 3 (Our fourth-order method)

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	4.44×10^{-6}	1.28×10^{-5}	3.11×10^{-5}	5.74×10^{-5}	8.78×10^{-5}	1.79×10^{-4}
32	4.11×10^{-7}	1.44×10^{-6}	4.62×10^{-6}	1.29×10^{-5}	3.11×10^{-5}	5.74×10^{-5}
64	2.93×10^{-8}	1.12×10^{-7}	4.16×10^{-7}	1.44×10^{-6}	4.62×10^{-6}	1.29×10^{-5}
128	1.90×10^{-9}	7.58×10^{-9}	2.95×10^{-8}	1.13×10^{-7}	4.16×10^{-7}	1.44×10^{-6}
256	1.20×10^{-10}	4.83×10^{-10}	1.92×10^{-9}	7.58×10^{-9}	2.96×10^{-8}	1.13×10^{-7}
512	7.54×10^{-12}	3.03×10^{-11}	1.21×10^{-10}	4.83×10^{-10}	1.92×10^{-9}	7.58×10^{-9}
1024	4.71×10^{-13}	1.90×10^{-12}	7.60×10^{-12}	3.03×10^{-11}	1.21×10^{-10}	4.83×10^{-10}

Table 15: Maximum absolute errors of example 3 (Eighth-order method in [18])

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	1.27×10^{-10}	2.84×10^{-9}	5.39×10^{-8}	8.31×10^{-7}	9.83×10^{-6}	8.43×10^{-5}
32	1.47×10^{-13}	4.96×10^{-12}	1.26×10^{-10}	2.84×10^{-9}	5.39×10^{-8}	8.31×10^{-7}
64	8.32×10^{-15}	7.27×10^{-15}	1.60×10^{-13}	4.96×10^{-12}	1.26×10^{-10}	2.84×10^{-9}
128	2.12×10^{-15}	5.99×10^{-15}	1.38×10^{-14}	8.60×10^{-15}	1.61×10^{-13}	4.96×10^{-12}

Table 16: Maximum absolute errors of example 3 (Fourth-order method in [18])

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	3.95×10^{-3}	7.01×10^{-3}	1.19×10^{-2}	1.94×10^{-2}	2.93×10^{-2}	4.27×10^{-2}
32	1.13×10^{-3}	2.12×10^{-3}	3.92×10^{-3}	7.02×10^{-3}	1.20×10^{-2}	1.96×10^{-2}
64	3.04×10^{-4}	5.84×10^{-4}	1.12×10^{-3}	2.12×10^{-3}	3.93×10^{-3}	7.05×10^{-3}
128	7.88×10^{-5}	1.53×10^{-4}	3.00×10^{-4}	5.35×10^{-4}	1.12×10^{-3}	2.13×10^{-3}

Table 17: Maximum absolute errors of example 4 (Our sixteen-order method)

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	1.71×10^{-14}	2.73×10^{-12}	3.23×10^{-10}	2.58×10^{-8}	1.25×10^{-6}	3.31×10^{-5}
32	1.22×10^{-19}	2.96×10^{-17}	6.40×10^{-15}	1.15×10^{-12}	1.59×10^{-10}	1.51×10^{-8}
64	7.40×10^{-25}	2.26×10^{-22}	6.64×10^{-20}	1.79×10^{-17}	4.23×10^{-15}	8.23×10^{-13}
128	3.60×10^{-30}	1.23×10^{-27}	4.22×10^{-25}	1.39×10^{-22}	4.38×10^{-20}	1.25×10^{-17}
256	1.55×10^{-35}	5.66×10^{-33}	2.08×10^{-30}	7.65×10^{-28}	2.77×10^{-25}	9.75×10^{-23}
512	6.33×10^{-41}	2.36×10^{-38}	9.05×10^{-36}	3.50×10^{-33}	1.36×10^{-30}	5.32×10^{-28}
1024	2.49×10^{-46}	9.45×10^{-44}	3.68×10^{-41}	1.46×10^{-38}	5.92×10^{-36}	2.43×10^{-33}

Table 18: Maximum absolute errors of example 4 (Our twelve-order method)

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	1.76×10^{-14}	2.75×10^{-12}	3.24×10^{-10}	2.59×10^{-8}	1.25×10^{-6}	3.32×10^{-5}
32	8.45×10^{-19}	3.86×10^{-17}	6.63×10^{-15}	1.16×10^{-12}	1.60×10^{-10}	1.51×10^{-8}
64	2.12×10^{-22}	8.38×10^{-21}	3.94×10^{-19}	2.37×10^{-17}	4.39×10^{-15}	8.32×10^{-13}
128	5.29×10^{-26}	2.14×10^{-24}	1.01×10^{-22}	5.18×10^{-21}	2.76×10^{-19}	1.74×10^{-17}
256	1.30×10^{-29}	5.33×10^{-28}	2.57×10^{-26}	1.33×10^{-24}	7.15×10^{-23}	3.88×10^{-21}
512	3.17×10^{-33}	1.30×10^{-31}	6.34×10^{-30}	3.32×10^{-28}	1.80×10^{-26}	1.00×10^{-24}
1024	7.76×10^{-37}	3.19×10^{-35}	1.55×10^{-33}	8.16×10^{-32}	4.44×10^{-30}	2.48×10^{-28}

Table 19: Maximum absolute errors of example 4 (Our eighth-order method)

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	6.81×10^{-12}	5.86×10^{-11}	4.77×10^{-10}	1.45×10^{-8}	1.16×10^{-6}	3.36×10^{-5}
32	3.18×10^{-14}	2.90×10^{-13}	3.04×10^{-12}	3.55×10^{-11}	3.45×10^{-10}	8.43×10^{-9}
64	1.33×10^{-16}	1.34×10^{-15}	1.56×10^{-14}	1.90×10^{-13}	2.27×10^{-12}	2.85×10^{-11}
128	5.33×10^{-19}	5.58×10^{-18}	6.83×10^{-17}	8.91×10^{-16}	1.17×10^{-14}	1.51×10^{-13}
256	2.09×10^{-21}	2.21×10^{-20}	2.76×10^{-19}	3.70×10^{-18}	5.11×10^{-17}	7.10×10^{-16}
512	8.19×10^{-24}	8.70×10^{-23}	1.08×10^{-21}	1.47×10^{-20}	2.06×10^{-19}	2.94×10^{-18}
1024	3.20×10^{-26}	3.40×10^{-35}	4.26×10^{-24}	5.78×10^{-23}	8.13×10^{-22}	1.17×10^{-20}

Table 20: Maximum absolute errors of example 4 (Our fourth-order method)

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	1.44×10^{-5}	2.72×10^{-5}	4.48×10^{-5}	6.54×10^{-5}	9.10×10^{-5}	2.18×10^{-4}
32	1.17×10^{-6}	2.70×10^{-6}	6.71×10^{-6}	1.62×10^{-5}	3.49×10^{-5}	5.99×10^{-5}
64	7.97×10^{-8}	2.10×10^{-7}	6.29×10^{-7}	1.92×10^{-6}	5.58×10^{-6}	1.45×10^{-5}
128	5.12×10^{-9}	1.41×10^{-8}	4.54×10^{-8}	1.54×10^{-7}	5.20×10^{-7}	1.68×10^{-6}
256	3.22×10^{-10}	9.04×10^{-10}	2.96×10^{-9}	1.04×10^{-8}	3.73×10^{-8}	1.33×10^{-7}
512	2.02×10^{-11}	5.68×10^{-11}	1.87×10^{-10}	6.66×10^{-10}	2.43×10^{-9}	9.01×10^{-9}
1024	1.26×10^{-12}	3.55×10^{-12}	1.17×10^{-11}	4.19×10^{-11}	1.53×10^{-10}	5.75×10^{-10}

Table 21: Maximum absolute errors of example 4 (Eighth-order method in [18])

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	4.40×10^{-10}	7.39×10^{-9}	1.10×10^{-7}	1.39×10^{-6}	3.51×10^{-5}	1.71×10^{-5}
32	6.00×10^{-13}	1.33×10^{-11}	2.74×10^{-10}	5.08×10^{-9}	2.20×10^{-6}	1.08×10^{-6}
64	4.77×10^{-15}	2.62×10^{-14}	3.97×10^{-13}	9.18×10^{-12}	1.38×10^{-9}	6.76×10^{-8}
128	1.22×10^{-16}	2.94×10^{-15}	9.10×10^{-15}	1.35×10^{-14}	8.64×10^{-11}	4.23×10^{-10}

Table 22: Maximum absolute errors of example 4 (Fourth-order method in [18])

n	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$	$\epsilon=\frac{1}{512}$
16	5.25×10^{-4}	1.20×10^{-3}	2.62×10^{-3}	5.37×10^{-3}	4.14×10^{-3}	6.36×10^{-3}
32	7.79×10^{-5}	1.91×10^{-4}	4.64×10^{-4}	1.08×10^{-4}	1.02×10^{-4}	1.61×10^{-4}
64	1.05×10^{-5}	2.70×10^{-5}	6.91×10^{-5}	1.74×10^{-4}	2.54×10^{-4}	4.03×10^{-4}
128	1.37×10^{-7}	3.59×10^{-6}	9.45×10^{-6}	2.48×10^{-5}	6.35×10^{-5}	9.01×10^{-5}

Table 23: Maximum absolute errors of example 5

n	20	80	320	1280
E_n	2.34×10^{-27}	4.47×10^{-38}	6.87×10^{-49}	1.01×10^{-59}

where f is nonlinear with respect to u and u_i is the approximation of the exact value $u(x_i)$ and $S_i(x)$ is non-polynomial spline function. Now by substituting (2.4) in the spline relation (2.3), we obtain the nonlinear system of equations in the following form

$$\begin{aligned}
&\beta_1(u_{i-4}+u_{i+4})+\alpha_1 h^2(f(x_{i-4},u_{i-4})+f(x_{i+4},u_{i+4}))+\\
&\beta_2(u_{i-3}+u_{i+3})+\alpha_2 h^2(f(x_{i-3},u_{i-3})+f(x_{i+3},u_{i+3}))+\\
&\beta_3(u_{i-2}+u_{i+2})+\alpha_3 h^2(f(x_{i-2},u_{i-2})+f(x_{i+2},u_{i+2}))+\\
&\beta_4(u_{i-1}+u_{i+1})+\alpha_4 h^2(f(x_{i-1},u_{i-1})+f(x_{i+1},u_{i+1}))+\\
&\beta_5 u_i+\alpha_5 h^2(f(x_i,u_i))=0, \quad i=4,...,(n-4).
\end{aligned} \tag{2.5}$$

By expanding (2.3) in Taylor series about x_i, we obtain the following local truncation error:

$$\begin{aligned}
t_i &= (2(\beta_1+\beta_2+\beta_3+\beta_4)+\beta_5)u_i+(2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_5+16\beta_1+9\beta_2+4\beta_3+\beta_4)h^2u_i^{(2)}+(16\alpha_1+9\alpha_2+4\alpha_3+\alpha_4+\tfrac{64}{3}\beta_1+\tfrac{27}{4}\beta_2+\tfrac{4}{3}\beta_3+\tfrac{1}{12}\beta_4)h^4u_i^{(4)}+\\
&(\tfrac{64}{3}\alpha_1+\tfrac{27}{4}\alpha_2+\tfrac{4}{3}\alpha_3+\tfrac{1}{12}\alpha_4+\tfrac{512}{45}\beta_1+\tfrac{81}{40}\beta_2+\tfrac{8}{45}\beta_3+\tfrac{1}{360}\beta_4)h^6u_i^{(6)}+(\tfrac{512}{45}\alpha_1+\tfrac{81}{40}\alpha_2+\tfrac{8}{45}\alpha_3+\tfrac{1}{360}\alpha_4+\tfrac{1024}{315}\beta_1+\tfrac{729}{2240}\beta_2+\tfrac{4}{315}\beta_3+\tfrac{1}{20160}\beta_4)h^8u_i^{(8)}+\\
&(\tfrac{1024\alpha_1}{315}+\tfrac{729\alpha_2}{2240}+\tfrac{4\alpha_3}{315}+\tfrac{\alpha_4}{20160}+\tfrac{8192\beta_1}{14175}+\tfrac{729\beta_2}{22400}+\tfrac{8\beta_3}{14175}+\tfrac{\beta_4}{1814400})h^{10}u_i^{(10)}+(\tfrac{8192\alpha_1}{14175}+\tfrac{729\alpha_2}{22400}+\tfrac{8\alpha_3}{467775}+\tfrac{\alpha_4}{1814400}+\\
&\tfrac{32768\beta_1}{467775}+\tfrac{2187\beta_2}{985600}+\tfrac{8\beta_3}{985600}+\tfrac{\beta_4}{239500800})h^{12}u_i^{(12)}+(\tfrac{32768\alpha_1}{467775}+\tfrac{2187\alpha_2}{985600}+\tfrac{8\alpha_3}{985600}+\tfrac{\alpha_4}{239500800}+\\
&\tfrac{262144\beta_1}{42567525}+\tfrac{19683\beta_2}{179379200}+\tfrac{16\beta_3}{42567525}+\tfrac{\beta_4}{435891456 00})h^{14}u_i^{(14)}+(\tfrac{262144\alpha_1}{42567525}+\tfrac{19683\alpha_2}{179379200}+\tfrac{16\alpha_3}{42567525}+\\
&\tfrac{\alpha_4}{43589145600}+\tfrac{262144\beta_1}{638512875}+\tfrac{59049\beta_2}{14350336000}+\tfrac{4\beta_3}{638512875}+\tfrac{\beta_4}{10461394944000})h^{16}u_i^{(16)}+(\tfrac{262144\alpha_1}{638512875}+\\
&\tfrac{59049\alpha_2}{14350336000}+\tfrac{4\alpha_3}{638512875}+\tfrac{\alpha_4}{10461394944000}+\tfrac{2097152\beta_1}{97692469875}+\tfrac{59049\beta_2}{487911424000}+\tfrac{8\beta_3}{97692469875}+\\
&\tfrac{\beta_4}{3201186852864000})h^{18}u_i^{(18)}+...,
\end{aligned}$$

$$i=4,5,...,n-4. \tag{2.6}$$

Remarks:

(I) By choosing $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0, \alpha_5=-\frac{1}{2}, \beta_1=\beta_2=\beta_3=0, \beta_4=\frac{1}{2}$, and $\beta_5=-1$ we obtain the second-order method with truncation error:

$$t_i=\frac{1}{24}h^4u_i^{(4)}+O(h^6).$$

(II) By choosing $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0, \alpha_5=\frac{3}{8}, \beta_1=\beta_2=0, \beta_3=\frac{1}{32}, \beta_4=\frac{-1}{2}$, and $\beta_5=\frac{15}{16}$ we obtain the fourth-order method with truncation error:

$$t_i=\frac{1}{240}h^6u_i^{(6)}+O(h^8).$$

(III) By choosing $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0, \alpha_5=-1, \beta_1=0, \beta_2=\frac{1}{90}, \beta_3=-\frac{3}{20}, \beta_4=\frac{3}{2}$, and $\beta_5=-\frac{49}{18}$ we obtain the sixth-order method with truncation error:

$$t_i=\frac{1}{560}h^8u_i^{(8)}+O(h^{10}).$$

(IV) By choosing $\alpha_1 = \frac{14}{5}, \alpha_2 = \frac{7028}{5}, \alpha_3 = \frac{204512}{5}, \alpha_4 = \frac{1235276}{5}, \alpha_5 = 437332,$ $\beta_1 = -\frac{1008}{5}, \beta_2 = -\frac{118944}{5}, \beta_3 = -\frac{959616}{5}, \beta_4 = -\frac{155232}{5}$, and $\beta_5 = 493920$ we obtain the eighth-order method with truncation error:

$$t_i = -\frac{21}{25} h^{10} u_i^{(10)} + O(h^{12}).$$

(V) By choosing $\alpha_1 = \alpha_2 = 0, \alpha_3 = 9, \alpha_4 = -\frac{34308}{43}, \alpha_5 = -\frac{129978}{43}, \beta_1 = 1, \beta_2 = -\frac{1263}{43}, \beta_3 = \frac{22993}{43}, \beta_4 = \frac{116527}{43}$, and $\beta_5 = -\frac{276600}{43}$ we obtain the tenth-order method with truncation error:

$$t_i = \frac{14837}{794640} h^{12} u_i^{(12)} + O(h^{14}).$$

(VI) By choosing $\alpha_1 = 0, \alpha_2 = \frac{619}{263655}, \alpha_3 = -\frac{454}{3255}, \alpha_4 = -\frac{4073}{3255}, \alpha_5 = -\frac{725308}{263655},$ $\beta_1 = 0, \beta_2 = \frac{7069}{112995}, \beta_3 = \frac{154}{155}, \beta_4 = 1$, and $\beta_5 = -\frac{92932}{22599}$ we obtain the twelve-order method with truncation error:

$$t_i = \frac{-137}{378378000} h^{14} u_i^{(14)} + O(h^{16}).$$

(VII) By choosing $\alpha_1 = 137, \alpha_2 = -\frac{131918192}{2511}, \alpha_3 = -\frac{54457608}{31}, \alpha_4 = -\frac{372129872}{31},$ $\alpha_5 = -\frac{59218139870}{2511}, \beta_1 = 4247, \beta_2 = \frac{7375290512}{7533}, \beta_3 = \frac{287518196}{31}, \beta_4 = 5231408$, and $\beta_5 = -\frac{233364802510}{7533}$ we obtain the fourteen-order method with truncation error:

$$t_i = \frac{250150819}{1005404400} h^{16} u_i^{(16)} + O(h^{18}).$$

(VIII) By choosing $\alpha_1 = 330907, \alpha_2 = 32535424, \alpha_3 = 543878896, \alpha_4 = 2750389888, \alpha_5 = 4824096670, \beta_1 = -\frac{127053415}{12}, \beta_2 = -\frac{1059629440}{3}, \beta_3 = -\frac{5809420960}{3}, \beta_4 = -\frac{1152538240}{3}$, and $\beta_5 = \frac{32213407975}{6}$ we obtain the sixteen-order method with truncation error:

$$t_i = \frac{5016301}{1701700} h^{18} u_i^{(18)} + O(h^{20}).$$

To obtain unique solution for the nonlinear system (2.3) we need six more equations. we define the following identities:

$$\begin{cases} (i) \ \sum_{i=0}^{5} \delta_{1,i} u_i - h^2 \sum_{i=0}^{15} \eta_{1,i} u_i^{(2)} - t_1 h^{18} u_0^{(18)} = 0, \\ (ii) \ \sum_{i=0}^{6} \delta_{2,i} u_i - h^2 \sum_{i=0}^{15} \eta_{2,i} u_i^{(2)} - t_2 h^{18} u_0^{(18)} = 0, \\ (iii) \ \sum_{i=0}^{7} \delta_{3,i} u_i - h^2 \sum_{i=0}^{15} \eta_{3,i} u_i^{(2)} - t_3 h^{18} u_0^{(18)} = 0, \\ (iv) \ \sum_{i=0}^{7} \delta_{3,i} u_{n-i} - h^2 \sum_{i=0}^{15} \eta_{3,i} u_{n-i}^{(2)} - t_{n-3} h^{18} u_n^{(18)} = 0, \\ (v) \ \sum_{i=0}^{6} \delta_{2,i} u_{n-i} - h^2 \sum_{i=0}^{15} \eta_{2,i} u_{n-i}^{(2)} - t_{n-2} h^{18} u_n^{(18)} = 0, \\ (vi) \ \sum_{i=0}^{5} \delta_{1,i} u_{n-i} - h^2 \sum_{i=0}^{15} \eta_{1,i} u_{n-i}^{(2)} - t_{n-1} h^{18} u_n^{(18)} = 0, \end{cases} \tag{2.7}$$

using Taylor's expansion we can obtain the unknown coefficients in (2.7) by the following algorithm in mathematica:

$$\begin{cases} \sum_{i=0}^{j} \delta_{k,i} = 0, \\ \sum_{i=1}^{j} i(\delta_{k,i}) = 0, \\ \frac{1}{2!} \sum_{i=1}^{j} i^2(\delta_{k,i}) = \sum_{i=0}^{15} \eta_{k,i}, \\ \frac{1}{l!} \sum_{i=1}^{j} i^l(\delta_{k,i}) = \frac{1}{r!} \sum_{i=1}^{15} i^r(\eta_{k,i}), \ \text{for } r = 1, 2, ..., 15, \ \text{and } l = r + 2. \end{cases}$$

$$\tag{2.8}$$

(I) If me choose $j = 5, k = 1$ and $\delta_{1,2} = 36, \delta_{1,3} = 951, \delta_{1,4} = 118, \delta_{1,5} = 1$ for system (2.8) we obtain the unknown coefficients equations $(i), (vi)$ in (2.7).
(II) If me choose $j = 6, k = 2$ and $\delta_{2,2} = -2451, \delta_{2,3} = 154, \delta_{2,4} = 952, \delta_{2,5} = 118, \delta_{2,6} = 1$ for system (2.8) we obtain the unknown coefficients equations $(ii), (v)$ in (2.7).
(III) If me choose $j = 7, k = 3$ and $\delta_{3,2} = 154, \delta_{3,3} = -2450, \delta_{3,4} = 154, \delta_{3,5} = 952, \delta_{3,6} = 118, \delta_{3,7} = 1$ for system (2.8) we obtain the unknown coefficients equations $(iii), (iv)$ in (2.7).

For sake of briefness we not rewrite the coefficients here.
and
$(t_1 = t_{n-1} = \frac{1738869905586403}{-711374856192000}, t_2 = t_{n-2} = -\frac{11882318830801}{10778406912000}, t_3 = t_{n-3} = \frac{-103014110604593}{2134124568576000})$

3. Convergence analysis

In this section, we investigate the convergence analysis of the eighth-order method and also in the same way we can prove the convergence analysis for any of the other methods. The equations (2.3) along with boundary condition (2.7) yields nonlinear system of equations, and may be written in a matrix form as

$$\epsilon A_0 U + h^2 B \mathbf{f}(U) = R, \tag{3.1}$$

where

$$\mathbf{f}(U) = (f_1, ..., f_{n-1})^t,$$

the matrices A_0 is diagonally dominant and B are an $(n-1) \times (n-1)$-dimensional which have the following forms:

$$\begin{gathered}(P_{n-1}(1,2,1))^4 - 41(P_{n-1}(1,2,1))^3 + 400(P_{n-1}(-1,4,-1))^2 \\ +3723P_{n-1}(1,2,1) - 14404I_{n-1} \le A_0 \le (P_{n-1}(1,2,1))^4 - 42(P_{n-1}(1,2,1))^3 \\ +407(P_{n-1}(-1,4,-1))^2 + 3733P_{n-1}(1,2,1) - 14529I_{n-1}.\end{gathered}$$

It is easy to see that A_0 is diagonally dominant and symmetric. The matrix I_{n-1} is identity matrix and $P_{n-1}(x, z, y)$ has the following form:

$$P_{n-1}(x,z,y) = \begin{pmatrix} z & -y & & & \\ -x & z & -y & & \\ & \ddots & \ddots & \ddots & \\ & & -x & z & -y \\ & & & -x & z \end{pmatrix}, \tag{3.2}$$

$$
B=\begin{pmatrix}
\eta_{1,1} & \eta_{1,2} & \eta_{1,3} & \cdots & \eta_{1,15} & & & & & & \\
\eta_{2,1} & \eta_{2,2} & \eta_{2,3} & \cdots & \eta_{2,15} & & & & & & \\
\eta_{3,1} & \eta_{3,2} & \eta_{3,3} & \cdots & \eta_{3,15} & & & & & & \\
-\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 & & & \\
-\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 \\
 & & & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_4 & -\alpha_3 & -\alpha_2 \\
 & & & & & & \eta_{3,15} & \cdots & \eta_{3,3} & \eta_{3,2} & \eta_{3,1} \\
 & & & & & & \eta_{2,15} & \cdots & \eta_{2,3} & \eta_{2,2} & \eta_{2,1} \\
 & & & & & & \eta_{1,15} & \cdots & \eta_{1,3} & \eta_{1,2} & \eta_{1,1}
\end{pmatrix}. \tag{3.3}
$$

and $\mathbf{f}(U)=\mathrm{diag}(f(x_i,u_i)),(i=1,2,...,n-1)$, is a diagonal matrix of order $n-1$.

$$
R=\begin{pmatrix}
-\epsilon\delta_{1,0}u_0+\epsilon h^2\eta_{1,0}u_0^{(2)}, \\
-\epsilon\delta_{2,0}u_0+\epsilon h^2\eta_{2,0}u_0^{(2)}, \\
-\epsilon\delta_{3,0}u_0+\epsilon h^2\eta_{3,0}u_0^{(2)}, \\
-\epsilon\beta_1 u_0-\epsilon h^2\alpha_1 u_0^{(2)}, \\
0 \\
\vdots \\
0 \\
-\epsilon\beta_1 u_n-\epsilon h^2\alpha_1 u_n^{(2)}, \\
-\epsilon\delta_{3,0}u_n+\epsilon h^2\eta_{3,0}u_n^{(2)}, \\
-\epsilon\delta_{2,0}u_n+\epsilon h^2\eta_{2,0}u_n^{(2)}, \\
-\epsilon\delta_{1,0}u_n+\epsilon h^2\eta_{1,0}u_n^{(2)},
\end{pmatrix}. \tag{3.4}
$$

We assume that

$$\epsilon A_0\overline{U}+h^2B\mathbf{f}(\overline{U})=R+t, \tag{3.5}$$

where the vector $\overline{U}=u(x_i),(i=1,2,...,n-1)$, is the exact solution and $t=[t_1,t_2,...,t_{n-1}]^T$, is the vector of local truncation error.
By using (3.1) and (3.5) we get

$$AE=[\epsilon A_0+h^2BF_k(U)]E=t, \tag{3.6}$$

where

$$E=\overline{U}-U \qquad \mathbf{f}(\overline{U})-\mathbf{f}(U)=F_k(U)E, \tag{3.7}$$

and $F_k(U)=\mathrm{diag}\{\frac{\partial f_i}{\partial u_i}\},(i=1,2,...,n-1)$, is a diagonal matrix of order $n-1$. To prove the existence of A^{-1}, since $A=\epsilon A_0+h^2BF_k(U)$, where A_0 is diagonally dominant and symmetric then A_0 is nonsingular.

Following Henrici [7] we have

$$\|(P_{n-1}(1,2,1))^{-1}\| \leq \frac{(b-a)^2}{8h^2}. \tag{3.8}$$

and also by using Usmani and Warsi [26] we get

$$\|(P_{n-1}(-1,4,-1))^{-1}\| \leq \frac{1}{2}. \tag{3.9}$$

It is clear that the matrix A_0 is nonsingular and also $\|A_0^{-1}\| < \omega$ where ω is a positive number ($\|.\|$ is the L_∞ norm).

Lemma 4.1 If M is a square matrix of order N and $\|\mathrm{M}\| < 1$, then $(I+\mathrm{M})^{-1}$ exist and $\|(I+\mathrm{M})^{-1}\| \leq \frac{1}{(1-\|\mathrm{M}\|)}$.

Lemma 4.2 The matrix $[\epsilon A_0 + h^2 BF_k(U)]$ in (3.6) is nonsingular, provided $Y < \frac{630(12!)\epsilon}{58763152900593469\omega h^2}$, where $Y = \max|\frac{\partial f_l}{\partial u_l}|, l = 1,2,...,n-1$. (The norm referred to is the L_∞ norm).

Proof:

We know that $[\epsilon A_0 + h^2 BF_k(U)] = \epsilon A_0[I + \frac{h^2}{\epsilon}A_0^{-1}BF_k(U)]$, we need to show that inverse of $[I + \frac{h^2}{\epsilon}A_0^{-1}BF_k(U)]$ exist. By using lemma 4.1, we have

$$\frac{h^2}{\epsilon}\|A_0^{-1}BF_k(U)\| \leq \frac{h^2}{\epsilon}\|A_0^{-1}\|\|B\|\|F_k(U)\| < 1, \tag{3.10}$$

by using (3.3) we obtain $\|B\| \leq \frac{58763152900593469}{630(12!)} = \varpi, (say)$ and also we have $\|F_k(U)\| \leq Y = \max|\frac{\partial f_l}{\partial u_l}|, l = 1,2,...,n-1$, and then by using (3.10) we obtain

$$Y < \frac{\epsilon}{\varpi\omega h^2}.$$

As a consequence of Lemmas 4.2 and 4.1 the nonlinear system (3.1) has a unique solution if $Y < \frac{\epsilon}{\varpi\omega h^2}$.

Theorem Let $u(x)$ be the exact solution of the boundary value problem (1.1)-(1.2) and assume u_i, $i = 1,2,...,n-1$, be the numerical solution obtained by solving the system (2.3), and (2.7). Then we have

$$\|E\| \equiv O(h^8),$$

provided $Y < \frac{\epsilon}{\varpi\omega h^2}$, where

$$\alpha_1 = \frac{1}{72}, \alpha_2 = \frac{251}{36}, \alpha_3 = \frac{1826}{9}, \alpha_4 = \frac{44117}{36}, \alpha_5 = \frac{78095}{36},$$

$$\beta_1 = -1, \beta_2 = -118, \beta_3 = -952, \beta_4 = -154, \beta_5 = 2450.$$

proof: We can write the error equation (3.6) in the following form

$$E = (\epsilon A_0 + h^2 BF_k(U))^{-1}t = \frac{1}{\epsilon}(I + \frac{h^2}{\epsilon}A_0^{-1}BF_k(U))^{-1}A_0^{-1}t,$$

$$\|E\| \leq \frac{1}{\epsilon}\|(I+\frac{h^2}{\epsilon}A_0^{-1}BF_k(U))^{-1}\|\|A_0^{-1}\|\|t\|,$$

it follows that

$$\|E\| \leq \frac{\|A_0^{-1}\|\|t\|}{\epsilon - h^2\|A_0^{-1}\|\|B\|\|F_k(U)\|}, \tag{3.11}$$

provided that $h^2\|A_0^{-1}\|\|B\|\|F_k(U)\| < 1$. We have

$$\|t\| \leq \frac{1738869905586403h^{10}M_{10}}{2(17!)}, \tag{3.12}$$

where $M_{10} = \max|u^{(10)}(\xi)|, a \leq \xi \leq b$.
From inequalities (3.11), (3.12), $\|A_0^{-1}\| < \omega$, $\|F_k(U)\| \leq Y$ and $\|B\| \leq \varpi$, we obtain

$$\|E\| \leq \frac{1738869905586403\omega h^{10}M_{10}}{2(17!)(\epsilon - h^2\varpi\omega Y)} \equiv O(h^8), \tag{3.13}$$

provided that

$$Y < \frac{\epsilon}{h^2\varpi\omega}. \tag{3.14}$$

Corollary

In the same manner we can prove the convergence analysis of the other methods and it follows $\|E\| \longrightarrow 0$ as $h \longrightarrow 0$.

Second-order method

If we choose $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0, \alpha_5 = -\frac{1}{2}, \beta_1 = \beta_2 = \beta_3 = 0, \beta_4 = \frac{1}{2}$, and $\beta_5 = -1$ we get $\|E\| \equiv O(h^2)$.

Fourth-order method

If we choose $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0, \alpha_5 = \frac{3}{8}, \beta_1 = \beta_2 = 0, \beta_3 = \frac{1}{32}, \beta_4 = \frac{1}{2}$, and $\beta_5 = \frac{15}{16}$ then we get $\|E\| \equiv O(h^4)$.

Sixth-order method

For $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0, \alpha_5 = -1, \beta_1 = 0, \beta_2 = \frac{1}{90}, \beta_3 = -\frac{3}{20}, \beta_4 = \frac{3}{2}$, and $\beta_5 = -\frac{49}{18}$ we have $\|E\| \equiv O(h^6)$.

Eighth-order method

For $\alpha_1 = \frac{14}{5}, \alpha_2 = \frac{7028}{5}, \alpha_3 = \frac{204512}{5}, \alpha_4 = \frac{1235276}{5}, \alpha_5 = 437332, \beta_1 = -\frac{1008}{5}$, $\beta_2 = -\frac{118944}{5}, \beta_3 = -\frac{959616}{5}, \beta_4 = -\frac{155232}{5}$, and $\beta_5 = 493920$ we have $\|E\| \equiv O(h^8)$.

Tenth-order method

For $\alpha_1 = \alpha_2 = 0, \alpha_3 = 9, \alpha_4 = -\frac{34308}{43}, \alpha_5 = -\frac{129978}{43}, \beta_1 = 1, \beta_2 = -\frac{1263}{43}, \beta_3 = \frac{22993}{43}, \beta_4 = \frac{116527}{43}$, and $\beta_5 = -\frac{276600}{43}$ we get $\|E\| \equiv O(h^{10})$.

Twelve-order method

By choose $\alpha_1 = 0, \alpha_2 = \frac{619}{263655}, \alpha_3 = -\frac{454}{3255}, \alpha_4 = -\frac{4073}{3255}, \alpha_5 = -\frac{725308}{263655}, \beta_1 = 0, \beta_2 = \frac{7069}{112995}, \beta_3 = \frac{154}{155}, \beta_4 = 1$, and $\beta_5 = -\frac{92932}{22599}$ we have $\|E\| \equiv O(h^{12})$.

Fourteen-order method

If we choose $\alpha_1 = 137, \alpha_2 = -\frac{131918192}{2511}, \alpha_3 = -\frac{54457608}{31}, \alpha_4 = -\frac{372129872}{31}, \alpha_5 =$

$-\frac{59218139870}{2511}, \beta_1 = 4247, \beta_2 = \frac{7375290512}{7533}, \beta_3 = \frac{287518196}{31}, \beta_4 = 5231408$, and $\beta_5 = -\frac{233364802510}{7533}$ then we can obtain $\|E\| \equiv O(h^{14})$.

Sixteen-order method

If we choose $\alpha_1 = 330907, \alpha_2 = 32535424, \alpha_3 = 543878896, \alpha_4 = 2750389888$, $\alpha_5 = 4824096670, \beta_1 = -\frac{127053415}{12}, \beta_2 = -\frac{1059629440}{3}, \beta_3 = -\frac{5809420960}{3}, \beta_4 = -\frac{1152538240}{3}$, and $\beta_5 = \frac{32213407975}{6}$ we obtain the sixteen-order method $\|E\| \equiv O(h^{16})$.
Therefore the convergence of the methods have been established.

4. Numerical illustrations

In this section the presented methods of orders 4th, 8th, 12th and 16th have been applied to the following test problems. Examples 1-5 have been solved using our methods and also max error in the compared solutions are obtained. The maximum absolute errors in solutions of singularly perturbed boundary-value problems are tabulated in tables 1-23. The maximum absolute errors in solutions of example 1-5 is compared with method in [2, 16, 17, 18, 19, 25].

Example 1. Consider the following singularly perturbed boundary value problem

$$-\epsilon u^{''}(x) + (1+x)u(x) = -40(x(x^2-1) - 2\epsilon), \ \ 0 \leq x \leq 1$$

with boundary conditions

$$u(0) = 0, \ \ u(1) = 1.$$

The exact solution for this problem is $u(x) = 40x(1-x)$. The observed maximum absolute errors for different values of ϵ and n are tabulated in Tables 1-2 and compared with the methods in [16, 17, 19].

Example 2. Consider the following singularly perturbed boundary value problem

$$-\epsilon u^{''}(x) + u(x) = -cos^2(\pi x) - 2\epsilon\pi^2 cos(2\pi x), \ \ 0 \leq x \leq 1$$

with boundary conditions

$$u(0) = 0, \ \ u(1) = 0.$$

The exact solution for this problem is $u(x) = \frac{e^{\frac{x-1}{\sqrt{\epsilon}}} + e^{\frac{-x}{\sqrt{\epsilon}}}}{1+e^{\frac{-1}{\sqrt{\epsilon}}}} - cos^2(\pi x)$. The observed maximum absolute errors for different values of ϵ and n are tabulated in Tables 3-10 and compared with the methods in [2, 18, 19, 25].

Example 3. Consider the following singularly perturbed boundary value problem

$$-\epsilon u^{''}(x) + u(x) = x, \ \ 0 \leq x \leq 1$$

with boundary conditions

$$u(0) = 1, \ \ u(1) = 1 + e^{\frac{-1}{\sqrt{\epsilon}}}.$$

The exact solution for this problem is $u(x) = x + e^{\frac{-x}{\sqrt{\epsilon}}}$. The observed maximum absolute errors for different values of ϵ and n are tabulated in Tables 11-16 and compared with the method in [18].

Example 4. Consider the following singularly perturbed boundary value problem

$$-\epsilon u^{''}(x)+(1+x(1-x))u(x)=1+x(1-x)+(2\sqrt{\epsilon}-x^2(1-x))e^{-\frac{1-x}{\sqrt{\epsilon}}}$$
$$+(2\sqrt{\epsilon}-x(1-x)^2)e^{-\frac{-x}{\sqrt{\epsilon}}},\quad 0\le x\le 1$$

with boundary conditions

$$u(0)=0,\ \ u(1)=0$$

The exact solution for this problem is $u(x)=1+(x-1)e^{\frac{-x}{\sqrt{\epsilon}}}-xe^{\frac{(x-1)}{\sqrt{\epsilon}}}$. The observed maximum absolute errors for different values of ϵ and n are tabulated in Tables 17-22 and compared with the method in [18].

Example 5. Consider the following boundary value problem

$$u^{''}(x)+3u(x)-2u^3(x)=cos(x)sin(2x),\ \ 0\le x\le 1$$

with boundary conditions

$$u(0)=0,\ \ u(1)=sin(1)$$

The exact solution for this problem is $u(x)=sin(x)$, [28]
This example is regular type of B.V.P, we solve this type to show the ability of our method. The observed maximum absolute errors for different values of n are tabulated in Table 23.

Conclusion

We approximate solution of singularly perturbed boundary-value problems by using tension spline. We developed various classes of 2th, 4th, 6th, 8th, 10th, 12th, 14th and 16th orders of methods. The new methods enable us to approximate the solution at every point of the range of integration. Tables 1-23 illustrated that our methods produced accurate results. The maximum absolute errors max $\mid e_i\mid$ in the computed solutions are compared with the methods developed in [2, 16, 17, 18, 19, 25].

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