Quintic Spline functions and Fredholm integral equation

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Abstract
A new six order method developed for the approximation Fredholm integral equation of the second kind. This method is based on the quintic spline functions (QSF). In our approach, we first formulate the Quintic polynomial spline then the solution of integral equation approximated by this spline. But we need to develop the end conditions which can be associated with the quintic spline. To avoid the reduction accuracy, we formulate the end condition in such a way to obtain the band matrix and also to obtain the same order of accuracy. The convergence of the method is discussed by using matrix algebra. Finally, four test problems have been used for numerical illustration to demonstrate the practical ability of the new method.

Keywords. Fredholm integral equation, quintic spline function (QSF).
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1. INTRODUCTION

We consider the following Fredholm integral equation of the second kind

\[ y(t) = f(t) + \int_{a}^{b} k(t, x)y(x)dx, \]  

(1.1)

where the functions \( f \) and \( k \) are known and sufficiently smooth while \( y \) is the unknown function (for problems that are not perfectly smooth, other models of splines can be used in parallel to our proposed method, for example, trigonometric spline [31]). The existence and uniqueness of the smooth solution are given by many authors, (see, for example, [15, 34]). This study is an effort to propose a new method to obtain the numerical solution of linear Fredholm integral equations. Many methods have been developed for solving this equations such as, ([5, 32, 10, 19, 21]).
A modified approach to the numerical solution of Fredholm integral equations of the second kind had been developed by Panda et al. [26]. Maleknejad and Mahmoudi [21] proposed hybrid Taylor and block pulse functions for solving these equations. A Sinc-collocation scheme for Fredholm integral equations of the second kind was proposed by Rashidinia and Zarebnia [28] and Tomoaki Okayama et al. [32] improved this version. Moreover, in [9] discrete collocation iterative method has been proposed to obtain the numerical solution of nonlinear Fredholm integral equations.

In [1] convergence of integro quartic and sextic B-spline interpolation is presented. Haghighi, A.R and Roohi, M. proposed the fractional Cubic Spline Interpolation without using the derivative values for studying the curves and surfaces [13]. Various methods have been used by many authors to solve Optimal Control Problems such as, [14, 24, 27, 25]. The method involving expansion of the unknown function of a Fredholm integral equation of the second kind in terms of polynomials has been obtained by A. Chakrabarti and S.C. Martha [6].

Also, the fractional differential equations of various types, play important roles not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena, can be converted to Volterra integral equation. A.Akgul et al. [2, 3, 4] solved many important models of fractional differential equations by reproducing kernel method.

The high accuracy cubic spline finite difference approximation for the solution of one-space dimensional non-linear wave equations have been developed by Mahanty et al. [22]. the quintic polynomial and Non-polynomial Spline functions are used on differential equations repeatedly (see, [16, 17, 33, 29, 30, 31]) but have not been directly applied for the approximation of the integral equations so far. This work is an effort to this end.

The important relations in our method are equations (2.8) and (2.9). By using the Taylor series we obtained the truncation error of order $O(h^5)$. The better accuracy of our method compared to the existing methods such as [26] with the same number of grid points is indicated in tables.

The structure of this paper is organized as follows. Quintic polynomial spline formulation is derived in section 2. In section 3, the methodology and formulation QSF method for Fredholm integral equations are brought. Section 4, is devoted to convergence analysis. Finally, in section 5, Numerical examples are given to clarify the applications and efficiency of the method. At the end, we have a conclusion of our study.

2. QUINTIC SPLINE

The numerical scheme has been developed on the domain of integration $\Omega = [a, b]$ with partitions $\Delta: a = x_0 < x_1 < \ldots < x_n = b$, where $h = \frac{b - a}{n}$. Let $S_j(x)$ be the interpolating quintic spline function which interpolates $y$ at $x_j$, by following previous work in quintic spline such as ([33, 29, 8]), defined on $[x_j, x_{j+1}]$, $j = 0, ..., n - 1$ as

$$S_j(x) = a_j(x-x_j)^5 + b_j(x-x_j)^4 + c_j(x-x_j)^3 + d_j(x-x_j)^2 + e_j(x-x_j) + f_j \quad (2.1)$$
where \(a_j, b_j, c_j, d_j, e_j\) and \(f_j\) are real numbers. To derive the coefficients \(a_j, b_j, c_j, d_j, e_j\) and \(f_j\), we define:

\[
S_j(x_i) = y_i, S_j(x_{i+1}) = y_{i+1}, S_j''(x_i) = M_i, S_j''(x_{i+1}) = F_i,
\]

\(j = 1, 2, \ldots, n\) \hspace{1cm} (2.2)

By algebraic manipulation, we get the following expression:

\[
a_j = \frac{F_{j+1} - F_j}{120h}, \quad b_j = \frac{F_j}{24}, \quad c_j = \frac{M_{j+1} - M_j}{6h} - \frac{h(F_{j+1} + F_j)}{36}, \quad d_j = \frac{M_j}{2}, \quad e_j = \frac{y_{j+1} - y_j}{h} - \frac{h(M_{j+1} + 2M_j)}{6} + \frac{h^3(7F_{j+1} + 8F_j)}{360}, \quad f_j = y_j,
\]

\(j = 1, 2, \ldots, n\) \hspace{1cm} (2.3)

The continuity of the first derivative at \(x = x_j\) implies

\[
M_{j+1} + 4M_j + M_{j-1} = \frac{6}{h^2}(y_{j+1} - 2y_j + y_{j-1}) + \frac{h^2}{60}(7F_{j+1} + 16F_j + 7F_{j-1}),
\]

\(j = 1, 2, \ldots, n\) \hspace{1cm} (2.5)

and the continuity of the third derivative at \(x = x_j\) implies

\[
M_{j+1} + 2M_j + M_{j-1} = \frac{h^2}{6}(F_{j+1} + 4F_j + F_{j-1}), \quad j = 1, 2, \ldots, n - 1
\]

\(j = 1, 2, \ldots, n\) \hspace{1cm} (2.6)

subtracting (2.6) from (2.5) and dividing by 6 we obtain

\[
M_j = \frac{1}{h^2}(y_{j+1} - 2y_j + y_{j-1}) - \frac{h^2}{120}(F_{j+1} + 8F_j + F_{j-1}), \quad j = 1, 2, \ldots, n - 1
\]

\(j = 1, 2, \ldots, n\) \hspace{1cm} (2.7)

Eliminating of \(M_j\)’s between (2.5) and (2.6) leads to the following useful relation for \(j = 2, 3, \ldots, n\)

\[
F_{j+2} + 26F_{j+1} + 66F_j + 26F_{j-1} + F_{j-2} = \frac{120}{h^4}(y_{j+2} - 4y_{j+1} + 6y_j - 4y_{j-1} + y_{j-2}),
\]

\(j = 2, 3, \ldots, n\) \hspace{1cm} (2.8)

Using the similar manner the following spline relation may be obtained

\[
M_{j+2} + 26M_{j+1} + 66M_j + 26M_{j-1} + M_{j-2} = \frac{20}{h^4}(y_{j+2} + 2y_{j+1} - 6y_j + 2y_{j-1} + y_{j-2}),
\]

\(j = 2, 3, \ldots, n\) \hspace{1cm} (2.9)

3. **The end condition**

To obtain the unique solution of the linear systems (2.8) and (2.9) we need eight more equations. By using Taylor series and method of undetermined coefficients, the boundary formulas associated with boundary conditions for the fourth-order method can be determined as follows.
To obtain the fourth-order boundary formula, we define the following identities

\[
\begin{align*}
\sum_{k=0}^{2} \gamma_k y''_k &= \frac{20}{h^2} (\sum_{k=0}^{5} \eta_k y_k) + O(h^4), j = 1 \\
\sum_{k=0}^{2} \nu_k y''_k &= \frac{20}{h^2} (\sum_{k=0}^{5} \sigma_k y_k) + O(h^4), j = 2 \\
\sum_{k=0}^{2} \nu_k y''_{N-k} &= \frac{20}{h^2} (\sum_{k=0}^{5} \eta_k y_{N-k}) + O(h^4), j = N - 1 \\
\sum_{k=0}^{2} \gamma_k y''_{N-k} &= \frac{20}{h^2} (\sum_{k=0}^{5} \sigma_k y_{N-k}) + O(h^4), j = N \\
\sum_{k=0}^{4} \nu_k y'''_{N-k} &= \frac{120}{h^2} (\sum_{k=0}^{7} \alpha_k y_{N-k}) + O(h^4), j = 2 \\
\sum_{k=0}^{4} \nu_k y'''_{N-k} &= \frac{120}{h^2} (\sum_{k=0}^{7} \beta_k y_{N-k}) + O(h^4), j = N - 1 \\
\sum_{k=0}^{4} \gamma_k y'''_{N-k} &= \frac{120}{h^2} (\sum_{k=0}^{7} \sigma_k y_{N-k}) + O(h^4), j = N
\end{align*}
\]  

(3.1)

By using Taylor’s expansion, we obtain the unknown coefficients in (3.1) as follows:

\[
\begin{align*}
(\gamma_0, \gamma_1, \gamma_2, \nu_0, \nu_1, \nu_2, \nu_3) &= (65, 26, 1, 26, 66, 26, 1), \\
(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5) &= \left(\frac{199}{60}, -\frac{649}{4579}, \frac{287}{189}, -\frac{122}{1372}, \frac{238}{59}, -\frac{13}{12}, \frac{1}{16}\right), \\
(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) &= \left(\frac{411}{240}, 10, -\frac{4579}{120}, 9, -\frac{1372}{32}, -\frac{59}{4}, -\frac{13}{12}, \frac{1}{16}\right), \\
(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) &= \left(\frac{289}{120}, -\frac{491}{189}, \frac{4327}{180}, -\frac{6527}{90}, -\frac{899}{70}, -\frac{6233}{1151}, -\frac{367}{159}\right), \\
(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7) &= \left(\frac{269}{120}, -\frac{491}{189}, \frac{4327}{180}, -\frac{6527}{90}, -\frac{899}{70}, -\frac{6233}{1151}, -\frac{367}{159}\right).
\end{align*}
\]

It can be seen that systems (2.8) and (2.9) with the above boundary condition are strictly diagonally dominant that have a unique solution to obtain \(M_0, M_1, ..., M_N\) and \(F_0, F_1, ..., F_N\). In the matrix notation, the relations (2.8) and (2.9) can be expressed as follows:

\[
PM = \frac{20}{h^2} T Y, \quad PF = \frac{120}{h^2} U Y,
\]

where \(T\) and \(U\) are five-diagonal matrices, \(Y = (y_0, y_1, y_2, ..., y_N)^T\), \(M = (M_0, M_1, M_2, ..., M_N)^T\), \(F = (F_0, F_1, F_2, ..., F_N)^T\) and

\[
P = \begin{bmatrix}
65 & 26 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
26 & 66 & 26 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
1 & 26 & 66 & 26 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
& & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & 26 & 66 & 26 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 26 & 66 & 26 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 26 & 65 \\
\end{bmatrix}
\]  

(3.2)

In (3.2) the matrix \(P\) is given by

\[
P = Q^2 + 18Q - 24I,
\]
where \( Q = (q_{ij}) \) is a three-diagonal matrix defined by
\[
q_{ij} = \begin{cases} 
4 & \text{if } i = j, \\
1 & \text{if } |i - j| = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

(3.3)

4. Methodology

Consider the following Fredholm integral equation of the second kind
\[
y(t) = f(t) + \int_a^b k(t, x)y(x)dx
\]

(4.1)

from (2.1), (2.3) and (2.4) we have
\[
S_j(x) = F_{j+1} - F_j - \frac{F_j}{120h} (x - x_j)^5 + \frac{F_j}{24} (x - x_j)^4 + \left[ \frac{M_{j+1} - M_j}{6h} + h\frac{F_{j+1} + F_j}{36} \right] (x - x_j)^3 + \frac{M_j}{2} (x - x_j)^2 \\
+ \left[ \frac{y_{j+1} - y_j}{h} - h\frac{M_{j+1} + 2M_j}{6} + h^3 \frac{7F_{j+1} + 8F_j}{360} \right] (x - x_j) + y_j,
\]

(4.2)

As we will show in Theorem (5.2), that for \( y \), solution of the Eq. (1.1), smooth enough and for all \( t \in [a, b] \) we have \( \| y - S \| = O(h^6) \). Hence, from (1.1), we obtain for all \( i = 0, \ldots, n \)
\[
y_i = f_i + \int_a^b k(t_i, x)y(x)dx \\
= f_i + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} k(t_i, x)y(x)dx \\
\approx f_i + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} k(t_i, x)S_j(x)dx + O(h^6)
\]

(4.3)
The above relation is equivalent to

\[
y_i = f_i + \sum_{j=0}^{n-1} \frac{F_{j+1}}{120h} \int_{x_j}^{x_{j+1}} k(t, x)(x - x_j)^5 \, dt - \sum_{j=0}^{n-1} \frac{F_j}{120h} \int_{x_j}^{x_{j+1}} k(t, x)(x - x_j)^5 \, dt
\]

\[
+ \sum_{j=0}^{n-1} \frac{F_j}{24} \int_{x_j}^{x_{j+1}} k(t, x)(x - x_j)^4 \, dt + \sum_{j=0}^{n-1} \left[ \frac{M_{j+1}}{6h} - \frac{F_{j+1}}{36} \right] \int_{x_j}^{x_{j+1}} k(t, x)(x - x_j)^3 \, dt
\]

\[
- \sum_{j=0}^{n-1} \left[ \frac{M_j}{6h} + \frac{F_j}{36} \right] \int_{x_j}^{x_{j+1}} k(t, x)(x - x_j)^3 \, dt + \sum_{j=0}^{n-1} \frac{M_j}{2} \int_{x_j}^{x_{j+1}} k(t, x)(x - x_j)^2 \, dt
\]

\[
+ \sum_{j=0}^{n-1} \left[ \frac{y_{j+1}}{h} - \sum_{j=0}^{n-1} \frac{M_{j+1}}{6h} + \sum_{j=0}^{n-1} \frac{7h^3 F_{j+1}}{360} \right] \int_{x_j}^{x_{j+1}} k(t, x)(x - x_j) \, dt + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} k(t, x)y_j \, dx
\]

\[
= f_i + \frac{1}{120h} \sum_{j=0}^{n} F_j a_{i,j} - \frac{1}{120h} \sum_{j=0}^{n} F_j b_{i,j} + \frac{1}{24} \sum_{j=0}^{n} F_j c_{i,j} + \frac{1}{6h} \sum_{j=0}^{n} M_j d_{i,j}
\]

\[
- \frac{h}{36} \sum_{j=0}^{n} F_j d_{i,j} - \frac{1}{6h} \sum_{j=0}^{n} M_j e_{i,j} - \frac{h}{36} \sum_{j=0}^{n} F_j e_{i,j} + \frac{1}{2} \sum_{j=0}^{n} M_j g_{i,j}
\]

\[
+ \frac{1}{h} \sum_{j=0}^{n} y_{j} l_{i,j} - \frac{h}{6} \sum_{j=0}^{n} M_j l_{i,j} + \frac{7h^3}{360} \sum_{j=0}^{n} F_j l_{i,j} + \frac{8h^3}{360} \sum_{j=0}^{n} F_j o_{i,j}
\]

\[
+ \frac{1}{h} \sum_{j=0}^{n} y_{j} o_{i,j} - \frac{h}{3} \sum_{j=0}^{n} M_j o_{i,j} + \sum_{j=0}^{n} y_{j} v_{i,j}, \ i = 0(1)n \quad (4.4)
\]

where, \( a_{i0} = b_{i0} = c_{i0} = d_{i0} = e_{i0} = g_{i0} = l_{i0} = o_{i0} = v_{i0} = 0 \).

It is supposed that \( y_i, M_i \) and \( F_i \) is approximated by \( \hat{y}_i, \hat{M}_i \) and \( \hat{F}_i \) such that satisfy in (2.8) and (2.9) for \( i = 0, 1, \ldots, n \) then we have
The matrix form of the relation (4.4) can be expressed as follows:

\[
\dot{y} = f_i + \frac{1}{120h} \sum_{j=0}^{n} \hat{F}_ja_{i,j} - \frac{1}{120h} \sum_{j=0}^{n} \hat{F}_jb_{i,j} + \frac{1}{24} \sum_{j=0}^{n} \hat{F}_jc_{i,j}
\]

\[
+ \frac{1}{6h} \sum_{j=0}^{n} \hat{M}_jd_{i,j} - \frac{h}{36} \sum_{j=0}^{n} \hat{F}_jd_{i,j} - \frac{1}{6h} \sum_{j=0}^{n} \hat{M}_je_{i,j} - \frac{h}{36} \sum_{j=0}^{n} \hat{F}_je_{i,j}
\]

\[
+ \frac{1}{2} \sum_{j=0}^{n} \hat{M}_j\dot{y}_{i,j} + \frac{1}{h} \sum_{j=0}^{n} \hat{M}_j\dot{y}_{i,j} - \frac{h}{6} \sum_{j=0}^{n} \hat{M}_jk_{i,j} + \frac{7h^3}{360} \sum_{j=0}^{n} \hat{F}_jk_{i,j}
\]

\[
+ \frac{8h^3}{360} \sum_{j=0}^{n} \hat{F}_j\dot{o}_{i,j} - \frac{1}{h} \sum_{j=0}^{n} \hat{M}_j\dot{\hat{o}}_{i,j} - \frac{h}{3} \sum_{j=0}^{n} \hat{M}_j\dot{\hat{\hat{o}}}_{i,j} + \sum_{j=0}^{n} \hat{y}_{j}\dot{v}_{i,j}, \quad i = 0(1)n
\]

The matrix form of the relation (4.4) can be expressed as follows:

\[
\hat{Y} = f + \left( \frac{1}{120h} A - \frac{1}{120h} B + \frac{1}{24} C - \frac{h}{36} D - \frac{h}{6} E + \frac{7h^3}{360} L + \frac{8h^3}{360} O \right) \hat{F}
\]

\[
+ \left( \frac{1}{6h} D - \frac{1}{6h} E + \frac{1}{2} G - \frac{h}{6} L - \frac{h}{3} O \right) \hat{M} + \left( \frac{1}{h} L - \frac{1}{h} O + V \right) \hat{Y}
\]

where, \( A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij}), E = (e_{ij}), L = (l_{ij}), O = (o_{ij}), G = (g_{ij}), V = (v_{ij}) \) and \( f = (f_0, f_1, f_2, \ldots, f_{i-1}, f_i) \). Finally,

\[
\left[ I - (X + \frac{120}{h^4} WP^{-1}U + \frac{20}{h^2} ZP^{-1}T) \right] \hat{Y} = f
\]

(4.5)

Hence, we can approximate the exact solution \( y \) using quintic spline \( \hat{S} = \hat{S}_i \) on \( \Omega_i = [x_i, x_{i+1}], \) for \( i = 0, 1, \ldots, n - 1 \) where,

\[
\hat{S}_j(x) = \frac{\hat{F}_{j+1} - \hat{F}_j}{120h}(x - x_j)^5
\]

\[
+ \frac{\hat{F}_j}{24}(x - x_j)^4 + \frac{\hat{M}_j + \hat{M}_{j+1}}{6h} + \frac{\hat{F}_{j+1} + \hat{F}_j}{36}(x - x_j)^3 + \frac{\hat{M}_j + \hat{M}_{j+1}}{2}(x - x_j)^2
\]

\[
+ \frac{\hat{y}_{j+1} - \hat{y}_j}{h} - \frac{\hat{M}_{j+1} + 2\hat{M}_j}{6} + \frac{h^3}{360}(7\hat{F}_{j+1} + 8\hat{F}_j)(x - x_j) + \hat{y}_j,
\]

(4.7)

5. Analysis of convergence

**Lemma 5.1.** Assume \( \Omega \) be a \( n \times n \) matrix with \( \|\Omega\|_\infty < 1 \), then, the matrix \( (I - \Omega) \) is invertible. In addition to \( \|(I - \Omega)^{-1}\|_\infty \leq \frac{1}{1 - \|\Omega\|_\infty} \).

**Theorem 5.2.** Let \( S(x) \) be the unique quintic spline satisfying (2.5,2.6,2.7) and (2.8), for a given function \( y \in C^{10}[a, b] \). Then the following error estimates hold:

\[ \|S - y\| = O(h^6), \]
Proof. Assuming the continuity of derivatives of $y$ of sufficiently high order, by using (2.5, 2.6, 2.7) and (2.8), we obtain

\begin{align*}
S'(x_j) &= y'_j + \frac{1}{5!}h^6 y_j^5 - \frac{1}{240}h^8 y_j^3 + O(h^{10}) \\
S''(x_j) &= y''_j + \frac{1}{240}h^4 y_j^3 + \frac{1}{31500}h^8 y_j^2 + O(h^{10}) \\
S'''(x_j) &= y'''_j - \frac{1}{240}h^4 y_j^3 + \frac{1}{31500}h^8 y_j^2 + O(h^8) \\
S^{(4)}(x_j) &= y^{(4)}_j - \frac{1}{12}h^2 y_j^1 + \frac{1}{120}h^4 y_j^2 - \frac{1}{7560}h^6 y_j^3 + O(h^8)
\end{align*}

(5.1)

Now, let $e(x) = y(x) - S(x)$, then for $0 \leq \theta \leq 1$,

\begin{align*}
&\frac{1}{\theta} \left[ \frac{1}{\theta} \left[ S^{(5)}(x_j^+^+) + S^{(5)}(x_j^-) \right] - S^{(5)}(x_j) \right] = y^{(5)}_j + \frac{1}{120}h^2 y_j^2 - \frac{1}{7560}h^6 y_j^3 + O(h^6) \\
&\frac{5}{6} \left[ S^{(5)}(x_j^+) - S^{(5)}(x_j^-) \right] = y^{(5)}_j + \frac{1}{120}h^2 y_j^2 + O(h^6)
\end{align*}

Substituting from (5.1) in (5.2), we obtain

\begin{align*}
e(x_j + \theta h) &= e(x_j) + \theta he'(x_j) + \frac{\theta^2 h^2}{2!} e''(x_j) + \frac{\theta^3 h^3}{3!} e'''(x_j) \\
&\quad + \frac{\theta^4 h^4}{4!} e^{(4)}(x_j) + \frac{\theta^5 h^5}{5!} e^{(5)}(x_j) + \ldots
\end{align*}

(5.2)

(5.3)

Lemma 5.3. Suppose $Q_{N \times N}$ is the matrix defined in equation (3.3), then $\|Q^{-1}\| \leq \frac{1 - \text{sech}^2 \frac{\theta}{2}}{\text{sech}^2 \frac{\theta}{2}}, \cos \theta = 2$.

Proof. Following [33].

Lemma 5.4. $\|P^{-1}\| \leq \frac{1 - \text{sech} \frac{\theta}{2}}{2 \text{sech} \frac{\theta}{2}}$.

Proof. Based on the definition of the matrix $Q$ and also using Lemma (5.1) and (5.3), we have:

\begin{align*}
\|P^{-1}\| &= \|Q^2 + 18Q - 24I\|^{-1} \\
&= \|\left( 18Q(I - [\frac{1}{18} Q - \frac{1}{18} Q])^{-1}\right) - 1\|^{-1} \|I - [\frac{24}{18} Q^{-1} - \frac{1}{18} Q]^{-1} Q^{-1}\|. \quad \text{Hence,}
\|P^{-1}\| \leq \frac{1}{18} \|I - [\frac{24}{18} Q^{-1} - \frac{1}{18} Q]^{-1}\| \|Q^{-1}\| \leq \frac{1 - \text{sech} \frac{\theta}{2}}{2 \text{sech} \frac{\theta}{2}} \text{□}.
\end{align*}

(5.4)

Theorem 5.5. Let $y \in C^{10}(I)$ and $k \in C^{10}(I \times I)$ such that,

\begin{align*}
\|k\|_{\infty} < \frac{2 - 1.7280 \text{sech} \frac{\theta}{2}}{3.6777 \text{sech} \frac{\theta}{2}}
\end{align*}

where $\theta = \cosh^{-1}(2)$, then (4.7) define a unique approximation and the resulting error $\hat{e} := y - \hat{s}$ satisfies

\begin{align*}
\|\hat{e}\|_{\infty} \leq \alpha h^6, \forall \psi \subset I,
\end{align*}

where $\alpha$ is a constant.
Proof. It is easy to verify that \( \|A\|_\infty, \|B\|_\infty \leq \frac{\|k\|_\infty (b-a)h^5}{6}, \|C\|_\infty \leq \frac{\|k\|_\infty (b-a)h^4}{6} \)
\( \|D\|_\infty, \|E\|_\infty \leq \frac{\|k\|_\infty (b-a)h^5}{4}, \|G\|_\infty \leq \frac{\|k\|_\infty (b-a)h^2}{3} \)
and also \( \|V\|_\infty \leq \|k\|_\infty (b-a) \). Hence, \( \|X\|_\infty \leq 2\|k\|_\infty (b-a), \|W\|_\infty \leq \frac{33}{420} \|k\|_\infty (b-a)h^2 \). Then we have:
\[
\|X + \frac{120}{h^4} WP^{-1}U + \frac{20}{h^2} ZP^{-1}T\|_\infty \\
\leq \|k\|_\infty (b-a) \left[ 2 + \frac{2834777(1 - sech \frac{\sqrt{\theta}}{2})}{17280sech \frac{\sqrt{\theta}}{2}} \right] < 1.
\]
Using lemma (5.1), the system (4.6) has a unique solution \( \hat{y} \). It follows that the Eq(4.7) defines a unique solution \( \hat{S} \). Now, let \( \hat{e} = y - \hat{y} = (y_0 - \hat{y}_0, ..., y_n - \hat{y}_n)^T \).
Then from (4.3) we get \( [I - (X + \frac{120}{h^4} WP^{-1}U + \frac{20}{h^2} ZP^{-1}T)] \hat{e} = O(h^6) \). Therefore, \( \hat{e} = [I - (X + \frac{120}{h^4} WP^{-1}U + \frac{20}{h^2} ZP^{-1}T)]^{-1} O(h^6) \), which implies by lemma (5.1), that there exist \( \alpha_1 > 0 \) such that
\[
\|\hat{e}\|_\infty \leq \frac{\alpha_1}{1 - \|k\|_\infty (b-a)} \left[ 2 + \frac{2834777(1 - sech \frac{\sqrt{\theta}}{2})}{17280sech \frac{\sqrt{\theta}}{2}} \right] h^6.
\]
On the other hand, from (2.8), we have \( (M - \hat{M}) = \frac{20}{h^2} P^{-1} T \hat{e} \) and from (2.9), we have \( (F - \hat{F}) = \frac{120}{h^4} P^{-1} U \hat{e} \). Therefore, there exist \( \alpha_3, \alpha_4 > 0 \) such that \( \|M - \hat{M}\|_\infty \leq \alpha_3 h^4 \)
and \( \|F - \hat{F}\|_\infty \leq \alpha_4 h^2 \).
In consequence, for all \( i = 0, 1, ..., n - 1 \) and \( x \in \Omega_i \), we have
\[
|S_i(x) - \hat{S}_i(x)| \\
\leq \frac{|F_{j+1} - \hat{F}_{j+1}| h^4}{18} + \frac{|E_j - \hat{E}_j| h^4}{10} + \frac{|M_{j+1} - \hat{M}_{j+1}| h^2}{3} + |M_j - \hat{M}_j| h^2 \\
+ |y_j - \hat{y}_j| + 2 |y_j - \hat{y}_j| \leq \frac{28}{180} \alpha_4 h^6 + \frac{4}{3} \alpha_3 h^6 + 3 \alpha_2 h^6 \\
\Rightarrow |S_i(x) - \hat{S}_i(x)| \leq \alpha_5 h^6,
\]
where \( \alpha_5 = \frac{28}{180} \alpha_4 + \frac{4}{3} \alpha_3 + 3 \alpha_2 \). It follows that
\[
\|y - \hat{S}\|_\infty \leq \|y - S\|_\infty + \|S - \hat{S}\|_\infty \leq \alpha_1 h^6 + \alpha_5 h^6.
\]
Thus, the proof is completed by taking \( \alpha = \alpha_1 + \alpha_5 \).

6. Computational illustrations

In this section, we have implemented our method (QSF) for solving examples of the Fredholm integral equations. To compare the presented method with the other methods, some examples are selected from other papers. The absolute error in the solution is compared with the methods in [23, 26, 28]. Comparison between results confirms the efficiency and simplicity of the presented method is considerable. It should be noted that the associated computations with the examples were performed.
Table 1. Absolute errors for Example 1.

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<td>QSF method</td>
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using MATLAB R2013a.

**Example 1.** Consider the following integral equation

$$y(t) = f(t) + \int_{0}^{1} k(t, x)y(x)dx, \quad t \in [0, 1],$$

where, $k(t, x) = e^{x-t-12}, f(t) = \cos(t) - \frac{1}{2}e^{-t-12}(e\sin 1 + e\cos 1 - 1)$ and the exact solution is given by the relation $y(t) = \cos t$. The absolute errors in the solution presented in Table 1.

**Example 2.** Consider the following integral equation

$$y(t) = f(t) + \int_{-1}^{1} k(t, x)y(x)dx, \quad t \in [-1, 1],$$

where, $k(t, x) = e^{t-x-3}, f = 1 - e^{t-2} + e^{t-4}$ and the exact solution is given by the relation $y(t) = 1$. The absolute errors in the solution presented in Table 2.

**Example 3.** Consider the following integral equation

$$y(t) = e^{-t^2}(1 - e^{-2} + e^4) + \int_{-1}^{1} e^{x^2-x-t^2-3} y(x)dx,$$

The unique solution is given by $y(t) = e^{-t^2}$. The absolute error in the solution is presented in Table 3.

**Example 4.** Consider the following Lichtenstein-Gersghorin integral equation for $\nu = 1.2$ and $-1 \leq t \leq 1$,

$$y(t) = 2\tan^{-1}\left[\frac{\nu}{\nu^2 + 1}\right] + \int_{-1}^{1} \frac{y(x)}{(\nu^2 + 1)\cos \pi(x+t)} dx.$$

This equation has been used for the determination of conformal mapping of an ellipse onto a circle as mentioned in [12]. Since the exact solution of this integral equation is unknown, the approximate solution compared with the available result of [26]. The absolute error in the solution is presented in Table 4. From this table, we find that the computed solution of this integral equation is in a good agreement with the
Table 2. Absolute errors for Example 2.

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Table 3. Absolute errors for Example 3.

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available result.

7. Conclusion

This work is an effort to obtain the numerical solution of the integral equation of the second kind. Analysis of convergence is investigated. Four test examples are considered from previous work in [23, 26, 28]. The computational solutions are compared with the exact solution. The absolute errors in the solutions by our QSF method are considerably accurate compared to [23, 26, 28].

References

Table 4. Absolute errors for Example 4.

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