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The numerical solution of Fisher Equation: A nonstandard finite difference in conjunction with Richtmyer formula

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Abstract A nonstandard finite-difference (NSFD) scheme for Fisher's Equation by using Richtmyer's (3, 1, 1) implicit formula has been presented, in this work. On nonstandard finite-difference scheme, two special cases of Richtmyer formula have been applied. The suitable functions in the denominator fraction of our NSFD scheme have been replaced to guarantee the highly accurate of the approximation. Furthermore, the analyses of stability, convergence, consistency for the NSFD method, have been provided. By calculating the absolute error, the comparison of these methods has been presented in some Examples and the results have shown that the error of our NSFD scheme is lower than the others. Finally, a comparison of these methods and the differential quadrature method (DQM) to solve the Fisher's Equation reveals that these techniques work better and give highly accurate results.

Keywords. Fisher's Equation, Finite- difference scheme, Nonstandard Finite-difference scheme, Consistency, Convergence, Stability.

2010 Mathematics Subject Classification. 35J05, 35J10, 35K05, 35L05.

1. Introduction

The Fisher Equation describes the process of interaction between diffusion and reaction. This Equation is encountered in chemistry [2] and population dynamics [22], which includes problems such as nonlinear evolution of a population in an unclear

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reaction and engineering [3], neurophysiology [20], mathematical biology [11]. This Equation is defined by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \alpha u(1-u), \tag{1.1}$$

where k is diffusion coefficient and α is reactive factor, t is time, x is distance and $u(x,t)$ is population density. This Equation will support traveling waves of the form $u(x-ct)$ moving in the positive x-direction, provided that the speed $c > 2\sqrt{k\alpha}$. There are many papers of the numerical solution for the Fisher's Equation such as using finite-element method [6, 20], Moving mesh method [16, 18], differential quadrature method [10], Adomian method [7], Since collocation method [1, 23]. We study the Equation (1.1) with k equal 1, i.e.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u). \tag{1.2}$$

This paper is organized as follows: Section 2, introduces the computational techniques to approximate solutions of the model under study, here we prove that our NSFD method is consistency and convergence to the exact solution and using RR2 and OR4 formula on this NSFD scheme [5, 13], an analysis of nonlinear stability is presented [12, 13]. Section 3, for some Examples the numerical results that illustrate the efficiency of proposed methods are reported and are compared with the results of [DQM] method used in [10]. Finally, a conclusion is given in Section 4, all the numerical experiments presented in this section were computed using the MATLAB 10 on a pc with a 2.5 GHz, 64-bit processor, 4 GB memory.

2. Numerical methods

2.1. **Finite-difference Scheme.** The main idea behind the Finite difference methods for obtaining the solution of a given partial differential Equation is to approximate the derivatives appearing in the Equation by the function at a selected number of points. The most usual way to generate these approximations is through the use of Taylor series. Let M and N be positive integers. In order to approximate the Equation (1.2) over the real line, we restrict our attention to a bounded spatial domain $[a_s, b_s]$ and impose appropriate boundary conditions. In order to, approximate the solution of the Fisher problem under study over a temporal interval $[0,T]$, we set $0 = t_0 < t_1 < \ldots < t_M = T$ and $a_s = x_0 < x_1 < \ldots < x_n = b_s$ of $[0,T]$, $[a_s, b_s]$, $\Delta x = \frac{b_s - a_s}{N}$ and $\Delta t = \frac{T}{M}$. u_n^k, is the approximation provided by the numerical method for the exact value of $u(x_n, t_k)$ for $n = 0, \ldots, N$ and $k = 0, 1, 2, \ldots, M$.

2.2. **Nonstandard finite difference scheme.** We construct a general NSFD scheme for the Equation (1.2) by using the Richtmyer's (3, 1, 1) implicit formula [13, 14]. This formula is a three-point and three-level formula. In this scheme, a weighted average of finite difference approximation to the time derivative is used. The following nomenclatures are introduced to approximate the partial derivatives, u with respect to t and x at the point (x_n, t_k).

$$\frac{du}{dt}\mid_n^k = (1-\theta)\frac{u_n^{k+1}-u_n^k}{\phi(\Delta t)} + \theta\frac{u_n^k - u_n^{k-1}}{\phi(\Delta t)} + O((1+2\theta)\Delta t, (\Delta x)^2), \tag{2.1}$$

$$\frac{d^2u}{dx^2}\mid_n{}^k = \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\psi(\Delta x)} + O(\Delta(x)^2), \tag{2.2}$$

where,

$$\phi(\Delta t) = \frac{1-e^{\frac{-\Delta t}{2}}}{\frac{1}{2}} \quad \text{and} \quad \psi(\Delta x) = 4\sinh^2\left(\frac{\Delta x}{2}\right).$$

By these conventions in hand, we will approximate solutions of Equation (1.2) in the $[a_s, b_s]$ and $[0, T]$, through the finite-difference scheme

$$\frac{du}{dt}\mid_n^k = \frac{d^2u}{dx^2}\mid_n^{k+1} + \alpha f(u_n^{k+1}), \tag{2.3}$$

where

$$f(u_n^{k+1}) = u_n^{k+1}(1-u_n^k). \tag{2.4}$$

The Nonstandard finite-difference scheme (2.3) may be conveniently rewritten as

$$A_1u_{n+1}^{k+1} + A_2u_n^{k+1} + A_1u_{n-1}^{k+1} = A_3u_n^k + A_4u_n^{k-1} + \alpha\Delta t f(u_n^{k+1}), \tag{2.5}$$

with

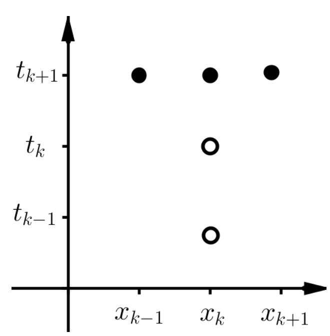
$$A_1 = -R,\ A_2 = 2R+1-\theta,\ A_3 = 1-2\theta,\ A_4 = \theta,$$

where

$$R = \frac{\phi(\Delta t)}{\psi(\Delta x)}, \tag{2.6}$$

is the Fourier number of the NSFD scheme (2.3), the coefficients A_1, A_2, A_3, A_4 depend on u_n^k. The computational stencil of our method is shown in Figure 1.

FIGURE 1. Computational stencil for the approximation to the partial differential Equation (1.2) at the time t_k by using the NSFD scheme (2.5).



2.3. **Matrix representation.** In this work, we will impose constraints on the form

$$u(a_s,t) = a_0(t) \quad and \quad u(b_s,t) = a_1(t), \tag{2.7}$$

satisfied for every $t \geq 0$. Here, a_0, a_1 are non-negative, real function which is less than or equal to 1. Let M_n be the vector space of all matrices over $\mathbb{R}$ of size $(n*n)$, for each positive integer n. The numerical method (2.5) can be presented in matrix form as the following

$$Au^{k+1} = b^k, \tag{2.8}$$

for $k \in \{1,\dots,M-1\}$, u^k is the $(N+1)$-dimensional vector

$$(u_0^k, u_1^k, \cdots, u_N^k),$$

for $k \in \{0,1,\dots,M\}$. We let

$$\begin{aligned} b^k &= Bu^k + Cu^{k-1} + d^k \\ &\quad + \alpha\Delta t diag\{0, f(u_1^{k+1}),\, f(u_2^{k+1}), \cdots,\, f(u_{N-1}^{k+1}), 0\}, \end{aligned} \tag{2.9}$$

for every $k \in \{0,1,\dots M\}$, where B and C are the diagonal matrices M_{N+1} given by

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & \mathrm{A}_3 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{A}_3 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & \mathrm{A}_4 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{A}_4 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{2.10}$$

The matrix A is a matrix of M_{N+1}, the vector d^k is an $(n+1)$-dimensional vector. The system (2.8) can be solved under the method in [19].

By employing discrete Dirichlet constraints in the form of

$$u_0^k = a_0(t_k),$$

and

$$u_N^k = a_1(t_k),$$

for $k \in \{0,1,\dots M\}$ we have the following presentations of A and d^k.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_1 & A_2 & A_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_1 & A_2 & A_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_1 & A_2 & A_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}, \; d^k = \begin{bmatrix} a_0(t_k) \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ a_1(t_k) \end{bmatrix}. \tag{2.11}$$

2.4. **Numerical properties.** In this Section, we show the properties stability and convergence and consistency for NSFD scheme (2.5).

2.4.1. *Convergence Analysis.* The local truncation error of our scheme at (x_n, t_k) is

$$\ell_n^k = (1-\theta)\frac{u_n^{k+1}-u_n^k}{\varphi(\Delta t)} + \theta\frac{u_n^{k+1}-u_n^{k-1}}{\varphi(\Delta t)} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{\psi(\Delta x^2)}$$
$$-\alpha f(\ u_n^{k+1}), \tag{2.12}$$

considering u is the exact solution of (1.2) and using Taylor's series expansion, we have

$$\ell_n^k = (1-2\theta)\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}\Big|_n^k - \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}\Big|_n^{k+1}. \tag{2.13}$$

we assume that $u_{tt},\ u_{xxxx}$ are continuous in $[0,T]\times[a_s,b_s]$, so there are constant $K_1,\ K_2$ such that

$$\left|\ell_n^k\right| \leq K_1\Delta t + K_2(\Delta x)^2 \equiv E.$$

Rearranging the terms of (2.12) we have

$$A_1u_{n+1}^{k+1} + A_2u_n^{k+1} + A_1u_{n-1}^{k+1} = A_3u_n^k + A_4u_n^{k-1} + \alpha\Delta t f(\ u_n^{k+1}) + \ell_n^k\Delta t, \tag{2.14}$$

Let

$$e_n^k\ = u_n^k\ - U_n^k,$$

that U is numerical solution at (x_n, t_k). We subtract (2.4) from (2.14) and assume $1-\theta \geq 0$.

After taking magnitudes of both sides of the new Equation, the following inequality is then obtained

$$A_1\left|e_{n+1}^{k+1}\right| + A_2\left|e_n^{k+1}\right| + A_1\left|e_{n-1}^{k+1}\right| \leq A_3\left|e_n^k\right| + A_4\left|e_n^{k-1}\right|$$
$$+\alpha\Delta t K_4\left|(\ e_n^{k+1}\ )\right| + E\Delta t, \tag{2.15}$$

where K_4 is the maximum magnitude of $f'(u)$, if we let

$$e^k\ = \max_{0\leq n\leq N}\left|e_n^k\right|,$$

then, the above inequality becomes

$$Me^{k+1} \leq A_3e^k + A_4e^{k-1} + E\Delta t, \tag{2.16}$$

that,

$$M = 1-\theta-K_4\alpha\Delta t.$$

Since $e^0 = e^{-1} = 0$, from Equation (2.16) we have

$$e^kM^k \leq ((A_3+M)^{k-1} - [M(A_3-A_4) + M^2(A_3-A_4)^2$$
$$+\cdots+M^{k-2}(A_3-A_4)^{k-2}])E\Delta t, \tag{2.17}$$

so $e^k \to 0$ as $\Delta t,\ \Delta x \to 0$. Thus we have proved the following theorem.

Theorem 2.1. *If the solution of (1.2) has continuous u_{tt}, u_{xxxx} in $[a_s,b_s]\times[0,T]$ then the approximation solution generated by the NSFD scheme (2.5) convergence to the exact one as $\Delta t,\ \Delta x \to 0$ keeping $0\leq\theta\leq 1$.*

2.4.2. *Stability analysis.* Applying the mean value Theorem to the differential operator (2.9) we get

$$\begin{aligned}(A - \alpha\Delta t diag\{0, f'(\tilde{u}_1^{k+1}),\ f'(\tilde{u}_2^{k+1}), \cdots, f'(\tilde{u}_{N-1}^{k+1}), 0\})\vec{u}^{k+1} \\ = B\vec{u}^k + C\vec{u}^{k-1} + d^k,\end{aligned} \tag{2.18}$$

or

$$\rho(A_k) \leq 1, \tag{2.19}$$

where

$$\begin{aligned}A_K &= A - \Delta A_k \\ &= A - \alpha\Delta t diag\{0, f'(\tilde{u}_1^{k+1}),\ f'(\tilde{u}_2^{k+1}), \cdots, f'(\tilde{u}_{N-1}^{k+1}), 0\},\end{aligned} \tag{2.20}$$

it is easy to verify that if $0 \leq \theta \leq 1$ then $\|A\|_\infty = 1$. Since

$$\rho(\Delta A_k) \leq 1 + \alpha K_4 \Delta t,$$

we have the following inequalities [4].

$$\rho(A_k) \leq 1 + \alpha K_4 \Delta t, \tag{2.21}$$

the following theorem is then proved.

Theorem 2.2. *The NSFD scheme (2.5) is stable if $0 \leq \theta \leq 1$. The essence of stability is that there should be a limit to the extent to which any initial error can be amplified in the numerical procedure. It is easy to see that if the scheme satisfies a stronger condition*

$$\rho(A_k) \leq 1, \tag{2.22}$$

then the initial error will not be amplified at all, in fact, it tends to zero if $\rho(A_k)$ is strictly less than one. This kind of stability was first studied by O'Brein, Hyman, and Kaplan [15].

The following theorem shows that the simple implicit scheme is also conditionally stable in the sense of B.H.K.

Theorem 2.3. *If $0 \leq \theta \leq 1$ and $\Delta t \leq \frac{(1-\rho(A))}{\alpha K_4}$, then $\rho(A_k) \leq 1$.*

Proof. The key point of the proof is to show that $\rho(A)$ is strictly less than 1. Let L be one positive integer less than n and v^L and $N+1$ dimensional vector having $\sin \dfrac{Li\pi}{N}$ as the ith component for $i = 0, 1, \cdots, N$. It is easy to see that the ith component Av^L is

$$-R \sin \frac{L(i-1)\pi}{N} + (2R + 1 - \theta) \sin \frac{Li\pi}{N} - R \sin \frac{L(i+1)\pi}{N}, \tag{2.23}$$

which can be simplified to

$$(1 - \theta - 4R\sin^2 \frac{L\pi}{2N}) \sin \frac{Li\pi}{N}, \tag{2.24}$$

thus

$$Av^L = \mu^L v^L,$$

showing that v^L is an eigenvector of A with eigenvalue μ^L, where

$$\mu^L = 1 - \theta - 4R\sin^2\frac{L\pi}{N}, \tag{2.25}$$

for $0 \le L \le N$. Therefore if $0 \le \theta \le 1$ then

$$\rho(A) = \mu^L \prec 1. \tag{2.26}$$

The rest of proof is just reapplying the inequality

$$\rho(A_K) \le \rho(A) + \rho(\Delta A_K),$$

which is used in the proof of Theorem 2.2. □

2.4.3. *Consistency analysis.* The local truncation error (LTE) of a numerical method is an estimate of the error introduced in a single iteration of the method, assuming that everything fed into the method was perfectly accurate.

Expanding the coefficients

$$u_n^{k+1}, u_{n+1}^{k+1}, u_{n-1}^{k+1}, u_n^{k-1},$$

by Taylor series method.

$$u_n^{k+1} = u_n^k + \frac{(\Delta t)}{1!}\frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3} + \cdots + O(\Delta t^4), \tag{2.27}$$

$$\begin{aligned} u_{n+1}^{k+1} = u_n^k &+ \frac{(\Delta x)}{1!}\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots + O(\Delta x^4) \\ &+ \frac{(\Delta t)}{1!}\frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3} + \cdots + O(\Delta t^4), \end{aligned} \tag{2.28}$$

$$\begin{aligned} u_{n-1}^{k+1} = u_n^k &- \frac{(\Delta x)}{1!}\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots + O(\Delta x^4) \\ &+ \frac{(\Delta t)}{1!}\frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3} + \cdots + O(\Delta t^4), \end{aligned} \tag{2.29}$$

$$u_n^{k-1} = u_n^k - \frac{(\Delta t)}{1!}\frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2} - \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3} + \cdots + O(\Delta t^4). \tag{2.30}$$

Now substituting the value of Equation (2.27), (2.28), (2.29), (2.30) in Equation (2.5), we get

$$\frac{1}{\Delta t}[\frac{\partial u}{\partial t}(\Delta t) + O(\Delta t^2)] = \frac{1}{(\Delta x^2)}[\frac{\partial^2 u}{\partial (x^2)} + O(\Delta x^4)] + \alpha f(u_n^{k+1}). \tag{2.31}$$

Local Truncation error for above Equation can be written as,

$$\begin{aligned} LTE = \lim_{\Delta x, \Delta t \to 0} (1-2\theta)\frac{(\Delta t^2)}{2!}\frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t^3)}{3!}\frac{\partial^3 u}{\partial t^3} \\ + \cdots + 2\frac{(\Delta x^4)}{4!}\frac{\partial^4 u}{\partial x^4} + 2\frac{(\Delta x^6)}{6!}\frac{\partial^6 u}{\partial t^6} + \cdots = 0. \end{aligned} \tag{2.32}$$

A finite difference representation of PDE is said to be consistent if we can show that the difference between PDE and it's FDE representation vanishes as meh is refined. So we can write as,

$$\lim_{mesh\to 0}(PDE - FDE) = \lim_{mesh\to 0}(LTE) = 0. \tag{2.33}$$

Since $\Delta t, \Delta x$ approaches to zero, so from Equation (2.32), local truncation error becomes zero, therefore the NSFD scheme (2.5) is consistent. And solving the Equation (2.31) and comparing with Equation (2.5) we can say order of our proposed scheme is first in time and second order in space.

2.4.4. *NSFD scheme by using RR2 Formula.* By choosing $\theta = 1/2$ in (2.5), we obtain Richtmyer's formula, denoted to RR2 [13], which is

$$-2Ru_{n+1}^{k+1} + (4R + 2 - \alpha\Delta t f(u_n^k))u_n^{k+1} - 2Ru_{n-1}^{k+1} = u_n^{k-1}. \tag{2.34}$$

This formula is a special case with a greater apparent order of convergence than (2.5) since it has a truncation error of $O((\Delta x^2), (\Delta t)^2)$.

2.4.5. *OR4 Formula.* By setting

$$\theta = 1/2 - 1/12R, \tag{2.35}$$

in Equation (2.5), the following fourth-order from of Richtmyer's weighted formula is obtained

$$\begin{aligned}-12R^2u_{n+1}^{k+1} + (24R^2 + 12R - (1-6R)\alpha\Delta t f(u_n^k))u_n^{k+1} - 12R^2u_{n-1}^{k+1}\\ = 2u_n^k + (6R-1)u_n^{k-1},\end{aligned} \tag{2.36}$$

this formula is denoted OR4.

3. Examples

3.1. **Example1.** In this Section, two Examples are provided to illustrate the validity and effectiveness of the proposed methods. The initial and boundary conditions are directly obtained from analytical solution. Consider the following generalized Fisher's Equation in the domain $[0, 1]$

$$u_t = u_{xx} + \alpha u(1-u), \tag{3.1}$$

with the initial condition

$$u(x,0) = \left\{\frac{1}{2}\tanh\left(-\frac{\alpha}{2\sqrt{2\alpha+4}}x\right) + \frac{1}{2}\right\}^{\frac{2}{\alpha}}, \tag{3.2}$$

the exact solution is presented in [8, 9] by

$$u(x,t) = \left\{\frac{1}{2}\tanh\left\{-\frac{\alpha}{2\sqrt{2\alpha+4}}\left(x - \frac{\alpha+4}{\sqrt{2\alpha+4}}t\right)\right\} + \frac{1}{2}\right\}^{\frac{2}{\alpha}}. \tag{3.3}$$

In Table 1, we compared the relative errors of the numerical approximations obtained by NSFD scheme (2.5), (2.34), (2.36) and the exact solution (3.3) with $\alpha = 1$. The results show that in most points the error of NSFD scheme (2.5) and the CPUTIME for this method is lowest. Note that the formula for error is

$$\|u_{numerical} - u_{exact}\|_{1,2,\infty}.$$

TABLE 1. Shows the three standard norms of errors for the numerical approximations are obtained by NSFD scheme (2.5), RR2, OR4 methods respect to the exact solution in Example 1 with $\alpha = 1$ and $\theta = 0.01$ the CPU time for each method.

Methods	$Norm$	$\Delta x = 0.01$ $\Delta t = 0.01$	$\Delta t = 0.0001$ $\Delta x = 0.01$	$\Delta t = 0.0002$ $\Delta x = 0.01$
$NSFD$ $model$(2.5)	$\|.\|_1$	8.9592×10^{-4}	2.1071×10^{-7}	3.9300×10^{-7}
	$\|.\|_2$	9.5475×10^{-5}	2.7962×10^{-8}	4.5973×10^{-8}
	$\|.\|_\infty$	1.2022×10^{-5}	6.1344×10^{-9}	9.1929×10^{-9}
CPUTIME(s)	•	0.000784	0.004750	0.002469
$RR2$ $model$(2.34)	$\|.\|_1$	0.0017	2.6213×10^{-5}	5.0308×10^{-5}
	$\|.\|_2$	1.8759×10^{-4}	3.4794×10^{-6}	6.6719×10^{-6}
	$\|.\|_\infty$	2.8235×10^{-5}	7.0457×10^{-7}	1.3230×10^{-6}
CPUTIME(s)		0.001025	0.005089	0.002548
$OR4$ $Model$(2.36)	$\|.\|_1$	0.0017	1.4834×10^{-5}	3.6860×10^{-5}
	$\|.\|_2$	1.8794×10^{-4}	4.9759×10^{-6}	4.9033×10^{-6}
	$\|.\|_\infty$	2.8253×10^{-5}	4.017×10^{-7}	9.7622×10^{-7}
CPUTIME(s)	•	0.000762	0.005140	0.002487

In Table 2, the obtained results with $\alpha = 1$, $\Delta t = 0.0001$, $\Delta x = 0.01$, are compared with the exact solutions and the results of DQM [10] for $t = 0.5$, $x = 0.25, 0.5, 0.75$. The results illustrate that our numerical method to solve the Fisher Equation is more accurate than the DQM [10] method.

TABLE 2. Shows the comparison of results for the last Example 1 with $\alpha = 1$, $\Delta t = 0.0001$, $\Delta x = 0.01$.

t	x	$NSFD$ $Model$(2.5)	DQM $Model$[10]	exact	error
0.5	0.25	0.33409	0.33412	0.33409	0.1823×10^{-7}
	0.5	0.30574	0.30576	0.30574	0.1620×10^{-7}
	0.75	0.27835	0.27838	0.27835	0.1307×10^{-7}

Figure 2, shows the graphs for the numerical results of the method. The figure shows that the numerical and exact solutions are exactly coincident together.

FIGURE 2. The graphs of the approximate and exact solutions of the partial differential Equation (1.2) in Example (1.1) by the model NSFD (2.5) for $x \in [0, 1]$ and several times. $t = 0.25, 0.5, 0.75$, the model parameter $\theta = 0.01$ and $\alpha = 1$ along with the discrete steps $\Delta t = 0.001$, $\Delta x = 0.02$.

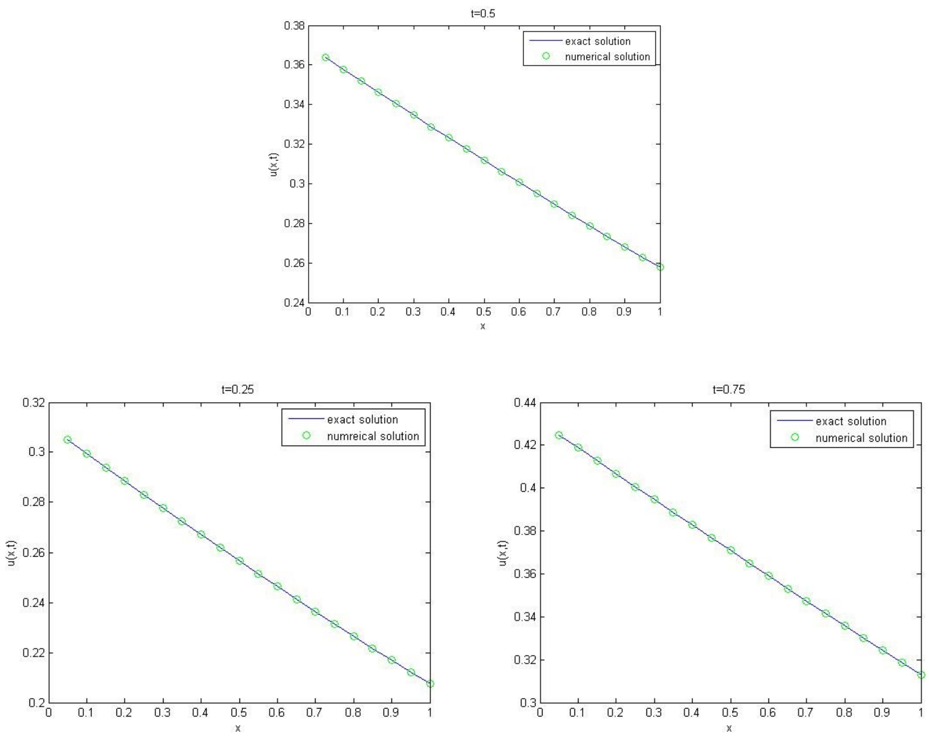


Figure 3, illustrates the graphs of the absolute errors of numerical solutions of NSFD scheme (2.5), RR2, OR4 in Example1 with $\alpha = 1$, and $\Delta t = 0.01$, $\Delta x = 0.05$. This Figure shows that the absolute error for the NSFD scheme of RR2 method has the most error and the absolute error of NSFD (2.5) is lowest.

Figure 4, illustrates the graph of the absolute error for the numerical method NSFD (2.5), with $\alpha = 1$ at different time level, for $\Delta t = 0.0001$, $\Delta x = 0.01$.

3.2. **Example2.** We now consider the Fisher's Equation

$$u_t = u_{xx} + 6u(1 - u), \tag{3.4}$$

subject to initial condition

$$u(x, 0) = \frac{1}{(1 + e^x)^2}, \tag{3.5}$$

FIGURE 3. The graphs of absolute error of NSFD (2.5), OR4and RR2 model in Example 1 with, $\alpha = 1$ and $\Delta t = 0.01$, $\Delta x = 0.05$ at $t = 0.5$.

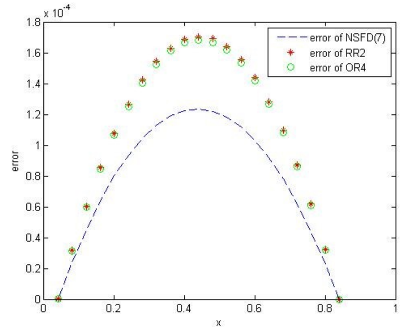
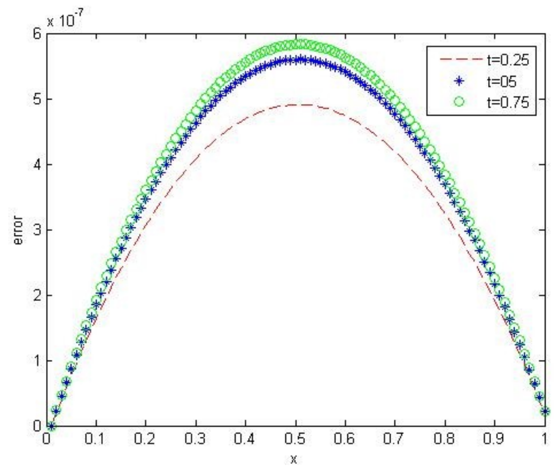


FIGURE 4. The absolute error of NSFD (2.5) model in Example 1 with $\alpha = 1$ at different time levels using $\Delta t = 0.0001$, $\Delta x = 0.01$.



that the exact solution is given by

$$u(x,t) = \frac{1}{(1 + e^{x-5t})^2}. \tag{3.6}$$

In Table 3, we give the absolute error between the exact and numerical results obtained by the NSFD (2.5) for Equation (3.4) with $\Delta t = 0.0001$, $\Delta x = 0.01, \theta = 0.01$ at $t = 0.5, t = 0.75$ and $x = 0.25, 0.5, 0.75$.

Table 4, shows the absolute error between the exact and numerical results obtained by NSFD (2.5) for the Equation (1.2) with various α.

TABLE 3. Shows the absolute error and CPU TIME of Example 2 with $\Delta t = 0.0001$, $\Delta x = 0.01, \theta = 0.01$.

t	x	error	CPUTIME(s)
0.5	0.25	0.3823×10^{-6}	0.005085
	0.5	0.2726×10^{-6}	
	0.75	0.1854×10^{-6}	
0.75	0.25	0.5075×10^{-5}	0.004486
	0.5	0.4176×10^{-5}	
	0.75	0.2110×10^{-5}	

TABLE 4. Shows the absolute error and CPUTIME for the results obtained by NSFD (2.5) with $\Delta t = 0.0001$, $\Delta x = 0.01$, $\theta = 0.01$ at t=0.5 for $\alpha = 2, 3, 4, 5$.

α	2	3	4	5
error	0.92490×10^{-7}	0.12804×10^{-6}	0.23619×10^{-6}	0.32994×10^{-6}
CPUTIME(s)	0.004577	0.004861	0.004392	0.005083

Figure 5, shows the (3D) graph of the numerical solution and exact solution of NSFD (2.5) for Example 2. The figure shows that the numerical and exact solutions are exactly coincident together.

FIGURE 5. 3(D) graphs of the approximate and exact solutions of the partial differential Equation (3.4) in Example (1.2) obtained by the model NSFD (2.5) with the discrete steps $\Delta t = 0.0001$, $\Delta x = 0.01$.

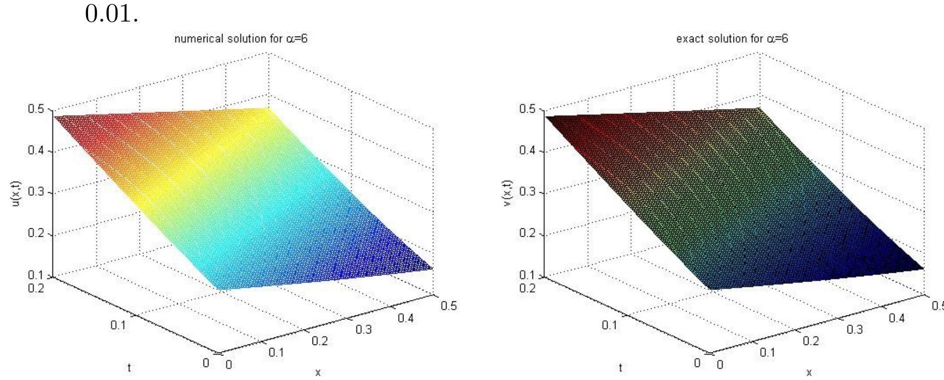
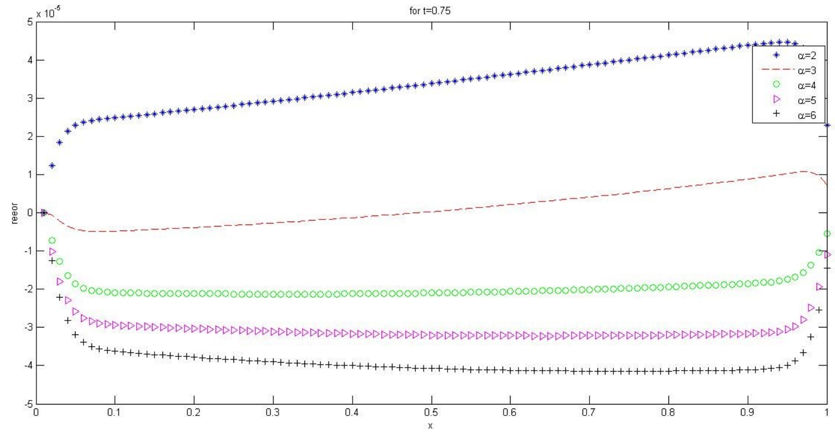


Figure 6 illustrates the graphs of the absolute errors of numerical solutions of the NSFD (2.5) with $\alpha = 2, 3, 4, 5, 6$, for $\Delta t = 0.0001$, $\Delta x = 0.01, \theta = 0.01$, $t = 0.75$ in Example 2.

Figure 6. The graphs of absolute error of NSFD (2.5) for $\alpha = 2, 3, 4, 5, 6$ with $\Delta t = 0.0001$, $\Delta x = 0.01$ at $t = 0.75$ in Example2.



3.3. **Example3.** In this Example, we consider a particular case of the Burgers-Huxley Equation [21].

$$u_t = ku_{xx} + \alpha^2 uf(u), \tag{3.7}$$

where

$$f(u) = (1 - u)(u - a). \tag{3.8}$$

We tacitly let k, α be both equal to 1 in Equation (3.7).

The exact solution for this Equation is given by

$$u(x,t) = \frac{A\exp(z_1) + aB\exp(z_2)}{A\exp(z_1) + B\exp(z_2) + C}, \tag{3.9}$$

where

$$z_1 = \pm\frac{1}{\sqrt{2}}x + (\frac{1}{2} - a)t, \tag{3.10}$$

$$z_2 = \pm\frac{1}{\sqrt{2}}ax + a(\frac{1}{2}a - 1)t, \tag{3.11}$$

and initial condition

$$u(x,0) = \frac{A\exp(\pm\frac{1}{\sqrt{2}}x) + aB\exp(\pm\frac{1}{\sqrt{2}}ax)}{A\exp(\pm\frac{1}{\sqrt{2}}x) + B\exp(\pm\frac{1}{\sqrt{2}}ax) + C}, \tag{3.12}$$

and A, B, and C are arbitrary constants [17].

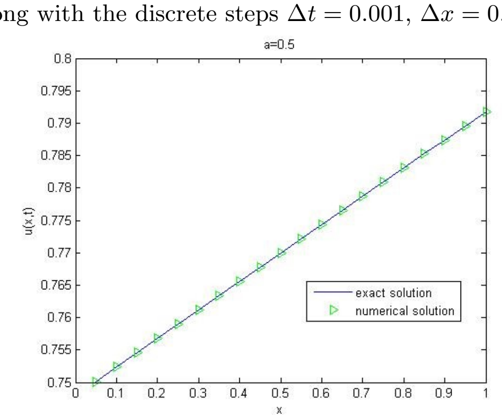
In Table 5, we compared the relative errors of the numerical approximations obtained by NSFD Scheme (2.5), (2.34), (2.36) and the exact solution (3.9) for Equation (3.7) with $\alpha = 1$, $A = B = 1, C = 0$, $a = 0.999, \theta = 0.01$, at $t = 0.5$. The results show that in most points the error of NSFD scheme (2.5) for this method is lowest. Note that the formula for error is $\|u_{numerical} - u_{exact}\|_{1,2,\infty}$.

TABLE 5. Shows the three standard norms of errors for the numerical approximations are obtained by NSFD scheme (2.5), RR2, OR4 methods respect to the exact solution in Example 3 for Equation (3.7) with $\alpha = 1, \theta = 0.01, a = 0.999, A = 1, B = 1, C = 0$, at $t = 0.5$ and the CPU time for each method.

Methods	$Norm$	$\Delta x = 0.01$ $\Delta t = 0.01$	$\Delta t = 0.0001$ $\Delta x = 0.01$	$\Delta t = 0.0002$ $\Delta x = 0.01$
$NSFD$ $model$(2.5)	$\|.\|_1$	2.2286×10^{-9}	1.4088×10^{-11}	2.7943×10^{-11}
	$\|.\|_2$	2.3789×10^{-10}	1.4244×10^{-12}	2.8366×10^{-12}
	$\|.\|_\infty$	3.0743×10^{-11}	1.4688×10^{-13}	2.9576×10^{-13}
CPUTIME(s)	•	0.000389	0.005057	0.002744
$RR2$ $model$(2.34)	$\|.\|_1$	0.0064	9.6271×10^{-5}	1.8969×10^{-4}
	$\|.\|_2$	6.8972×10^{-4}	9.7236×10^{-6}	1.9224×10^{-5}
	$\|.\|_\infty$	8.8432×10^{-5}	9.9935×10^{-7}	1.9987×10^{-6}
CPUTIME(s)	•	0.000246	0.004656	0.002546
$OR4$ $Model$(2.36)	$\|.\|_1$	8.2422×10^{-8}	7.8127×10^{-10}	1.9861×10^{-9}
	$\|.\|_2$	8.9698×10^{-9}	7.9033×10^{-11}	2.0182×10^{-10}
	$\|.\|_\infty$	1.1958×10^{-9}	8.1490×10^{-12}	2.1099×10^{-11}
CPUTIME(s)	•	0.000248	0.005140	0.002494

Figure 7, shows the graph of the numerical solution and exact solution of NSFD (2.5) for Example 3 with $a = 0.3, 0.5, 0.9$ and $\alpha = 1, \Delta t = 0.001, \Delta x = 0.5$. The figures show that the numerical and exact solutions are exactly coincident together.

FIGURE 7. The graphs of the approximate and exact solutions of the partial differential Equation (3.7) in Example 3 by the model NSFD (2.5) for $x \in [0, 1]$ and $t = 0.5, a = 0.3, 0.5, 0.9$, $\theta = 0.01$ and $\alpha = 1$, $A = B = 1, C = 0$ along with the discrete steps $\Delta t = 0.001$, $\Delta x = 0.02$.



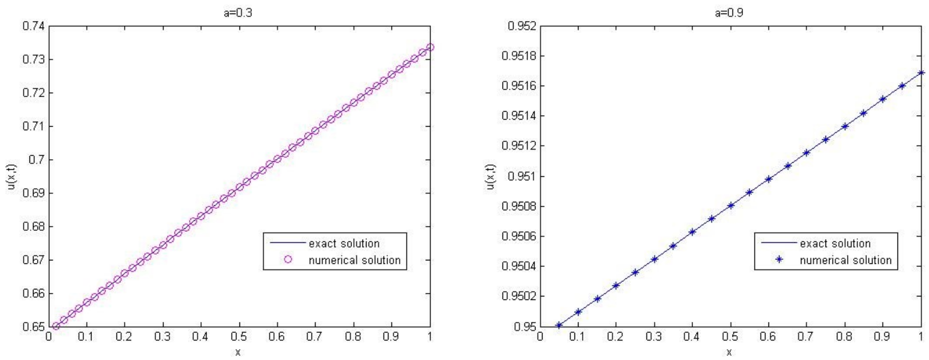
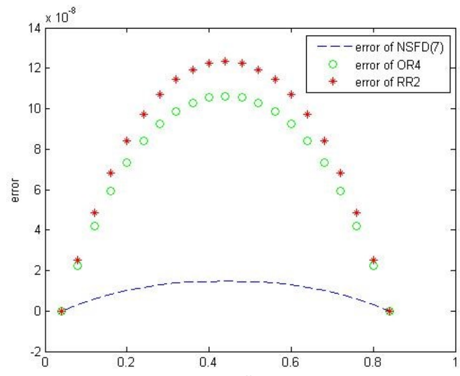


Figure 8, illustrates the graphs of the absolute errors of numerical solutions of NSFD scheme (2.5), RR2, OR4 in Example3 with $\alpha = 1$, $a = 0.99$ and $\Delta x = 0.05, \theta = 0.01$ at $t = 0.5$. This Figure shows that the absolute error for the NSFD scheme of RR2 method has the most error and the absolute error of NSFD (2.5) is lowest.

FIGURE 8. The graphs of absolute error of NSFD (2.5), OR4 and RR2 model in Example 3 with $\alpha = 1$, $a = 0.99$ and $\Delta t = 0.01$, $\Delta x = 0.05, \theta = 0.01$ at $t = 0.5, A = B = 1, C = 0$.



Tables 6, 7, illustrate the absolute error between the exact and numerical results obtained by NSFD (2.5), for the Equation (1.2) and (3.7) with some amount of θ, these results show that the absolute error of these NSFD schemes depend on θ, when θ close to zero the NSFD scheme has the lowest absolute error and when θ close to 1, the NSFD scheme has the most absolute error.

As all the Figures and Tables show, the proposed methods give very accurate results.

Table 6. Shows the absolute error for the results obtained by NSFD (2.5) in Example 1 with $\Delta t = 0.001$, $\Delta x = 0.01$, $\theta = 0.001, 0.1, 0.5, 0.6, 0.8, 0.9$, $\alpha = 1$ at $t = 0.5$.

θ	0.001	0.1	0.5	0.6	0.8	0.9
eror	2.0991×10^{-7}	2.7539×10^{-7}	4.5118×10^{-6}	1.9663×10^{-4}	0.0027	0.1229

Table 7. Shows the absolute error for the results obtained by NSFD (2.5) in Example3 with $\Delta t = 0.001$, $\Delta x = 0.01$, $\theta = 0.001, 0.1, 0.5, 0.6, 0.8, 0.9$, $\alpha = 1$, $a = 0.9$ at $t = 0.5$.

θ	0.001	0.1	0.5	0.6	0.8	0.9
error	3.3062×10^{-9}	1.5508×10^{-7}	1.5324×10^{-6}	4.0175×10^{-6}	7.8328×10^{-5}	9.1389×10^{-4}

4. Conclusion

In this paper, the solution of the Fisher's Equation was successfully approximated by a high-order numerical NSFD method. The convergence, consistency and stability analysis for this NSFD scheme has been proved. We applied two special cases of the NSFD scheme, (OR4) and (RR2) methods for the Fisher's Equation and showed that the original NSFD method has the lowest absolute error and the absolute error of these NSFD schemes depends on θ, when θ close to zero the NSFD scheme has the lowest absolute error The numerical results from the method have been compared with the exact solution and the results [10]. As the numerical results showed, performance of the methods is in excellent agreement with the exact solution.

It may be concluded that the NSFD method (2.5) and it's special cases are very powerful and efficient techniques for finding an approximate solution for various kind of linear/nonlinear problems.

Note: The data used to support the findings of this study are included in the article. We have not used any extra data in this article. We have solved the Equation by a mathematical technique and all the results are inside the paper.

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