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A New Error Evaluation for Singularly Perturbed Problem with Multi-Point Boundary Condition

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Abstract We consider finite difference method to find best approximation of nonlinear singularly perturbed problem which contains multi-point boundary conditions. We surveyed the asymptotic estimates of the corresponding problem that needs to be solved with maximum principle. We constructed a finite difference scheme by using Bakhvalov mesh. Based on the error estimation, we proved that this method was first-order, uniformly convergent method with in the discrete maximum norm. Finally, we conducted a numerical experiment in order to check the theoretical results.

Keywords. Singular perturbation, finite difference scheme, Bakhvalov mesh, uniformly convergence, multi-point boundary conditions.

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1. Introduction

The present study focuses the following nonlinear singularly perturbed problem that with multi-point boundary conditions:

$$-\varepsilon u''(x) + f(x,u) = 0, \qquad 0 < x < 1, \tag{1.1}$$

$$u'(0) = \frac{A}{\varepsilon}, \tag{1.2}$$

$$u(1) - \varphi(u(l_1)) = 0, \qquad 0 < l_1 < 1, \tag{1.3}$$

where $0 < \varepsilon << 1$ is a small perturbation parameter. A is given constant, and $\varphi(u)$ is sufficiently smooth on $[0,1]$. Furthermore

$$|\frac{\partial \varphi}{\partial u}| \leq m < 1, \tag{1.4}$$

$$\frac{\partial f(x,u)}{\partial u} \geq \alpha > 0, \tag{1.5}$$

and the solution, $u(x)$, for the problem (1.1)-(1.3) has boundary layers near $x = 0$ and $x = 1$ points.

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A singularly perturbed equation contains a very small parameter ε which is multiplied by the highest derivative. Certain classical numerical methods lead to difficulties in solving such problems due to the small parameter ε. Therefore, it is considered appropriate to apply numerical methods such as the finite difference method, the finite element method, etc.

Singularly perturbed equations with a multi-point boundary are widely applied in fluid mechanics and other branches of applied mathematics [1, 10, 12, 13, 25, 29]. Examples include human pupil light reflex, problems in neurobiology, modeling physiological processes and diseases, optimal control theory, optically biostable devices, signal transmission, Reynold's number in fluid dynamics, heat transfer problem, hydrodynamics, transonic gas dynamics, chemical-reactor theory, control theory, oceanography, fluid mechanics, quantum mechanics, hydro mechanical problems, meteorology, electrical networks and other physical models. For further information, please refer to [9, 16, 21, 29, 31]. In recent years, various researchers developed existence-uniqueness solutions and the different numerical methods for the solution of these problems [4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25, 26, 27, 28, 29, 32, 33].

The aim of the present paper was to propose a new perspective using Bakhvalov mesh in the finite difference approach for numerical solution nonlinear singularly perturbed problem with multi-point boundary conditions. In particular, we know that it is hard to solve singularly perturbed problems exactly. Therefore, appropriate approximate and numerical methods should be used such as finite difference method, finite elements method. Also, many kinds of equations can be solved easily by finite difference method [2, 3, 20, 24, 30, 34]. The paper was constructed as follows: The Section 2 of the present study provides the asymptotic estimates of the solution with respect to the numerical solution analyses included in later sections. In Section 3, finite difference scheme was constructed with integral identities and quadrature rules. Furthermore, the remainder terms pertained the integral form. In section 4, the error analysis was provided to demonstrate the uniformly convergent first-order on Bakhvalov mesh. Finally, the numerical experiment is presented to support the theory.

In the following sections, C and C_0 were used to refer to the positive constants independent of ε and the mesh parameter, respectively.

2. Asymptotic behaviour estimates of the exact solution

In this section, a priori estimates for the exact solution of the problem (1.1)-(1.3) were provided for the analysis of the numerical solution in the next section.

Lemma 2.1. *If $f(x,u) \in C^1[0,1]$ and $\frac{\partial f(x,u)}{\partial u} \geq \alpha > 0$, then, the solution of the problem (1.1)-(1.3) fulfills the following constraints:*

$$\|u\,(x)\| \leq C_0, \tag{2.1}$$

where,

$$C_0 = \beta^{-1}(1 + m(1-m)^{-1})|A| + (1-m)^{-1}|B| + \alpha^{-1}m(1-m)^{-1}\|f\|_\infty,$$

$$B = \varphi(0)), \quad \beta > 0, \quad |\gamma| \leq m < 1,$$

and

$$|u'(x)| \leq C\left\{1+\frac{1}{\sqrt{\varepsilon}}\left[\exp(-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}})+\exp(-\frac{\sqrt{\alpha}(1-x)}{\sqrt{\varepsilon}})\right]\right\}. \tag{2.2}$$

Proof. Using the mean value theorem for nonlinear function $f(x,u)$ in Eq. (1.1), we acquired,

$$\frac{f(x,u)-f(x,0)}{u-0}=\frac{\partial f}{\partial u}(x,u_*), \quad u_*=\gamma u, \quad 0<\gamma<1,$$

and,

$$F(x)=-f(x,0), a(x)=\frac{\partial f}{\partial u}(x,u_*).$$

Hence, the following linear equation was obtained,

$$-\varepsilon u''(x)+a(x)u(x)=F(x), \tag{2.3}$$

where $a(x)\geq\alpha>0$ and $F(x)$ are sufficiently smooth functions.

Here in, it was also possible to use the mean value theorem for Equation (3.1) and the following form was obtained:

$$\gamma=\frac{d\varphi}{du}(\eta u(l_1)), 0<\eta<1, u(1)-\gamma u(l_1)=B. \tag{2.4}$$

After using the maximum principle for the evaluation of the function $u(x)$ in Eq. (2.3), we obtained,

$$|u(x)|\leq\beta^{-1}|A|+|u(1)|+\alpha^{-1}\|F\|_\infty. \tag{2.5}$$

From boundary condition (2.4), we have

$$|u(1)|\leq|B|+m|u(l_1)|. \tag{2.6}$$

By setting $x=l_1$ in (2.5), we obtain

$$|u(l_1)|\leq\beta^{-1}|A|+|u(1)|+\alpha^{-1}\|F\|_\infty. \tag{2.7}$$

By substituting into (2.6), (2.7), we have

$$|u(1)|\leq|B|+m\left(\beta^{-1}|A|+|u(1)|+\alpha^{-1}\|F\|_\infty\right), \tag{2.8}$$

the estimation (2.8) is written in Eq. (2.5) and so, we have

$$|u(x)|\leq\beta^{-1}(1+m(1-m)^{-1})|A|+(1-m)^{-1}|B|+\alpha^{-1}m(1-m)^{-1}\|f\|_\infty,$$

where,

$$C_0=\beta^{-1}(1+m(1-m)^{-1})|A|+(1-m)^{-1}|B|+\alpha^{-1}m(1-m)^{-1}\|f\|_\infty,$$

which proves the inequality (2.1).

The proof of inequality (2.2) is almost identical to that of [13, 15] □

3. Difference approximation of the problem

Bakhvalov mesh was defined in the present section and the problem (1.1)-(1.3) was discretized with the finite difference method on Bakhvalov type mesh.

Notations: It is essential to remind that the following finite difference for any mesh function $g_i = g(x_i)$ given on $\bar{\omega}_N$ was,

$$g_{\bar{x},i} = \frac{g_i - g_{i-1}}{h_i}, \quad g_{x,i} = \frac{g_{i+1} - g_i}{h_{i+1}}, \quad g_{\overset{0}{x},i} = \frac{g_{x,i} + g_{\bar{x},i}}{2},$$

$$g_{\widehat{x},i} = \frac{g_{i+1} - g_i}{\hbar_i}, \quad g_{\bar{x}\widehat{x},i} = \frac{g_{x,i} - g_{\bar{x},i}}{\hbar_i}, \quad \hbar_i = \frac{h_i + h_{i+1}}{2},$$

$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{\omega}_N} := \max_{0\le i\le N} |g_i| .$$

3.1. Bakhvalov mesh.

The difference scheme should be $\varepsilon-$ uniform; therefore, Bakhvalov mesh was used for the problem (1.1)-(1.3). For a positive integer N, the interval $[0,1]$ was divided into three subintervals $[0,\sigma]$, $[\sigma, 1-\sigma]$ and $[1-\sigma,1]$. In practice, $\sigma << 1$ was usually employed to make the mesh fine on the intervals $[0,\sigma]$ and $[1-\sigma,1]$ and coarse on the interval $[\sigma,1-\sigma]$. Here σ depicts the transition point and could be written as follows:

$$\sigma = \min\left\{\frac{1}{4}, -(\sqrt{\alpha})^{-1}\varepsilon\ln\varepsilon\right\}.$$

A set of the mesh points $\bar{\omega}_N = \{x_i\}_{i=0}^N$ were introduced,

$$x_i = \begin{cases} -(\sqrt{\alpha})^{-1}\varepsilon\ln\left[1-(1-\varepsilon)\frac{4i}{N}\right], \quad x_i\in[0,\sigma], \quad i=0,...,\frac{N}{4}; \\ -(\sqrt{\alpha})^{-1}\varepsilon\ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4i}{N}\right], \quad x_i\in[0,\sigma], \quad i=0,...,\frac{N}{4}; \\ \sigma+(i-\frac{N}{4})h^{(1)}, \quad i=\frac{N}{4}+1,...,\frac{3N}{4}, \quad h^{(1)}=\frac{2(1-2\sigma)}{N}; \\ 1-\sigma-(\sqrt{\alpha})^{-1}\varepsilon\ln\left[1-(1-\varepsilon)\frac{4(i-\frac{3N}{4})}{N}\right], \quad x_i\in[1-\sigma,1], \\ \qquad i=\frac{3N}{4}+1,...,N; \\ 1-\sigma=\frac{3}{4}, \quad 1-\sigma-(\sqrt{\alpha})^{-1}\varepsilon\ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4(i-\frac{3N}{4})}{N}\right], \\ \qquad x_i\in[1-\sigma,1], \quad i=\frac{3N}{4}+1,...,N. \end{cases}$$

The step-size was set as $h_i = x_i - x_{i-1}$ for $i = 1, 2, ..., N$.

3.2. Construction of the difference scheme on Bakhvalov mesh.

Non-uniform mesh on the interval $[0,1]$ was denoted by ω_N in the following equations,

$$\omega_N = \{0 < x_1 < x_2 < ... < x_{N-1} < 1\},$$

and

$$\bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = 1\}.$$

Eq. (1.1) was integrated over (x_{i-1}, x_{i+1}) to construct the difference scheme for this equation as follows,

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} -\varepsilon u''(x)\varphi_i(x)dx + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x,u)\varphi_i(x)dx = 0,\ \ i = 1, ..., N-1, \tag{3.1}$$

here $\{\varphi_i(x)\}_{i=1}^{N-1}$ represent the basis functions,

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) = \frac{x-x_{i-1}}{h_i}, & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) = \frac{x_{i+1}-x}{h_{i+1}}, & x_i < x < x_{i+1}, \\ \quad 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

and the solution of the problem for $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ were determined respectively as,

$$\begin{aligned} -\varepsilon\varphi_i^{(1)\prime\prime} &= 0, \quad x_{i-1} < x < x_i, \\ \varphi_i(x_{i-1}) &= 0, \quad \varphi_i(x_i) = 1, \end{aligned}$$

$$\begin{aligned} -\varepsilon\varphi_i^{(2)\prime\prime} &= 0, \quad x_i < x < x_{i+1}, \\ \varphi_i(x_i) &= 1, \quad \varphi_i(x_{i+1}) = 0. \end{aligned}$$

Due to several calculations in Eq. (3.1), using the interpolating quadrature rules from [4] with weight functions $\varphi_i(x)$ on subintervals (x_{i-1}, x_{i+1}), it was possible to obtain,

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi'(x)dx + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x,u)\varphi_i(x)dx = 0, \tag{3.2}$$

and

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi'(x)dx + f(x_i, u_i) + R_i = 0, \tag{3.3}$$

where R_i was the local truncation error and it was in the following form

$$R_i = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x)\int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx}f(x,u(\xi))K_0^*(x,\xi)d\xi. \tag{3.4}$$

In order to obtain the approximation for the boundary condition (1.2), we integrated (1.1) over (x_0, x_1)

$$\int_{x_0}^{x_1} -\varepsilon u''(x)\varphi_0(x)dx + \int_{x_0}^{x_1} f(x_0,u_0)\varphi_0(x)dx = 0,\ \ i = 1, ..., N-1, \tag{3.5}$$

and

$$\varepsilon u'(x_0) + \int\limits_{x_0}^{x_1} [u'(x)\varphi_0'(x) + f(x_0, u_0)\varphi_0(x)]\, dx = 0. \tag{3.6}$$

Due to several arrangements in Eq. (3.6), we obtained,

$$\varepsilon u'(x_0) - \varepsilon u_{x,0} + \theta_0 f(x_0, u_0) + r_0 = 0, \tag{3.7}$$

with the reminder term

$$r_0 = \int\limits_{x_0}^{x_1} dx \varphi_0(x) \int\limits_{x_0}^{x_1} \frac{d}{dx} f(x, u(\xi)) K_0^*(x,\xi) d\xi. \tag{3.8}$$

The approximation for the boundary condition (1.3) was obtained as

$$u_N - \varphi(u_{N_0}) + r_1 = 0, \tag{3.9}$$

with r_1 reminder term as follows

$$r_1 = [u(l_1) - u(x_{N_0})]\, \varphi'(\xi), \tag{3.10}$$

where ξ was the intermediate point between $u(x_{N_0})$ and $u(x_{l_1})$.

The following difference scheme for the problem (1.1)-(1.3) was proposed via neglecting R_i, r_0 and r_1 in Eqs. (3.4), (3.8) and (3.10), respectively.

$$-\varepsilon y_{\bar{x}\hat{x},i} + f(x_i y_i) = 0,\ i = 1, ..., N-1, \tag{3.11}$$

$$\varepsilon y_{x,0} - A - \theta_0 f(x_0, y_0) = 0, \tag{3.12}$$

$$y_N - \varphi(y_{N_0}) = 0. \tag{3.13}$$

4. Convergence analysis

The present section provides the convergence analysis of the method for the problem (1.1)-(1.3). Given the error function $z_i = y_i - u_i$, $i = 0, 1, ..., N$ and z_i as the solution of the discrete problem, it was possible to write down the following equations:

$$-\varepsilon z_{\bar{x}\hat{x},i} + [f(x_i, y_i) - f(x_i, u_i)] = R_i,\ i = 1, ..., N-1, \tag{4.1}$$

$$\varepsilon z_{x,0} - \theta_0 [f(x_0, y_0) - f(x_0, u_0)] = r_0, \tag{4.2}$$

$$z_N - [\varphi(y_{N_0}) - \varphi(u_{N_0})] = r_1. \tag{4.3}$$

Lemma 4.1. *The following estimate was valid for the solution of z_i in the discrete problem (4.1)-(4.3)*

$$\|z\|_{\infty,\bar{\omega}_N} \le C \left\{ \|R\|_{\infty,\omega_N} + |r_0| + |r_1| \right\}, \tag{4.4}$$

holds.

Proof. The discrete problem (4.1)-(4.3) could be rewritten as

$$-\varepsilon z_{\bar{x}\hat{x},i} + \bar{a}_i z_i = R_i, i = 1, ..., N-1, \tag{4.5}$$

$$\varepsilon z_{x,0} - \theta_0 \bar{b} z_0 = r_0, \tag{4.6}$$

$$z_N - \bar{\gamma} z_{N_0} = r_1, \tag{4.7}$$

where

$$\bar{a_i} = \frac{\partial f}{\partial u}(x_i, \bar{y_i}), \quad \bar{b} = \frac{\partial f}{\partial u}(x_0, \bar{y_0}), \quad \bar{\gamma} = \varphi'(y\bar{_{N_0}}),$$

$$\bar{y_i}, \quad \bar{y_0}, \quad y\bar{_{N_0}} - intermediate \;\; values.$$

Based on the maximum principle for (4.5)-(4.7), we obtained,

$$|z_i| \leq \beta^{-1}|r_0| + |z_N| + \alpha^{-1} \|R\|_{\infty,\omega_N}, \tag{4.8}$$

and

$$|z_N| \leq |r_1| + k|z_{N_0}|, \tag{4.9}$$

where

$$\bar{\gamma} \leq k < 1.$$

In Eq. (4.8), $i = N_0$ yielded,

$$|z_{N_0}| \leq \beta^{-1}|r_0| + |z_N| + \alpha^{-1} \|R\|_{\infty,\omega_N}, \tag{4.10}$$

and applying (4.8)-(4.10), it was possible to obtain,

$$|z_i| \leq \beta^{-1}|r_0| + (1-k)^{-1} \left\{ k\beta^{-1}|r_0| + |r_1| + k\alpha^{-1} \|R\|_{\infty,\omega_N} \right\}. \tag{4.11}$$

Due to several calculations in Eq. (4.11), we obtained,

$$|z_i| \leq C \left\{ \|R\|_{\infty,\omega_N} + |r_0| + |r_1| \right\}, \tag{4.12}$$

which, along with (4.12), proved Lemma 4.1. □

Lemma 4.2. *Based on the assumptions for Lemma 2.1 in Section 2, for the error functions R_i, r_0 and r_1 we obtained,*

$$\|R\|_{\infty,\omega_N} \leq CN^{-1}, \tag{4.13}$$

$$r_0 \leq CN^{-1}, \tag{4.14}$$

$$r_1 \leq CN^{-1}. \tag{4.15}$$

Proof. The error functions R_i, r_0 and r_1 were evaluated for the intervals $[0, \sigma]$, $[\sigma, 1-\sigma]$ and $[1-\sigma, 1]$ on Bakhvalov mesh, respectively.

From the previous expression (3.4) for R_i and (2.2), we deduced that

$$\begin{aligned} |R_i| &\leq \int\limits_{x_{i-1}}^{x_{i+1}} \varphi_i(x)dx \int\limits_{x_{i-1}}^{x_{i+1}} |\frac{\partial f(\xi, u(\xi))}{\partial \xi} + \frac{\partial f}{\partial u}\frac{du(\xi)}{d\xi}|d\xi \\ &\leq C \int\limits_{x_{i-1}}^{x_{i+1}} (1 + u'(\xi))d\xi \\ &\leq C\left\{ \int\limits_{x_{i-1}}^{x_{i+1}} (1 + \frac{1}{\sqrt{\varepsilon}}\left[\exp(-\frac{\sqrt{\alpha x}}{\sqrt{\varepsilon}}) + \exp(-\frac{\sqrt{\alpha x}(1-x)}{\sqrt{\varepsilon}})\right])dx \right\}. \end{aligned} \tag{4.16}$$

1) The error function R_i was evaluated for $\sigma < \frac{1}{4}$ and $x_i \in [0, \sigma]$:

$$x_{i-1} = -(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1 - (1-\varepsilon)\frac{4(i-1)}{N}\right], \tag{4.17}$$

$$h_i = -(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1 - (1-\varepsilon)\frac{4i}{N}\right] + (\sqrt{\alpha})^{-1}\varepsilon \ln\left[1 - (1-\varepsilon)\frac{4(i-1)}{N}\right]. \tag{4.18}$$

It was possible to write the following equations,

$$h_i = (\sqrt{\alpha})^{-1}\varepsilon\frac{4(1-\varepsilon)N^{-1}}{1 - 4i_1(1-\varepsilon)N^{-1}} \leq CN^{-1}, \tag{4.19}$$

and

$$\hbar_i \leq CN^{-1}. \tag{4.20}$$

Based on the mean value theorem for (4.18).

Also, a simple calculation (4.16) yielded,

$$\exp(-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}) + \exp(-\frac{\sqrt{\alpha}x_{i+1}}{\sqrt{\varepsilon}}) \leq CN^{-1}, \tag{4.21}$$

and

$$\exp(-\frac{\sqrt{\alpha}(1-x_{i+1})}{\sqrt{\varepsilon}}) + \exp(-\frac{\sqrt{\alpha}(1-x_{i-1})}{\sqrt{\varepsilon}}) \leq CN^{-1}. \tag{4.22}$$

Hence, (4.16), (4.21) and (4.22) led to,

$$|R_i| \leq CN^{-1}, \quad i = 0, ..., \frac{N}{4}.$$

2) The error funtion R_i was evaluated for $\sigma = \frac{1}{4}$ and $x_i \in [0, \sigma]$:

$$x_{i-1} = -(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1 - (1 - e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4(i-1)}{N}\right], \tag{4.23}$$

$$h_i = -(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4i}{N}\right] +(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4(i-1)}{N}\right]. \tag{4.24}$$

Once the mean value theorem in (4.24) was used, it was possible to obtain,

$$h_i = (\sqrt{\alpha})^{-1}\varepsilon \frac{\ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4(i-1)}{N}\right]}{\ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4i}{N}\right]} \leq CN^{-1}. \tag{4.25}$$

Therefore, based on (4.16) and (4.25), it was possible to write,

$$|R_i| \leq CN^{-1}, \quad i = 0, ..., \frac{N}{4}.$$

3) The error funtion R_i was evaluated for $x_i \in [\sigma, 1-\sigma]$:

$$h^{(1)} = \frac{2(1-2\sigma)}{N} \leq CN^{-1}, \tag{4.26}$$

and applying (4.16) and (4.26), it was possible to obtain,

$$|R_i| \leq Ch^{(1)} \leq CN^{-1}, \quad i = \frac{N}{4}+1, ... \frac{3N}{4}.$$

4) The error function R_i was evaluated for $x_i \in [1-\sigma, 1]$:

$$x_{i-1} = 1-\sigma-(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1-(1-\varepsilon)\frac{4(i-1-\frac{3N}{4})}{N}\right], \tag{4.27}$$

$$h_i = -(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1-(1-\varepsilon)\frac{4(i-\frac{3N}{4})}{N}\right] +(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1-(1-\varepsilon)\frac{4(i-1-\frac{3N}{4})}{N}\right]. \tag{4.28}$$

When the mean value theorem in (4.28) was applied, we obtained,

$$h_i \leq CN^{-1}. \tag{4.29}$$

Thus, the following inequality could be written based on (4.16) and (4.29):

$$|R_i| \leq CN^{-1}, \quad i = \frac{3N}{4}+1, ..., N.$$

5) The error function R_i was evaluated for $1-\sigma = \frac{3}{4}$:

$$x_{i-1} = 1\sigma-(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4(i-1)}{N}\right], \tag{4.30}$$

$$\begin{aligned} h_i &= -(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4(i-\frac{3N}{4})}{N}\right] \\ &\quad +(\sqrt{\alpha})^{-1}\varepsilon \ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4(i-1-\frac{3N}{4})}{N}\right]. \end{aligned} \tag{4.31}$$

Once the mean value theorem in (4.31) was used, it was possible to deduce that

$$h_i = (\sqrt{\alpha})^{-1}\varepsilon \frac{\ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4(i-1-\frac{3N}{4})}{N}\right]}{\ln\left[1-(1-e^{\frac{-\sqrt{\alpha}}{4\varepsilon}})\frac{4(i-\frac{3N}{4})}{N}\right]} \leq CN^{-1}. \tag{4.32}$$

And, together with (4.16), it yielded,

$$|R_i| \leq CN^{-1}, \quad i = \frac{3N}{4}+1, ..., N.$$

Herein, it was as well possible to estimate the error function r_0 based on the explicit expression in (3.8), we obtained,

$$\begin{aligned} |r_0| &\leq \int_{x_0}^{x_1} \varphi_0(x)dx \int_{x_0}^{x_1} |\frac{\partial f(x,u}{\partial x} + \frac{\partial f}{\partial u}\frac{du}{dx}|dx \\ &\leq C\int_{x_0}^{x_1}(1+u'(x))dx \\ &\leq Ch_1\int_{x_0}^{x_1}(1+\frac{1}{\sqrt{\varepsilon}}\left[\exp(-\frac{\sqrt{\alpha x}}{\sqrt{\varepsilon}})+\exp(-\frac{\sqrt{\alpha x}(1-x)}{\sqrt{\varepsilon}})\right])dx, \end{aligned} \tag{4.33}$$

in which

$$\exp(-\frac{\sqrt{\alpha}x_0}{\sqrt{\varepsilon}})+\exp(-\frac{\sqrt{\alpha}x_1}{\sqrt{\varepsilon}}) \leq CN^{-1}, \tag{4.34}$$

and

$$\exp(-\frac{\sqrt{\alpha}(1-x_1)}{\sqrt{\varepsilon}})+\exp(-\frac{\sqrt{\alpha}(1-x_0)}{\sqrt{\varepsilon}}) \leq CN^{-1}. \tag{4.35}$$

The inequalities (4.33), (4.34) and (4.35) enabled the following equation:

$$|r_0| \leq Ch^{(1)}\left\{Ch^{(1)}+2CN^{-1}\right\} \leq CN^{-1}.$$

Finally, it was possible to estimate the error function r_1. Based on (3.10), (4.34) and (4.35), we obtained

$$\begin{aligned}|r_1| &\leq |\varphi'(\xi)| \int\limits_{x_{N_0}}^{l_1} u'(\eta)d\eta \\ &\leq l_1 - x_{N_0} + \exp(-\frac{\sqrt{\alpha}x_{N_0}}{\sqrt{\varepsilon}}) + \exp(-\frac{\sqrt{\alpha}l_1}{\sqrt{\varepsilon}}) \\ &\quad + \exp(-\frac{\sqrt{\alpha}(1-l_1)}{\sqrt{\varepsilon}}) + \exp(-\frac{\sqrt{\alpha}(1-x_{N_0})}{\sqrt{\varepsilon}}) \\ &\leq \hbar_i + CN^{-1} \\ &\leq CN^{-1}. \end{aligned} \tag{4.36}$$

Thus, it would be possible to confidently yield the numerical solution of the problem (1.1)-(1.3), using the following theorem on $\varepsilon-$ uniform convergence for the presented method. With these expressions the proof of Lemma 4.2 could be obtained. □

The following theorem demonstrates the convergence result of the present study.

Theorem 4.3. *Let $u\left(x\right)$ be the exact solution of the problem (1.1)-(1.3) and y_i be the discrete solution of the difference scheme (3.11)-(3.13). Then the following estimates yields,*

$$\|y-u\|_{\infty,\bar{\omega}_N} \leq CN^{-1}.$$

Proof. The proof of this theorem could be obtained by using the Lemma 4.1 and Lemma 4.2. The proof is highly clear and easy. □

5. Algorithm and numerical results

This section gives on the demonstration of the following quasi-linearization procedure for the difference scheme (3.11)-(3.13). Moreover, the effectiveness of the presented method was confirmed by applying it to a nonlinear problem (1.1)-(1.3) for $i = 1, ..., N-1$:

$$\left(\frac{\varepsilon}{\hbar_i h_i}\right) y_{i-1} - \left(\frac{2\varepsilon}{h_i h_{i+1}} - \frac{\partial f}{\partial y}(x_i, y_i^{(n-1)})\right) y_i + \left(\frac{\varepsilon}{\hbar_i h_{i+1}}\right) y_{i+1} = -f_i,$$

$$\varepsilon y_{x,0}^{(n)} = A + f(x_0 - y_0^{(n-1)} + \frac{\partial f}{\partial y}(x_0, y_0^{(n-1)})(y_0^{(n)} - y_0^{(n-1)})),$$

$$y_N^{(n)} = \varphi(y_{N_0}^{(n-1)} + \frac{\partial \varphi}{\partial y}(y_{N_0}^{(n-1)})(y_{N_0}^{(n)} - y_{N_0}^{(n-1)})),$$

$$A_i = \frac{\varepsilon}{\hbar_i h_i}, B_i = \frac{\varepsilon}{\hbar_i h_{i+1}}, C_i = \frac{2\varepsilon}{\hbar_i h_{i+1}} - \frac{\partial f}{\partial y}(x_i, y_i^{(n-1)}),$$

$$F_i = f(x_i - y_i^{(n-1)}) - \frac{\partial f}{\partial y}(x_i, y_i^{(n-1)}),$$

$$\alpha_1 = 1,\ \beta_1 = \varepsilon^{-1} h_1 (A + f(x_0, y_0^{(n-1)}) + \frac{\partial f}{\partial y}(x_0, y_0^{(n-1)})(y_0^{(n)} - y_0^{(n-1)})),$$

$$\alpha_{i+1} = \frac{B_i}{C_i - A_i \alpha_i},\ \beta_{i+1} = \frac{F_i + A_i \beta_i}{C_i - A_i \alpha_i},\quad i = 1, ..., N-1,$$

$$y_i^{(n)} = \alpha_{i+1} y_{i+1}^{(n)} + \beta_{i+1},\quad i = N-1, ..., 2, 1.$$

This algorithm was stable due to $A_i > 0$, $B_i > 0$, $C_i > A_i + B_i$, $i = 1, 2, ..., N$.

Subsequently, the following problem was taken into consideration in order to prove that the presented method worked:

$$-\varepsilon u''(x) - \exp(-u(x)) = 0,\quad 0 < x < 1,$$

$$u'(0) = \frac{1}{\varepsilon},\quad u(1) - \varphi(u(\frac{1}{2})) = 0,\quad \varphi(u(x)) = \frac{1}{2} u(x) - 2.$$

The exact solution of this problem remained unavailable. Therefore, it was necessary to use the double mesh method [8] which was expected to provide the errors and rates of convergence for the present example.

The $\varepsilon-$ uniform convergence rates were calculated using the following expression:

$$P^N = \frac{\ln\left(e^N / e^{2N}\right)}{\ln 2}.$$

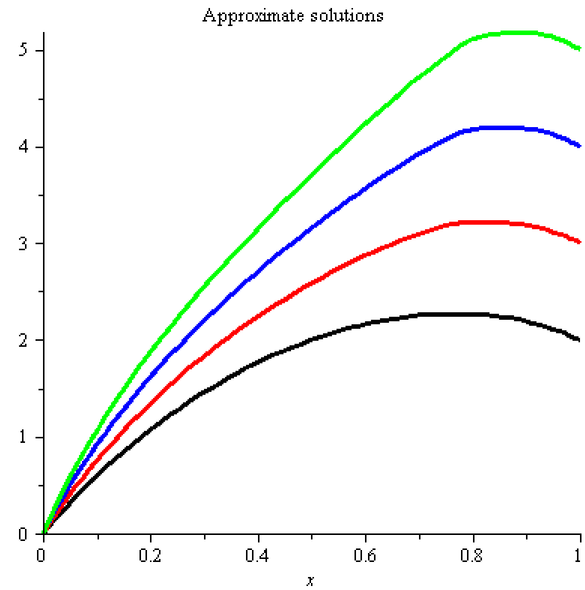
The error estimates were also denoted by

$$e_\varepsilon^N = \left\| y^{\varepsilon,N} - y^{\varepsilon,2N} \right\|_{\infty,\bar{\omega}_N},\quad e^N = \max_\varepsilon e_\varepsilon^N.$$

TABLE 1. Errors e^N and rates of convergence p^N for test problem

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-1}	0.2137822	0.1078726	0.0542131	0.0271136	0.0135880	0.0067903
	0.98	0.99	0.99	0.99	1.00	
3^{-1}	0.2514692	0.1305050	0.0665697	0.0336397	0.0168998	0.0084705
	0.94	0.97	0.98	0.99	0.99	
4^{-1}	0.3316944	0.1764517	0.0912917	0.0464764	0.0234520	0.0117883
	0.91	0.95	0.97	0.98	0.99	
5^{-1}	0.4114538	0.2181725	0.1143295	0.0586237	0.0296941	0.0150992
	0.91	0.93	0.96	0.98	0.97	
6^{-1}	0.4834303	0.2566565	0.1360977	0.0702530	0.0357047	0.0179924
	0.91	0.91	0.95	0.97	0.98	
7^{-1}	0.5383700	0.2925638	0.1568564	0.0814925	0.0416001	0.0204012
	0.88	0.89	0.94	0.97	1.02	
e^N	0.5383700	0.2925638	0.1568564	0.0814925	0.0416001	0.0204012
p^N	0.88	0.89	0.94	0.97	1.02	

FIGURE 1. Approximate solution distribution for $N = 32$, $\varepsilon = 2^{-1}, 3^{-1}, 5^{-1}, 7^{-1}$.



The method of this paper focused on was applied to the above the example. When N takes increasing values, it was observed in table and graphs that the convergence rate of the smooth convergence speed p^N was first order. When the ε is smaller, approximate solution curve is closer to the axes as presented in Figure 1. Hence, the numerical results indicated that the proposed scheme was working effectively.

6. Conclusion

The present study examined the nonlinear singularly perturbed problem that contains multi-point boundary conditions through the finite difference method. It was revealed that the method displayed uniform convergence with respect to the perturbation parameter ε in the discrete maximum norm. Furthermore, the results of the theory and the numerical experiment provided results consistent with each other. Another main finding of the study was the potential of the proposed method to be used for more complicated nonlinear singularly perturbed problems.
As a suggestion, with the help of this study, the convergence rate can be increased to higher degrees.

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References

[1] N. Adzic, *Spectral approximation and nonlocal boundary value problems*, Novi Sad J. Math, *30*(3) (2000), 1-10.

[2] P. Agarwal, S. S. Dragomir, M. Jleli, and B. Samet, *Advances in Mathematical Inequalities and Applications*, Trends in Mathematics, Singapore, 2018.

[3] P. Agarwal and A. A. El-Sayed, *Non-standard finite difference and Chebyshev collocation methods for solving fractional diffusion equation*, Physica A: Statistical Mechanics and Its Applications *500* (2018), 40-49.

[4] G. M. Amiraliyev, *Difference method for a singularly perturbed initial value problem*, Turkish J. Math, *22* (1998), 283-294.

[5] G. M. Amiraliyev and M. Cakir, *A uniformly convergent difference scheme for singularly perturbed problem with convective term and zeroth order reduced equation*, International Journal of Applied Mathematics, *2* (2000), 1407-1419.

[6] G. M. Amiraliyev and M. Cakir, *Numerical solution of the singularly perturbed problem with nonlocal condition*, Applied Mathematics and Mechanics (English Edition), *23* (2002), 755-764.

[7] D. Arslan, *Finite difference method for solving singularly perturbed multi-point boundary value problem*, Journal of the Institute of Natural and Applied Sciences, *22* (2017), 64-75.

[8] N. S. Bakhvalov, *On optimization of methods for solving boundary value problems in the presence of a boundary layer*, Zhurnal Vychislitel'noi Matematikii Matematicheskoi Fiziki, *9* (1969), 841-859.

[9] A. V. Bitsadze and A. A. Samarskii, *On some simpler generalization of linear elliptic boundary value problems*, Doklady Akademii Nauk SSSR, *185* (1969), 739-40.

[10] M. Cakir and G. M. Amiraliyev, *Numerical solution of the singularly perturbed three-point boundary value problem*, International Journal of Computer Mathematics, *84* (2007), 1465-1481.

[11] M. Cakir, *Uniform second-order difference method for a singularly perturbed three-point boundary value problem*, Advences in Difference Equations, *2010* (2010), 13 pages. doi:10.1155/2010/102484.

[12] M. Cakir and G. M. Amiraliyev, *A numerical method for a singularly perturbed three-point boundary value problem*, Journal of Applied Mathematics, *2010* (2010), 17 pages, doi:10.1155/2010/495184.

[13] M. Cakir and D. Arslan, *A numerical method for nonlinear singularly perturbed multi-point boundary value problem*, Journal of Applied Mathematics and Phyics, *4* (2016), 1143-1156.

[14] M. Cakir and D. Arslan, *Numerical solution of the nonlocal singularly perturbed problem*, International Journal of Modern Research in Engineering and Technology, *1* (2016), 13-24.

[15] M. Cakir and D. Arslan, *Finite difference method for nonlocal singularly perturbed problem*, International Journal of Modern Research in Engineering and Technology, *1* (2016), 25-39.

[16] R. Chegis, A. Stikonas, O. Stikoniene and O. Suboc, *A monotonic finite difference scheme for a parabolic problem with nonlocal conditions*, Differ. Equ, *38* (2002), 1027-1037.

[17] E. Cimen and G. M. Amiraliyev, *A uniform convergent method for singularly perturbed nonlinear differential-difference equation*, Int. J. Comput. Math, *9* (2017), 191-199.

[18] E. Cimen and M. Cakir, *Numerical treatment of nonlocal boundary value problem with layer behaviour*, Bull. Belg. Math. Soc. Simon Stevin, *24* (2017), 339-352.

[19] E. Cimen, *A priori estimates for solution of singularly perturbed boundary value problem with delay in convection term*, J. Math. Anal, *8* (2017), 202-211.

[20] H. Demiray, *Exact solution of perturbed KdV equation with variable dissipation coefficient*, Applied and Computational Mathematics, *16*(1) (2017), 12-16.

[21] P. A. Farell, J. J. H. Miller, E. O'Riordan, and G.I. Shishkin, *A uniformly convergent finite difference scheme for a singularly perturbed semilinear equation*, SIAM Journal on Numerical Analysis, *33* (1996), 1135-1149.

[22] W. Gao, G. Yel, H. M. Baskonus, and C. Cattani, *Complex Solitons in the Conformable (2+1)-dimensional Ablowitz-Kaup-Newell-Segur Equation*, Aims Mathematics, *5* (2019), 507-521.

[23] W. Gao, B. Ghanbari and H. M. Baskonus, *New numerical simulations for some real world problems with Atangana-Baleanu fractional derivative*, Chaos, Solitons and Fractals, *128* (2019), 34-43.

[24] F. A. Ghassabzade, J. S. Nadjafi, and A. R. Soheili, *A method based on the meshless approach for singularly perturbed differential-difference equations with Boundary layers*, Computational Methods for Differential Equations, *6* (2018), 295-311.

[25] C. P. Gupta and S. I. Trofimchuk, *A sharper condition for the solvability of a three-point second order boundary value problem*, Journal of Mathematical Analysis and Applications, *205* (1997), 586-597.

[26] D. Herceg and K. Surla, *Solving a nonlocal singularly perturbed nonlocal problem by splines in tension*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Math., *21* (1991) 119-132.

[27] T. Jankowski, *Existence of solutions of differential equations with nonlinear multipoint boundary conditions*, Comput. Math. Appl, *47* (2004), 1095-110.

[28] A. H. Nayfeh, *Introduction to perturbation techniques*, Wiley, New York, 1993.

[29] R. E. O'Malley, *Singular perturbation methods for ordinary differential equations*, Springer Verlag, New York, 1991.

[30] M. Rahimian and M. Nadjafikhah, *Approximate symmetry and exact solutions of the perturbed nonlinear Klein-Gordon equation*, Computational Methods for Differential Equations, *7* (2019), 266-275.

[31] H. G. Roos, M. Stynes, and L. Tobiska, *Robust numerical methods singularly perturbed differential equations: Convection-Diffusion and Flow Problems*, Springer-Verlag, Berlin, 2008.

[32] P. Veeresha, D. G. Prakasha, and H. M. Baskonus, *An efficient technique for a fractional-order system of equations describing the unsteady flow of a polytropic gas*, Pramana-Journal of Physic, *93* (2019), 1-11.

[33] P. Veeresha, D. G. Prakasha, and H. M. Baskonus, *Solving Smoking Epidemic Model of Fractional Order Using a Modified Homotopy Analysis Transform Method*, Mathematical Sciences, *13* (2019), 115-128.

[34] H. Zhou, L. Yang and P. Agarwal, *Solvability for fractional p-Laplacian differential equations with multipoint boundary conditions at resonance on infinite interval*, Journal of Applied Mathematics and Computing, *53* (2017), 51-76.