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First order linear fuzzy differential equations with fuzzy variable coefficients

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Abstract In this study, we investigate the first order linear fuzzy differential equations with fuzzy variable coefficients. Appearance of the multiplication of a fuzzy variable coefficient by an unknown fuzzy function in linear differential equations persuades us to employ the concept of the cross product of fuzzy numbers. Mentioned product overcomes to some difficulties we face to in the case of the usual product obtained by Zadeh's extension principle. Under the cross product, we obtain the explicit fuzzy solutions for a fuzzy initial value problem applying the concept of the strongly generalized differentiability. Finally, some examples are given to illustrate the theoretical results. The obtained numerical results are compared with other approaches in the literature for similar parameters.

Keywords. First order linear fuzzy differential equations, Fuzzy variable coefficients, Fuzzy initial value problem, The cross product of fuzzy numbers.

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1. Introduction

Many of the differential equations that describe real natural phenomena are linear differential equations. They arise in the field of biology, mechanics, heat, electricity, interaction between neurons, population models, and growth model. In the real world, some information on physical phenomena are uncertain and imprecise. The concept of interval and fuzzy differential equations are born when the uncertainty comes in modeling on a problem with differential equations. Many studies have been done by several authors in the theory of interval, fuzzy differential equations and fully fuzzy linear systems (see e.g. [1, 4, 5, 7, 12, 13, 17]).
There are different approaches to interpret the concept of a solution to first order linear fuzzy differential equations (see e.g. [8, 10, 11, 14, 15, 16, 18]).

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In [8], the authors presented a variation of constants formulas for fuzzy initial value problem of the following first order linear fuzzy differential equations

$$y'(t) = a(t) \cdot y(t) + b(t), \tag{1.1}$$

where a is a real function and b is a fuzzy function. In [16], the authors generalized the results of [8] and they investigated the analytic solutions of the equation (1.1) with more cases. In the mentioned papers, strongly generalized differentiability concept introduced in [6, 7], have been used. As we observe, the equation (1.1) involves the term $a \cdot y$. In the case that a is a fuzzy function, we face to the interpretation of the product of two fuzzy numbers. The definition of product of fuzzy numbers are based on Zadeh's extension principle. Some researches have concentrated on linear fuzzy differential equations with fuzzy variable coefficients under the usual product[10, 15]. Using the definition of the usual product of fuzzy numbers, we arrive at

$$[a \cdot y]_r = [\min\{a_r^- y_r^-, a_r^- y_r^+, a_r^+ y_r^-, a_r^+ y_r^+\}, \max\{a_r^- y_r^-, a_r^- y_r^+, a_r^+ y_r^-, a_r^+ y_r^+\}].$$

This shows that the usual product formula based on Zadeh's extension principle is not practical and applicable in this case. To apply this product to linear fuzzy differential equations, there are some difficulties which to overcome them it appears to have several limitations and to be very restrictive. Recently, a new definition of product so-called the cross product of fuzzy numbers, was introduced in [9] and studied in [2, 6]. This concept allows us to solve the above mentioned shortcomings. From the theoretical and numerical point of view, there are some studies which apply the concept of the cross product for some problems [2, 11, 18]. In [2], the authors have been applied the concept of the cross product to the fuzzy transport equation with fuzzy coefficients. In [11], a numerical solution (Euler method) and in [18] Runge-Kutta Fehlberg method for solving first order fully fuzzy differential equations in the form $y'(t) = a \cdot y(t), \quad y(0) = y_0$, were considered.

These motivate us to use the cross product instead of the product obtained by Zadeh's extension principle in order to gain analytical solutions for linear fuzzy differential equations (1.1) and other alternative formats of Problem (1.1).

The structure of the present study is as follows. In Section 2, we give a brief review of definitions and calculus related to fuzzy numbers. In Section 3, we present some results on the calculus of fuzzy function. We apply our approach to the first order linear fuzzy differential equations and construct the analytical solutions of the equation associated with the uncertainty of data in Section 4. Finally, the applicable examples and comparison results with other approaches in the literature are given in Section 5.

2. Preliminaries of fuzzy numbers

The space of fuzzy numbers is denoted by $\mathbb{R}_{\mathcal{F}}$. For $0 < \alpha \leq 1$, α-cuts of $u \in \mathbb{R}_{\mathcal{F}}$ is defined by

$$[u]_\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$$

with $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$. We denote $Core(u) = [u]_1$ and $Supp(u) = [u]_0 = \overline{\{x \in \mathbb{R}; u(x) > 0\}}$. Recall that the triangular and the trapezoidal fuzzy numbers u, v are denoted by $u =< a, b, c >$ and $v =< a, b, c, d >$ respectively. For $u, v \in \mathbb{R}_{\mathcal{F}}$, $\lambda \in \mathbb{R}$, we define the addition $u+v$ and scalar multiplication λu as $[u+v]_\alpha = [u]_\alpha + [v]_\alpha$ and $[\lambda u]_\alpha = \lambda [u]_\alpha$, where $[u]_\alpha + [v]_\alpha$ and $\lambda [u]_\alpha$ mean the usual addition of two intervals of $\mathbb{R}$ and the

usual product between a scalar and an interval of $\mathbb{R}$ respectively.
If u and v are two fuzzy numbers, then the usual product $w = u \cdot v$ is defined based on Zadeh's extension principle by $[w]_\alpha = [w_\alpha^-, w_\alpha^+]$, where for every $\alpha \in [0,1]$

$$w_\alpha^- = \min\{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\},$$

and

$$w_\alpha^+ = \max\{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\}.$$

If $u \in \mathbb{R}_\mathcal{F}$, then we have its length as $Diam(u) = u_0^+ - u_0^-$. Let $u, v \in \mathbb{R}_\mathcal{F}$, if there exists a unique fuzzy number $w \in \mathbb{R}_\mathcal{F}$ such that $v + w = u$, then w is called the H-difference of u, v and denoted by $u \ominus v$ (see e.g. [6]).

Definition 2.1. [6] Given two fuzzy numbers $u, v \in \mathbb{R}_\mathcal{F}$, the generalized Hukuhara difference (gH-difference, for short) is the fuzzy number w, if it exists, such that

$$u \ominus_{gH} v = w \Leftrightarrow \left\{ \begin{array}{ccc} & (i) & u = v + w \\ \text{or} & (ii) & v = u - w. \end{array} \right.$$

Definition 2.2. [6, 9] We will say that a fuzzy number u is positive if for the lower endpoint of its core we have $u_1^- > 0$. Also we call a fuzzy number negative if $u_1^+ < 0$. The set of positive (negative) fuzzy numbers is denoted by $\mathbb{R}_\mathcal{F}^+ (\mathbb{R}_\mathcal{F}^-)$.

Proposition 2.3. [6, 9] *If u and v are positive fuzzy numbers, then $w = u \odot v$ defined by $w_\alpha = [w_\alpha^-, w_\alpha^+]$, where*

$$w_\alpha^- = u_\alpha^- v_1^- + u_1^- v_\alpha^- - u_1^- v_1^-,$$

and

$$w_\alpha^+ = u_\alpha^+ v_1^+ + u_1^+ v_\alpha^+ - u_1^+ v_1^+$$

for every $\alpha \in [0,1]$, is a positive fuzzy number.

The above definition is extended to the negative fuzzy numbers as follows.

Proposition 2.4. [6, 9] *Let u and v be two fuzzy numbers.*

(1) *If u is positive and v is negative we define*

$$u \odot v = -(u \odot (-v)),$$

which is a negative fuzzy number.

(2) *If u is negative and v is positive we define*

$$u \odot v = -((-u) \odot v),$$

which is a negative fuzzy number.

(3) *If u and v are negative we define*

$$u \odot v = ((-u) \odot (-v)),$$

which is a positive fuzzy number.

Definition 2.5. [6, 9] The binary operation introduced as above is called the cross product of fuzzy numbers.

3. Calculus of fuzzy number valued functions

For convenience, we will hereafter recall a fuzzy number valued function $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ by a fuzzy function. In this paper, for the integral concept, we will use the fuzzy Riemann integral (see e.g. [6]). If $g : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is an integrable fuzzy function such that $[g(t)]_\alpha = [(g(t))^-_\alpha, (g(t))^+_\alpha]$, then the boundary functions $(g(t))^-_\alpha$ and $(g(t))^+_\alpha$ are integrable and $[\int_a^b g(t)dt]_\alpha = [\int_a^b (g(t))^-_\alpha dt, \int_a^b (g(t))^+_\alpha dt]$. In the special case, if we consider a triangular fuzzy function $g : (a, b) \to \mathbb{R}_\tau$ such that $g(x) =< g_l(t), g_c(t), g_r(t) >$, then

$$\int_a^b g(t)dt =< \int_a^b g_l(t)dt, \int_a^b g_c(t)dt, \int_a^b g_r(t)dt > .$$

Definition 3.1. [6, 7] Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ and $t_0 \in (a, b)$. We say g is generalized differentiable at t_0, if there exists an element $f'(t_0) \in \mathbb{R}_{\mathcal{F}}$, such that
(i) for all $h > 0$ sufficiently small, there exist $f(t_0 + h) \ominus f(t_0),\ f(t_0) \ominus f(t_0 - h)$ and the limits

$$\lim_{h \searrow 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \searrow 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0),$$

or
(ii) for all $h > 0$ sufficiently small, there exist $f(t_0) \ominus f(t_0 + h),\ f(t_0 - h) \ominus f(t_0)$ and the limits

$$\lim_{h \searrow 0} \frac{f(t_0) \ominus f(t_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(t_0 - h) \ominus f(t_0)}{-h} = f'(t_0),$$

or
(iii) for all $h > 0$ sufficiently small, there exist $f(t_0) \ominus f(t_0 + h),\ f(t_0) \ominus f(t_0 - h)$ and the limits

$$\lim_{h \searrow 0} \frac{f(t_0) \ominus f(t_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0),$$

or
(iv) for all $h > 0$ sufficiently small, there exist $f(t_0 + h) \ominus f(t_0),\ f(t_0 - h) \ominus f(t_0)$ and the limits

$$\lim_{h \searrow 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \searrow 0} \frac{f(t_0 - h) \ominus f(t_0)}{-h} = f'(t_0).$$

Lemma 3.2. [6] *Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy function. Then $F(x) = \int_a^x f(t)dt$ is (i)-differentiable and we have $F'(x) = f(x)$.*

The following lemma states the derivative of the summation and Hukuhara difference of f and g for which f and g are fuzzy function.

Lemma 3.3. [3, 6, 8, 16] *Let $f, g : (a, b) \to \mathbb{R}_{\mathcal{F}}$ are generalized differentiable at $x \in (a, b)$. Then the differentiability of $f + g$ and $f \ominus g$ are as in Table 1 provided that the involving H-differences exist.*

TABLE 1. The differentiability of $f + g$ and $f \ominus g$.

Case	*Diff of* f	*Diff of* g	*Diff of* $f+g$	$(f+g)'$	*Diff of* $f \ominus g$	$(f \ominus g)'$
1	*(i)*	*(i)*	*(i)*	$f'(x) + g'(x)$	*(i)*	$f'(x) \ominus g'(x)$
2	*(i)*	*(ii)*	*(i)*	$f'(x) \ominus (-1)g'(x)$	*(i)*	$f'(x) + (-1)g'(x)$
3	*(ii)*	*(i)*	*(ii)*	$f'(x) \ominus (-1)g'(x)$	*(ii)*	$f'(x) + (-1)g'(x)$
4	*(ii)*	*(ii)*	*(ii)*	$f'(x) + g'(x)$	*(ii)*	$f'(x) \ominus g'(x)$

The following lemma states the derivative of a fuzzy function multiplied by a crisp function. The interpretation of product is based on Zadeh's extension.

Lemma 3.4. [8] *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be two differentiable functions. Then the differentiability of fg is as in Table 2 provided that the involving H-differences exist.*

TABLE 2. The differentiability of $f \cdot g$.

Case	*The sign of* $f(x)f'(x)$	*Diff of* g	*Diff of* $f \cdot g$	$(f \cdot g)'$
1	> 0	*(i)*	*(i)*	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$
2	< 0	*(i)*	*(i)*	$f(x) \cdot g'(x) \ominus (-1)f'(x) \cdot g(x)$
3	< 0	*(i)*	*(ii)*	$f'(x) \cdot g(x) \ominus (-1)f(x) \cdot g'(x)$
4	> 0	*(ii)*	*(ii)*	$f(x) \cdot g'(x) \ominus (-1)f(x) \cdot g'(x)$
5	> 0	*(ii)*	*(i)*	$f'(x) \cdot g(x) \ominus (-1)f(x) \cdot g'(x)$
6	< 0	*(ii)*	*(ii)*	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$

4. Linear fuzzy differential equations with fuzzy coefficient

In this section, we consider the following three fuzzy initial value problems of linear fuzzy differential equations with fuzzy variable coefficients

$$\begin{cases} y'(t) = a(t) \odot y(t) + b(t), \\ y(t_0) = y_0, \end{cases} \qquad (I)$$

$$\begin{cases} y'(t) + (-b(t)) = a(t) \odot y(t), \\ y(t_0) = y_0, \end{cases} \qquad (II)$$

$$\begin{cases} y'(t) + (-a(t)) \odot y(t) = b(t), \\ y(t_0) = y_0, \end{cases} \qquad (III)$$

where $a, b : [t_0, T) \to \mathbb{R}_{\mathcal{F}}$, $T > t_0$ and $y_0 \in \mathbb{R}_{\mathcal{F}}$.
It is worth noting that although the above problems are equivalent in the crisp version, they are different in the fuzzy version. Moreover, the above problems are generalization of the problems mentioned in [8] for which $a(t)$ is a real function. In the above problems, we have applied the concept of the cross product instead of the usual product. Therefore, we investigate the above problems separately in the next sections.
Throughout this section, for convenience, we will use the following definition related to the definition of the cross product.

Definition 4.1. Let $u, v \in \mathbb{R}_{\mathcal{F}}^{+}$ (or $\mathbb{R}_{\mathcal{F}}^{-}$). If $Core(u)$ consists exactly one element, i.e. $[u]_1 = \{u_c\}$, we define

$$(u - u_c) \odot v = u \odot v \ominus u_c v.$$

4.1. **Investigation of solutions to Problem (I).** In this section, we study Problem (I) and give its explicit fuzzy solutions.

Theorem 4.2. *Let $a, b : [t_0, T) \to \mathbb{R}_{\mathcal{F}}$ be two fuzzy functions and the core $[a(t)]_1 = \{a_c(t)\}$ consists of exactly one element for any $t \in [t_0, T)$ and $y_0 \in \mathbb{R}_{\mathcal{F}}$.*

1. *If $a(t) \in \mathbb{R}_{\mathcal{F}}^{+}$ and $y_0 + \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a_c(r)dr} ds \in \mathbb{R}_{\mathcal{F}}^{+}$ (or $\mathbb{R}_{\mathcal{F}}^{-}$) for all $t \in (t_0, T)$, then y_1 defined as*

$$y_1(t) = e^{\int_{t_0}^{t} a_c(s)ds}(y_0 + \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a_c(r)dr} ds)$$
$$+ e^{\int_{t_0}^{t} a_c(s)ds} \int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr) ds,$$

   is (i)-differentiable w.r.t. t and satisfies Problem (I).

2. *If $a(t) \in \mathbb{R}_{\mathcal{F}}^{-}$ and $y_0 \ominus (-1) \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a_c(r)dr} ds \in \mathbb{R}_{\mathcal{F}}^{+}$ (or $\mathbb{R}_{\mathcal{F}}^{-}$) for all $t \in (t_0, T)$, then y_2 defined*

$$y_2(t) = e^{\int_{t_0}^{t} a_c(s)ds}(y_0 \ominus (-1) \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a_c(r)dr} ds)$$
$$\ominus (-1) e^{\int_{t_0}^{t} a_c(s)ds} \int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 \ominus (-1) \int_{t_0}^{s} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr) ds,$$

   is (ii)-differentiable w.r.t. t and satisfies Problem (I) provided that the H-differences involving y_2 and the following H-difference exist

$$e^{\int_{t_0}^{t} a_c(s)ds}(a(t) - a_c(t)) \odot (y_0 \ominus (-1) \int_{t_0}^{t} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr) \ominus (-1)$$
$$a_c(t) e^{\int_{t_0}^{t} a_c(s)ds} \int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 \ominus (-1) \int_{t_0}^{s} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr) ds.$$

Proof. Case 1. It follows from Lemma 3.2 and Case 1 of Lemma 3.3 that

$$y_0 + \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a_c(r)dr} ds,$$

and

$$\int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr) ds$$

are (i)-differentiable. Moreover, we have

$$(y_0 + \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a_c(r)dr} ds)' = b(t) e^{-\int_{t_0}^{t} a_c(r)dr},$$

and

$$\begin{aligned}&(\int_{t_0}^{t}(a(s)-a_c(s))\odot(y_0+\int_{t_0}^{s}b(r)e^{-\int_{t_0}^{r}a_c(u)du}dr)ds)'\\&=(a(t)-a_c(t))\odot(y_0+\int_{t_0}^{t}b(r)e^{-\int_{t_0}^{r}a_c(u)du}dr).\end{aligned}$$

Since $a(t)\in\mathbb{R}_{\mathcal{F}}^{+}$ for all $t\in(t_0,T)$, from Case 1 of Lemma 3.4 we deduce that the following fuzzy functions are (i)-differentiable

$$\begin{aligned}&(e^{\int_{t_0}^{t}a_c(s)ds}(y_0+\int_{t_0}^{t}b(s)e^{-\int_{t_0}^{s}a_c(r)dr}ds))'\\&=a_c(t)e^{\int_{t_0}^{t}a_c(s)ds}(y_0+\int_{t_0}^{t}b(s)e^{-\int_{t_0}^{s}a_c(r)dr}ds)+b(t),\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}&(e^{\int_{t_0}^{t}a_c(s)ds}\int_{t_0}^{t}(a(s)-a_c(s))\odot(y_0+\int_{t_0}^{s}b(r)e^{-\int_{t_0}^{r}a_c(u)du}dr)ds)'\\&=a_c(t)e^{\int_{t_0}^{t}a_c(s)ds}\int_{t_0}^{t}(a(s)-a_c(s))\odot(y_0+\int_{t_0}^{s}b(r)e^{-\int_{t_0}^{r}a_c(u)du}dr)ds\\&+e^{\int_{t_0}^{t}a_c(s)ds}(a(t)-a_c(t))\odot(y_0+\int_{t_0}^{t}b(r)e^{-\int_{t_0}^{r}a_c(u)du}dr).\end{aligned}\tag{4.2}$$

From Eqs. (4.1)-(4.2) and Case 1 of Lemma 3.3, we conclude that y_1 is (i)-differentiable and also from Definition 4.1, we have

$$y_1'(t)=a_c(t)y_1(t)+(a(t)-a_c(t))\odot y_1(t)+b(t)=a(t)\odot y_1(t)+b(t).$$

It means that y_1 is a solution of Problem (I).
Case 2. We suppose that all of the H-differences involving in y_2 exist. It follows from Lemma 3.2 and Case 3 of Lemma 3.3 that $y_0\ominus(-1)\int_{t_0}^{t}b(s)e^{-\int_{t_0}^{s}a_c(r)dr}ds$ is (ii)-differentiable and from Lemma 3.2 that

$$\int_{t_0}^{t}(a(s)-a_c(s))\odot(y_0\ominus(-1)\int_{t_0}^{s}b(r)e^{-\int_{t_0}^{r}a_c(u)du}dr)ds$$

is (i)-differentiable. Moreover, their derivatives are as follows

$$(y_0\ominus(-1)\int_{t_0}^{t}b(s)e^{-\int_{t_0}^{s}a_c(r)dr}ds)'=b(t)\cdot e^{-\int_{t_0}^{t}a_c(r)dr},$$

and

$$\begin{aligned}&(\int_{t_0}^{t}(a(s)-a_c(s))\odot(y_0\ominus(-1)\int_{t_0}^{s}b(r)e^{-\int_{t_0}^{r}a_c(u)du}dr)ds)'\\&=(a(t)-a_c(t))\odot(y_0\ominus(-1)\int_{t_0}^{t}b(r)e^{-\int_{t_0}^{r}a_c(u)du}dr).\end{aligned}$$

Since $a(t) \in \mathbb{R}_{\mathcal{F}}^{-}$ for all $t \in (t_0, T)$, from Case 6 of Lemma 3.4 we have (ii)-differentiability for the fuzzy function below

$$\begin{aligned}&(e^{\int_{t_0}^{t} a_c(s)ds}(y_0 \ominus (-1)\int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds))'\\&= a_c(t)e^{\int_{t_0}^{t} a_c(s)ds}(y_0 \ominus (-1)\int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds) + b(t).\end{aligned}$$

On the other hand, from Case 2 of Lemma 3.4 we have (i)-differentiability for the following fuzzy function

$$\begin{aligned}&(e^{\int_{t_0}^{t} a_c(s)ds}\int_{t_0}^{t}(a(s) - a_c(s)) \odot (y_0 \ominus (-1)\int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du}dr)ds)'\\&= e^{\int_{t_0}^{t} a_c(s)ds}(a(t) - a_c(t)) \odot (y_0 \ominus (-1)\int_{t_0}^{t} b(r)e^{-\int_{t_0}^{r} a_c(u)du}dr)\\&\ominus (-1)a_c(t)e^{\int_{t_0}^{t} a_c(s)ds}\int_{t_0}^{t}(a(s) - a_c(s)) \odot (y_0 \ominus (-1)\int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du}dr)ds.\end{aligned}$$

According to the assumptions of the present theorem, the above H-difference exists. It follows from two above equations and Case 3 of Lemma 3.3 that y_2 is (ii)-differentiable. Moreover, from Definition 4.1 and two above equations, we have

$$y_2'(t) = a_c(t)y_2(t) + (a(t) - a_c(t)) \odot y_2(t) + b(t) = a(t) \odot y_2(t) + b(t).$$

It means that y_2 is a solution of Problem (I). □

4.2. **Investigation of solutions to Problem (II).** In this section, we study Problem (II) and give its explicit fuzzy solutions.

Theorem 4.3. *Let $a, b : [t_0, T) \to \mathbb{R}_{\mathcal{F}}$ be two fuzzy functions and the core of $a(t)$ consists exactly one element for any $t \in [t_0, T)$, i.e. $[a(t)]_1 = \{a_c(t)\}$ and $y_0 \in \mathbb{R}_{\mathcal{F}}$.*

1. *If $a(t) \in \mathbb{R}_{\mathcal{F}}^{+}$ and $y_0 \ominus (-1)\int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds \in \mathbb{R}_{\mathcal{F}}^{+}$ (or $\mathbb{R}_{\mathcal{F}}^{-}$) for all $t \in (t_0, T)$, then y_1 defined as*

$$\begin{aligned}y_1(t) &= e^{\int_{t_0}^{t} a_c(s)ds}(y_0 \ominus (-1)\int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds)\\&+ e^{\int_{t_0}^{t} a_c(s)ds}\int_{t_0}^{t}(a(s) - a_c(s)) \odot (y_0 \ominus (-1)\int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du}dr)ds\end{aligned}$$

is (i)-differentiable w.r.t. t and satisfies Problem (II) provided that for $t \in (t_0, T)$ the H-difference involving y_1 and the following H-difference

$$a_c(t)e^{\int_{t_0}^{t} a_c(s)ds}(y_0 \ominus (-1)\int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds) \ominus (-1)b(t)$$

exist.

2. *If* $a(t) \in \mathbb{R}_{\mathcal{F}}^{-}$ *and* $y_0 + \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds \in \mathbb{R}_{\mathcal{F}}^{+}$ *(or* $\mathbb{R}_{\mathcal{F}}^{-}$*) for all* $t \in (t_0, T)$*, then* y_2 *defined as*

$$y_2(t) = e^{\int_{t_0}^{t} a_c(s)ds}(y_0 + \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds)$$
$$\ominus (-1)e^{\int_{t_0}^{t} a_c(s)ds} \int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du}dr)ds$$

is (ii)*-differentiable w.r.t.* t *and satisfies Problem (II) provided that for* $t \in (t_0, T)$ *the H-differences involving* y_2*, the following H-differences*

$$a_c(t)e^{\int_{t_0}^{t} a_c(s)ds}(y_0 + \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds) \ominus (-1)b(t),$$

and

$$e^{\int_{t_0}^{t} a_c(s)ds}(a(t) - a_c(t)) \odot (y_0 + \int_{t_0}^{t} b(r)e^{-\int_{t_0}^{r} a_c(u)du}dr) \ominus (-1)$$
$$a_c(t)e^{\int_{t_0}^{t} a_c(s)ds} \int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du}dr)ds$$

exist.

Proof. Case 1. From Lemma 3.2 and Case 3 of Lemma 3.3,

$$y_0 \ominus (-1) \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds$$

is (ii)-differentiable and

$$\int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du}dr)ds$$

is (i)-differentiable. Moreover, we have

$$(y_0 \ominus (-1) \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds)' = b(t)e^{-\int_{t_0}^{t} a_c(r)dr},$$

and

$$(\int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du}dr)ds)'$$
$$= (a(t) - a_c(t)) \odot (y_0 + \int_{t_0}^{t} b(r)e^{-\int_{t_0}^{r} a_c(u)du}dr).$$

Since $a(t) \in \mathbb{R}_{\mathcal{F}}^{+}$ for all $t \in (t_0, T)$, from Case 5 of Lemma 3.4, we have (i)-differentiability for the following fuzzy function

$$(e^{\int_{t_0}^{t} a_c(s)ds}(y_0 \ominus (-1) \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds))'$$
$$= a_c(t)e^{\int_{t_0}^{t} a_c(s)ds}(y_0 \ominus (-1) \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr}ds) \ominus (-1)b(t).$$

According to the assumptions of the present theorem, the H-differences appeared above exist. On the other hand, from Case 1 of Lemma 3.4, we have (i)-differentiability for the following fuzzy function as below

$$(e^{\int_{t_0}^t a_c(s)ds} \int_{t_0}^t (a(s)-a_c(s)) \odot (y_0 \ominus (-1) \int_{t_0}^s b(r)e^{-\int_{t_0}^r a_c(u)du} dr)ds)'$$
$$= a_c(t)e^{\int_{t_0}^t a_c(s)ds} \int_{t_0}^t (a(s)-a_c(s)) \odot (y_0 \ominus (-1) \int_{t_0}^s b(r)e^{-\int_{t_0}^r a_c(u)du} dr)ds$$
$$+ e^{\int_{t_0}^t a_c(s)ds}(a(t)-a_c(t)) \odot (y_0 + \int_{t_0}^t b(r)e^{-\int_{t_0}^r a_c(u)du} dr).$$

It follows from Case 1 of Lemma 3.3 that y_1 is (i)-differentiable and from two above equations and Definition 4.1 that

$$y_1'(t) + (-1)b(t) = a_c(t)y_1(t) + (a(t)-a_c(t)) \odot y_1(t) = a(t) \odot y_1(t).$$

It means that y_1 is a solution of Problem (II).

Case 2. We suppose that the H-difference involving in y_2 exists. From Lemma 3.2 and Case 1 of Lemma 3.3, $y_0 + \int_{t_0}^t b(s)e^{-\int_{t_0}^s a_c(r)dr} ds$ and $\int_{t_0}^t (a(s)-a_c(s)) \odot (y_0 + \int_{t_0}^s b(r)e^{-\int_{t_0}^r a_c(u)du} dr)ds$ are (i)-differentiable and we have

$$(y_0 + \int_{t_0}^t b(s)e^{-\int_{t_0}^s a_c(r)dr} ds)' = b(t)e^{-\int_{t_0}^t a_c(r)dr},$$

and

$$(\int_{t_0}^t (a(s)-a_c(s)) \odot (y_0 + \int_{t_0}^s b(r)e^{-\int_{t_0}^r a_c(u)du} dr)ds)'$$
$$= (a(t)-a_c(t)) \odot (y_0 + \int_{t_0}^t b(r)e^{-\int_{t_0}^r a_c(u)du} dr).$$

Since $a(t) \in \mathbb{R}_{\mathcal{F}}^-$ for all $t \in (t_0, T)$, from Case 3 of Lemma 3.4, we have (ii)-differentiability for the following function as below

$$(e^{\int_{t_0}^t a_c(s)ds}(y_0 + \int_{t_0}^t b(s)e^{-\int_{t_0}^s a_c(r)dr} ds))'$$
$$= a_c(t)e^{\int_{t_0}^t a_c(s)ds}(y_0 + \int_{t_0}^t b(s)e^{-\int_{t_0}^s a_c(r)dr} ds) \ominus (-1)b(t).$$

According to the assumptions of the present theorem, the above H-difference exists. On the other hand, we have (i)-differentiability for the following fuzzy function from Case 2 of Lemme 3.4

$$(e^{\int_{t_0}^t a_c(s)ds} \int_{t_0}^t (a(s)-a_c(s)) \odot (y_0 + \int_{t_0}^s b(r)e^{-\int_{t_0}^r a_c(u)du} dr)ds)'$$
$$= e^{\int_{t_0}^t a_c(s)ds}(a(t)-a_c(t)) \odot (y_0 + \int_{t_0}^t b(r)e^{-\int_{t_0}^r a_c(u)du} dr)$$
$$\ominus (-1)a_c(t)e^{\int_{t_0}^t a_c(s)ds} \int_{t_0}^t (a(s)-a_c(s)) \odot (y_0 + \int_{t_0}^s b(r)e^{-\int_{t_0}^r a_c(u)du} dr)ds.$$

According to the assumptions of the present theorem, the above H-difference exists. It follows from case 3 of Lemma 3.3 that y_2 is (ii)-differentiable and from two above equations and Definition 4.1 we have

$$y_2'(t) + (-1)b(t) = a_c(t)y_2(t) + (a(t) - a_c(t)) \odot y_2(t) = a(t) \odot y_2(t).$$

It means that y_2 is a solution of Problem (II). □

4.3. **Investigation of solutions to Problem (III).** In this section, we study Problem (III) and give its explicit fuzzy solutions.

Theorem 4.4. *Let $a, b : [t_0, T) \to \mathbb{R}_{\mathcal{F}}$ be two fuzzy functions and the core of $a(t)$ consists exactly one element for any $t \in [t_0, T)$, i.e. $[a(t)]_1 = \{a_c(t)\}$ and $y_0 \in \mathbb{R}_{\mathcal{F}}$.*

1. *If $a(t) \in \mathbb{R}_{\mathcal{F}}^+$ and $y_0 \ominus (-1)\int_{t_0}^t b(s)e^{-\int_{t_0}^s a_c(r)dr}ds \in \mathbb{R}_{\mathcal{F}}^+$ (or $\mathbb{R}_{\mathcal{F}}^-$) for all $t \in (t_0, T)$, then y_1 defined by*

$$y_1(t) = e^{\int_{t_0}^t a_c(s)ds}(y_0 \ominus (-1)\int_{t_0}^t b(s)e^{-\int_{t_0}^s a_c(r)dr}ds)$$
$$+ e^{\int_{t_0}^t a_c(s)ds}\int_{t_0}^t (a(s) - a_c(s)) \odot (y_0 \ominus (-1)\int_{t_0}^s b(r)e^{-\int_{t_0}^r a_c(u)du}dr)ds$$

is (ii)-differentiable w.r.t. t and satisfies Problem (III) provided that the H-differences involving y_1 and the H-differences

$$b(t) \ominus (-1)a_c(t)e^{\int_{t_0}^t a_c(s)ds}(y_0 \ominus (-1)\int_{t_0}^t b(s)e^{-\int_{t_0}^s a_c(r)dr}ds),$$

and

$$b(t) \ominus (-1)a(t) \odot y_1(t),$$

for all $t \in (t_0, T)$ exist.

2. *If $a(t) \in \mathbb{R}_{\mathcal{F}}^-$ and $y_0 + \int_{t_0}^t b(s)e^{-\int_{t_0}^s a_c(r)dr}ds \in \mathbb{R}_{\mathcal{F}}^+$ (or $\mathbb{R}_{\mathcal{F}}^-$) for all $t \in (t_0, T)$, then y_2 defined by*

$$y_2(t) = e^{\int_{t_0}^t a_c(s)ds}(y_0 + \int_{t_0}^t b(s)e^{-\int_{t_0}^s a_c(r)dr}ds)$$
$$\ominus (-1)e^{\int_{t_0}^t a_c(s)ds}\int_{t_0}^t (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^s b(r)e^{-\int_{t_0}^r a_c(u)du}dr)ds$$

is (i)-differentiable w.r.t. t and satisfies Problem (III) provided that the H-difference involving y_2 and the H-differences

$$b(t) \ominus (-1)a_c(t)e^{\int_{t_0}^t a_c(s)ds}(y_0 \ominus (-1)\int_{t_0}^t b(s)e^{-\int_{t_0}^s a_c(r)dr}ds),$$

and

$$a_c(t)e^{\int_{t_0}^t a_c(s)ds}\int_{t_0}^t (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^s b(r)e^{-\int_{t_0}^r a_c(u)du}dr)ds \ominus (-1)$$
$$e^{\int_{t_0}^t a_c(s)ds}(a(t) - a_c(t)) \odot (y_0 + \int_{t_0}^t b(r)e^{-\int_{t_0}^r a_c(u)du}dr),$$

for all $t \in (t_0, T)$ exist.

Proof. Case 1. From Lemma 3.2 and Case 4 of Lemma 3.3,

$$y_0 \ominus (-1) \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a_c(r)dr} ds$$

is (ii)-differentiable and $\int_{t_0}^{t} (a(s)-a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr) ds$ is (i)-differentiable. Moreover, we have

$$(y_0 \ominus (-1) \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a_c(r)dr} ds)' = b(t) e^{-\int_{t_0}^{t} a_c(r)dr},$$

and

$$\begin{aligned} &\left(\int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 \ominus (-1) \int_{t_0}^{s} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr) ds \right)' \\ &= (a(t) - a_c(t)) \odot \left(y_0 \ominus (-1) \int_{t_0}^{t} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr \right). \end{aligned}$$

Since $a(t) \in \mathbb{R}_{\mathcal{F}}^{+}$ for all $t \in (t_0, T)$, from Case 4 of Lemma 3.4 we have (ii)-differentiability for the following fuzzy function

$$\begin{aligned} &(e^{\int_{t_0}^{t} a_c(s)ds} (y_0 \ominus (-1) \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a_c(r)dr} ds))' \\ &= b(t) \ominus (-1) a_c(t) e^{\int_{t_0}^{t} a_c(s)ds} (y_0 \ominus (-1) \int_{t_0}^{t} b(s) e^{-\int_{t_0}^{s} a_c(r)dr} ds. \end{aligned}$$

According to the assumptions of the present theorem, the above H-differences exist. On the other hand, from Case 1 of Lemma 3.4 we have (i)-differentiability for the following fuzzy function as bellow

$$\begin{aligned} &(e^{\int_{t_0}^{t} a_c(s)ds} \int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 \ominus (-1) \int_{t_0}^{s} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr) ds)' \\ &= a_c(t) e^{\int_{t_0}^{t} a_c(s)ds} \int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 \ominus (-1) \int_{t_0}^{s} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr) ds \\ &+ e^{\int_{t_0}^{t} a_c(s)ds} (a(t) - a_c(t)) \odot (y_0 \ominus (-1) \int_{t_0}^{t} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr). \end{aligned}$$

It follows from Case 3 of Lemma 3.3 and the above equations that y_1 is (ii)-differentiable provided that the H-difference $b(t) \ominus (-1) a(t) \odot y_1(t)$ for all $t \in (t_0, T)$ exists and we have

$$\begin{aligned} &y_1'(t) = b(t) \ominus (-1) \\ &\quad \left(a_c(t) y_1(t) + e^{\int_{t_0}^{t} a_c(s)ds} (a(t) - a_c(t)) \left(y_0 \ominus (-1) \int_{t_0}^{t} b(r) e^{-\int_{t_0}^{r} a_c(u)du} dr \right) \right) \\ &\quad = b(t) \ominus (-1) a_c(t) y_1(t) + (a(t) - a_c(t)) \odot y_1(t) = b(t) \ominus (-1) a(t) \odot y_1(t). \end{aligned}$$

Then we have $y_1'(t) + (-1) a(t) \odot y_1(t) = b(t)$. It means that y_1 is a solution of Problem (III).

Case 2. We suppose that the H-difference involving in y_2 exists. We know that

$$y_0 + \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr} ds,$$

and

$$\int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du} dr)ds$$

are (i)-differentiable. Also, we have

$$(y_0 + \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr} ds)' = b(t) \cdot e^{-\int_{t_0}^{t} a_c(r)dr},$$

and

$$\begin{aligned}
&(\int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du} dr)ds)' \\
&= (a(t) - a_c(t)) \odot (y_0 + \int_{t_0}^{t} b(r)e^{-\int_{t_0}^{r} a_c(u)du} dr).
\end{aligned}$$

Since $a(t) \in \mathbb{R}_{\mathcal{F}}^{-}$ for all $t \in (t_0, T)$, from Case 2 of Lemma 3.4 we have (i)-differentiability for the following fuzzy function

$$\begin{aligned}
&(e^{\int_{t_0}^{t} a_c(s)ds}(y_0 + \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr} ds))' \\
&= b(t) \ominus (-1)a_c(t)e^{\int_{t_0}^{t} a_c(s)ds}(y_0 + \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a_c(r)dr} ds),
\end{aligned}$$

and (ii)-differentiability for the following function from Case 3 of Lemma 3.4

$$\begin{aligned}
&(e^{\int_{t_0}^{t} a_c(s)ds} \int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du} dr)ds)' \\
&= a_c(t)e^{\int_{t_0}^{t} a_c(s)ds} \int_{t_0}^{t} (a(s) - a_c(s)) \odot (y_0 + \int_{t_0}^{s} b(r)e^{-\int_{t_0}^{r} a_c(u)du} dr)ds \\
&\ominus (-1)e^{\int_{t_0}^{t} a_c(s)ds}(a(t) - a_c(t)) \odot (y_0 + \int_{t_0}^{t} b(r)e^{-\int_{t_0}^{r} a_c(u)du} dr).
\end{aligned}$$

It follows from two above equations and Case 2 of Lemma 3.3 that y_2 is (i)-differentiable and we have from Definition 4.1

$$y_2'(t) = b(t) \ominus (-1)(a_c(t)y_2(t) + (a(t) - a_c(t)) \odot y_2(t)).$$

Therefore, it means that $y_2'(t) + a(t) \odot y_2(t) = b(t)$ and y_2 is the solution of Problem (III). □

5. Examples and Comparison with other approaches

This section is devoted to some examples and comparing the new approach proposed in this paper to the existing ones in the literature.

Example 5.1. Consider the following initial value problem

$$\begin{cases} y'(t) =< t, 2t, 3t > \odot y(t)+ < \frac{t}{2}, t, \frac{3t}{2} >, \\ y(0) =< -\frac{3}{2}, -1, \frac{-1}{2} > . \end{cases} \tag{5.1}$$

Since for all $t > 0$, $a(t) \in \mathbb{R}^{+}_{\mathcal{F}}$ and

$$\begin{aligned} y_0 + \int_0^t b(s) e^{-\int_0^s a_c(r)dr} ds \qquad &=< -\tfrac{3}{2}, -1, -\tfrac{1}{2} > + \textstyle\int_0^t < \tfrac{1}{2}, 1, \tfrac{3}{2} > se^{-s^2} ds \\ &=< -\tfrac{3}{2}, -1, -\tfrac{1}{2} > + < \tfrac{1}{2}, 1, \tfrac{3}{2} > (-\tfrac{1}{2}e^{-t^2} + \tfrac{1}{2}) \\ &=< -\tfrac{1}{4}e^{-t^2} - \tfrac{5}{4}, -\tfrac{1}{2}e^{-t^2} - \tfrac{1}{2}, -\tfrac{3}{4}e^{-t^2} + \tfrac{1}{4} >\in \mathbb{R}^{-}_{\mathcal{F}}, \end{aligned} \tag{5.2}$$

we can apply Case 1 of Theorem 4.2 in order to obtain a solution for Problem (5.1) which is (i)-differentiable for all $t > 0$. To this purpose we derive

$$\begin{aligned} &\textstyle\int_0^t (a(s) - a_c(s)) \odot (y_0 + \int_0^s b(r) e^{-\int_0^r a_c(u)du}) ds = \\ &\textstyle\int_0^t < s, 2s, 3s > \odot < -\frac{1}{4}e^{-s^2} - \frac{5}{4}, -\frac{1}{2}e^{-s^2} - \frac{1}{2}, -\frac{3}{4}e^{-s^2} + \frac{1}{4} > \\ &\ominus (2s) < -\tfrac{1}{4}e^{-s^2} - \tfrac{5}{4}, -\tfrac{1}{2}e^{-s^2} - \tfrac{1}{2}, -\tfrac{3}{4}e^{-s^2} + \tfrac{1}{4} > ds \\ &= \textstyle\int_0^t < -s(\frac{1}{2} + \frac{1}{2}e^{-s^2}), 0, s(\frac{1}{2} + \frac{1}{2}e^{-s^2}) > ds \\ &=< -\tfrac{1}{4}t^2 + \tfrac{1}{4}e^{-t^2} - \tfrac{1}{4}, 0, \tfrac{1}{4}t^2 - \tfrac{1}{4}e^{-t^2} + \tfrac{1}{4} > . \end{aligned} \tag{5.3}$$

Utilizing Eqs. (5.2)-(5.3) and Case 1 of Theorem 4.2, we obtain the solution of Problem 5.1 as follows

$$u(t) =< u_l(t), u_c(t), u_r(t) >=< -\frac{1}{4}t^2e^{t^2} - \frac{3}{2}e^{t^2}, -\frac{1}{2}e^{t^2} - \frac{1}{2}, \frac{1}{4}t^2e^{t^2} + \frac{1}{2}e^{t^2} - 1 > .$$

Moreover, we have

$$Diam(u(t)) = \frac{1}{2}t^2e^{t^2} + 2e^{t^2} - 1.$$

As we can see in Figure 2(b), the diam of the solution of Problem(5.1) is increasing w.r.t. t. Also, three functions u_l, u_c, u_r can be seen in Figure 2(a).

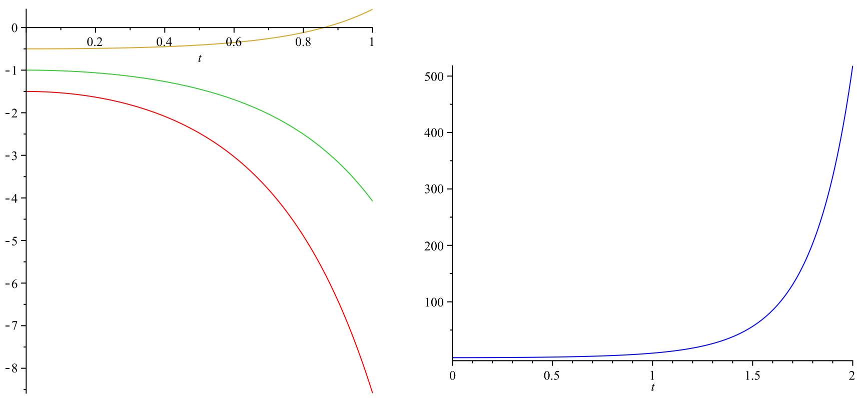
Example 5.2. Consider the following initial value problem

$$\begin{cases} y'(t) =< -3t, -2t, -t > \odot y(t)+ < \frac{t}{2}, t, \frac{3t}{2} >, \\ y(0) =< -\frac{3}{2}, -1, \frac{-1}{2} > . \end{cases} \tag{5.4}$$

Since $a(t) \in \mathbb{R}^{-}_{\mathcal{F}}$ for $t > 0$, we apply Case 2 of Theorem 4.2 and check if all of the H-differences appeared in this case exist. The following H-difference for $t \in [0, \sqrt{\ln 3}]$ exists, that is

$e^{t^2}, -\frac{3}{4} + \frac{1}{4}e^{t^2} >\in \mathbb{R}^{-}_{\mathcal{F}}$.

FIGURE 1. Plot of the solution of Problem (5.1).



(a) The solution of Problem (5.1) Based on the cross product

(b) Plot of the diam of the solutions of Problem (5.1)

On the other hand, we can derive

$$(a(t) - a_c(t)) \odot (y_0 \ominus (-1)\int_0^t b(s)e^{-\int_0^s a_c(r)dr})ds =$$
$$< -3t, -2t, -t > \odot < -\frac{9}{4} + \frac{3}{4}e^{t^2}, -\frac{3}{2} + \frac{1}{2}e^{t^2}, -\frac{3}{4} + \frac{1}{4}e^{t^2} >$$
$$\ominus (-2t) < -\frac{9}{4} + \frac{3}{4}e^{t^2}, -\frac{3}{2} + \frac{1}{2}e^{t^2}, -\frac{3}{4} + \frac{1}{4}e^{t^2} > = < -\frac{3}{2}t + \frac{1}{2}te^{t^2}, 0, \frac{3}{2}t - \frac{1}{2}te^{t^2} > .$$

Utilizing the above equation, it follows that the following H-difference for $t \in [0, \sqrt{\frac{2}{3}}]$ exists, that is

$$e^{\int_0^t a_c(s)ds}(a(t) - a_c(t)) \odot (y_0 \ominus (-1)\int_0^t b(r)e^{-\int_0^r a_c(u)du}dr) \ominus (-1)$$
$$a_c(t)e^{\int_0^t a_c(s)ds}\int_0^t (a(s) - a_c(s)) \odot (y_0 \ominus (-1)\int_0^s b(r)e^{-\int_0^r a_c(u)du}dr)ds$$
$$= e^{-t^2} < -t + \frac{3}{2}t^3, 0, t - \frac{3}{2}t^3 > .$$

Therefore, one can obtain the following solution for Problem (5.4) which is (ii)-differentiable for $0 \leq t \leq \sqrt{\frac{2}{3}}$

$$u(t) = \frac{1}{4}e^{-t^2} < 2e^{t^2} + 3t^2 - 8, 2e^{t^2} - 6, 2e^{t^2} - 3t^2 - 4 > .$$

Moreover, we have

$$Diam(u(t)) = \frac{1}{4}e^{-t^2}(-6t^2 + 4).$$

Now, we are going to compare this example with Example 5.3 in [10]. The proposed method in the mentioned paper is based on the usual product obtained by Zadeh's extension principle. The current example is the same as Example 5.3 in [10] for which we have the following α-cuts of the solution for $t \in (0, \frac{\sqrt{2}}{2})$ as follows

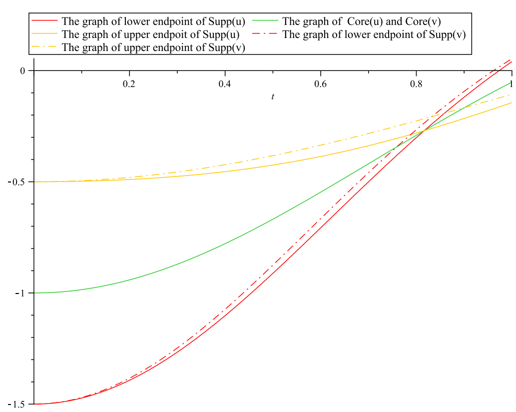
$$[v(t)]_\alpha = \frac{1}{2}\left[1 + (-4+\alpha)e^{\frac{\alpha-3}{2}t^2}, 1 + (-2-\alpha)e^{\frac{1}{2}(-1-\alpha)t^2}\right].$$

Moreover, the diam of this solution is as follows

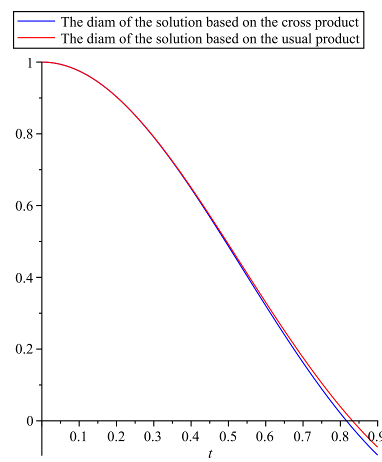
$$Diam(v(t)) = 2e^{\frac{-3t^2}{2}} - e^{\frac{-t^2}{2}}.$$

We observe in this example that the solution obtained by our approach is different from the solution based on the usual product in [10]. The comparison between these two solutions obtained by the different methods demonstrates that the uncertainty of the solution obtained by the method of the cross product is less than the uncertainty of the solution obtained by the method based on the usual product. This fact can be illustrated in Figures 3(a) and 3(b).

FIGURE 2. Plot of the solutions of Problem (5.4).



(a) The solutions u and v of Problem (5.4)



(b) Plot of the diam of the solutions u and v of Problem (5.4)

Example 5.3. Consider the following initial value problem

$$\begin{cases} y'(t) = < -\frac{3}{2}, -1, -\frac{1}{2} > \odot y(t), \\ y(0) = < \frac{1}{2}, 1, \frac{3}{2} > . \end{cases} \tag{5.5}$$

Since $a(t) \in \mathbb{R}^-_{\mathcal{F}}$ for $t > 0$, we apply Case 2 of Theorem 4.2 and check if all of the H-differences appeared in this case exist. The following H-difference for $t > 0$ exists, that is

$$y_0 \ominus (-1) \int_0^t b(s) e^{-\int_0^s a_c(r)dr} =< \frac{1}{2}, 1, \frac{3}{2} > .$$

On the other hand, we can derive

$$(a(t) - a_c(t)) \odot (y_0 \ominus (-1) \int_0^t b(s) e^{-\int_0^s a_c(r)dr}) =< -\frac{1}{2}, 0, \frac{1}{2} > .$$

Utilizing the above equation, it follows that the following H-difference for $t \in [0, 1]$ exists, that is

$$e^{\int_0^t a_c(s)ds}(a(t) - a_c(t)) \odot (y_0 \ominus (-1) \int_0^t b(r) e^{-\int_0^r a_c(u)du} dr) \ominus (-1)$$

$$a_c(t) e^{\int_0^t a_c(s)ds} \int_0^t (a(s) - a_c(s)) \odot (y_0 \ominus (-1) \int_0^s b(r) e^{-\int_0^r a_c(u)du} dr) ds$$

$$=< -\frac{1}{2}, 0, \frac{1}{2} > \ominus < -\frac{1}{2}t, 0, \frac{1}{2}t >=< -\frac{1}{2} + \frac{1}{2}t, 0, \frac{1}{2} - \frac{1}{2}t > .$$

Therefore, one can obtain the following solution for Problem (5.5) which is (ii)-differentiable for $0 < t < 1$

$$u(t) = \frac{1}{2} e^{-t} < 1 + t, 2, 3 - t > .$$

Moreover, we have

$$Diam(u(t)) = e^{-t}(1 - t).$$

The current example is the same as Example 5.1 in [10] for which we have the following α-cuts of the solution for $t \in (0, \frac{2}{3})$ as follows

$$[v(t)]_\alpha = \frac{1}{2} \left[(1 + \alpha) e^{\frac{1}{2}(-1-\alpha)t}, (3 - \alpha) e^{\frac{1}{2}(-3+\alpha)t} \right] .$$

Moreover, the diam of this solution is given by

$$Diam(v(t)) = \frac{3}{2} e^{\frac{-3t}{2}} - \frac{1}{2} e^{\frac{-t}{2}} .$$

Another interpretation of solution is Based on Zadeh's Extension Principle [6]. Under this interpretation, Problem (5.5) is solved as a crisp problem. Then a solution of Problem (5.5) is generated using Zadeh's extension principle on the classical solution. Therefore, we have the following solution to Problem (5.5)

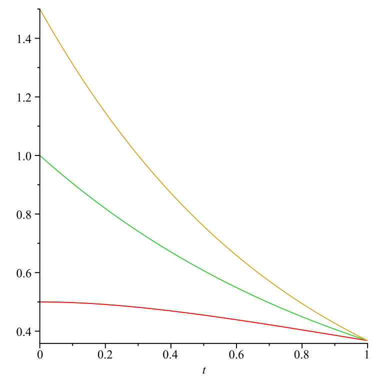
$$[w(t)]_\alpha = \frac{1}{2} \left[(1 + \alpha) e^{\frac{1}{2}(-3+\alpha)t}, (3 - \alpha) e^{\frac{1}{2}(-1-\alpha)t} \right] .$$

Moreover, the diam of this solution is as follows

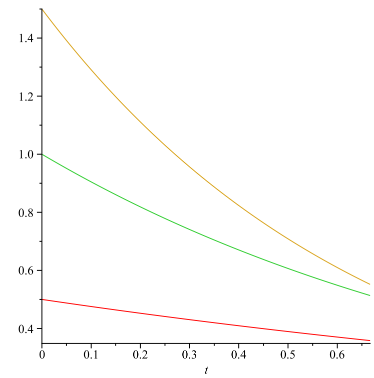
$$Diam(w(t)) = \frac{3}{2} e^{\frac{-t}{2}} - \frac{1}{2} e^{\frac{-3t}{2}} .$$

Here, we can observe that the solution obtained by our approach is different from the solutions based on the usual product in [10] and Zadeh's extension principle. The comparison between these three solutions obtained by the different methods demonstrates that the uncertainty of the solution obtained by the method of the cross

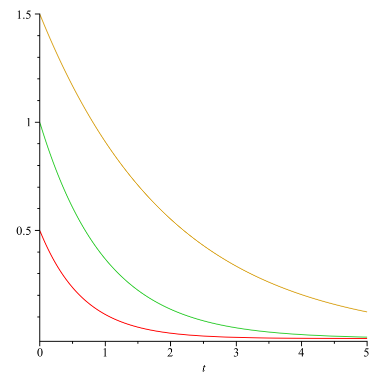
FIGURE 3. Plot of the solutions of Problem (5.5).



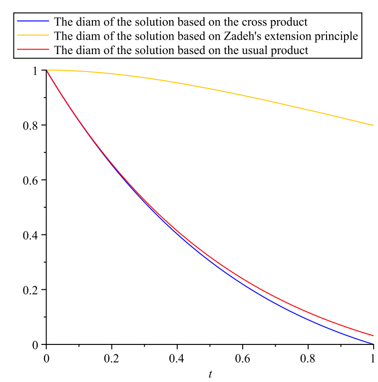
(a) The solution u of Problem (5.5)



(b) The solution v of Problem (5.5)



(c) The solution w of Problem (5.5)



(d) Plot of the diam of the solutions u, v and w of Problem (5.5)

product is less than the uncertainty of the solution obtained by the methods based on the usual product and Zadeh's extension principle. This fact has been illustrated in Figures 4(a), 4(b), 4(c) and 4(d).

Example 5.4. Consider the following initial value problem for $t > 0$

$$\begin{cases} y'(t) = \cos t < 1, 2, 3 > \odot y(t) + \cos t < \frac{1}{2}, 1, \frac{3}{2} >, \\ y(0) =< 1, 2, 3 > . \end{cases} \tag{5.6}$$

Since $a(t)$ has the different signs for $t > 0$, we can partition the interval $[0, +\infty)$ to some subintervals such that $a(t)$ is positive or negative on them. For the convenience, we just consider two of the subintervals $[0, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \pi]$ such that $a(t)$ is positive on the first one and negative on the second one. For $t \in [0, \frac{\pi}{2}]$, the following function is positive

$$
\begin{aligned}
y_0 + \int_0^t b(s)e^{-\int_0^s a_c(r)dr}ds =& < 1,2,3 > + \int_0^t < \frac{1}{2}, 1, \frac{3}{2} > \cos s e^{-2\sin s} ds \\
=& < 1,2,3 > + < \frac{1}{2}, 1, \frac{3}{2} > (-\frac{1}{2}e^{-2\sin t} + \frac{1}{2}) \\
=& < -\frac{1}{4}e^{-2\sin t} + \frac{5}{4}, -\frac{1}{2}e^{-2\sin t} + \frac{5}{2}, -\frac{3}{4}e^{-2\sin t} + \frac{15}{4} > . \qquad (5.7)
\end{aligned}
$$

On the other hand, utilizing Definition 4.1, we have

$$
\begin{aligned}
&\int_0^t (a(s) - a_c(s)) \odot (y_0 + \int_0^s b(r)e^{-\int_0^r a_c(u)du})ds = \\
&\int_0^t < \cos s, 2\cos s, 3\cos s > \odot < \frac{-1}{4}e^{-2\sin s} + \frac{5}{4}, \frac{-1}{2}e^{-2\sin s} + \frac{5}{2}, \frac{-3}{4}e^{-2\sin s} + \frac{15}{4} > \\
&\ominus (2\cos s) < -\frac{1}{4}e^{-2\sin s} + \frac{5}{4}, -\frac{1}{2}e^{-2\sin s} + \frac{5}{2}, -\frac{3}{4}e^{-2\sin s} + \frac{15}{4} > ds \\
&= \int_0^t < -\cos s(\frac{5}{2} - \frac{1}{2}e^{-2\sin s}), 0, \cos s(\frac{5}{2} - \frac{1}{2}e^{-2\sin s}) > ds \\
&= < -\frac{5}{2}\sin t - \frac{1}{4}e^{-2\sin t} + \frac{1}{4}, 0, \frac{5}{2}\sin t + \frac{1}{4}e^{-2\sin t} - \frac{1}{4} > . \qquad (5.8)
\end{aligned}
$$

We apply Case 1 of Theorem 4.2 together with Eqs. (5.7) and (5.8) in order to obtain the following solution for Problem (5.6) which is (i)-differentiable for $0 < t < \frac{\pi}{2}$

$$
y(t) = e^{2\sin t} < -\frac{5}{2}\sin t - \frac{1}{2}e^{-2\sin t} + \frac{3}{2}, -\frac{1}{2}e^{-2\sin t} + \frac{5}{2}, \frac{5}{2}\sin t - \frac{1}{2}e^{-2\sin t} + \frac{7}{2} > .
$$

In this step, one can consider a new fuzzy initial value problem to gain solution for Problem (5.6) on the interval $[\frac{\pi}{2}, \pi]$ (if it exists). To this end we consider the new initial value problem as follows

$$
\begin{cases} y'(t) = \cos t < 1,2,3 > \odot y(t) + \cos t < \frac{1}{2}, 1, \frac{3}{2} >, \\ y_0 = y(\frac{\pi}{2}) = e^2 < -1 - \frac{1}{2}e^{-2}, \frac{5}{2} - \frac{1}{2}e^{-2}, 6 - \frac{1}{2}e^{-2} > . \end{cases} \qquad (5.9)
$$

Since $a(t) \in \mathbb{R}_{\mathcal{F}}^-$ for $t \in [\frac{\pi}{2}, \pi]$, we apply Case 2 of Theorem 4.2 and check if all of H-differences appeared in this case exist. The following H-difference exists, that is

$$
\begin{aligned}
&y_0 \ominus (-1)\int_{\frac{\pi}{2}}^t b(s)e^{-\int_{\frac{\pi}{2}}^s a_c(r)dr} \\
&= e^2 < -1 - \frac{1}{2}e^{-2}, \frac{5}{2} - \frac{1}{2}e^{-2}, 6 - \frac{1}{2}e^{-2} > \ominus (-1)e^2 \int_{\frac{\pi}{2}}^t < \frac{1}{2}, 1, \frac{3}{2} > \cos s e^{-2\sin s} ds \\
&= e^2 < -1 - \frac{1}{4}e^{-2} - \frac{1}{4}e^{-2\sin t}, \frac{5}{2} - \frac{1}{2}e^{-2\sin t}, 6 + \frac{1}{4}e^{-2} - \frac{3}{4}e^{-2\sin t} > \in \mathbb{R}_{\mathcal{F}}^+.
\end{aligned}
$$

On the other hand, we can derive

$$
\begin{aligned}
&\mathrm{e}^{-2\sin t}, 6 + \tfrac{1}{4}e^{-2} - \tfrac{3}{4}e^{-2\sin t} > \\
&\ominus 2e^2\cos t < -1 - \tfrac{1}{4}e^{-2} - \tfrac{1}{4}e^{-2\sin t}, \tfrac{5}{2} - \tfrac{1}{2}e^{-2\sin t}, 6 + \tfrac{1}{4}e^{-2} - \tfrac{3}{4}e^{-2\sin t} > \\
&= e^2 < \cos t(\tfrac{5}{2} - \tfrac{1}{2}e^{-2\sin t}), 0, -\cos t(\tfrac{5}{2} - \tfrac{1}{2}e^{-2\sin t}) > .
\end{aligned}
$$

But the following H-difference dose not exists, that is

$$e^{\int_{\frac{\pi}{2}}^{t} a_c(s)ds}(a(t)-a_c(t))\odot(y_0\ominus(-1)\int_{\frac{\pi}{2}}^{t} b(r)e^{-\int_{\frac{\pi}{2}}^{r} a_c(u)du}dr)\ominus(-1)$$
$$a_c(t)e^{\int_{t_0}^{t} a_c(s)ds}\int_{\frac{\pi}{2}}^{t}(a(s)-a_c(s))\odot(y_0\ominus(-1)\int_{\frac{\pi}{2}}^{s} b(r)e^{-\int_{\frac{\pi}{2}}^{r} a_c(u)du}dr)ds$$
$$=< -\frac{5}{2}\cos t-\frac{1}{2}e^{-2}\cos t+\frac{5}{2}\sin 2t, 0, \frac{5}{2}\cos t+\frac{1}{2}e^{-2}\cos t-\frac{5}{2}\sin 2t >\notin \mathbb{R}_{\mathcal{F}}.$$

It follows that we are not able to find any solution on the interval $[\frac{\pi}{2},\pi]$.

Example 5.5. Consider the following initial value problem

$$\begin{cases} y'(t) =< t^2, t^2+1, t^2+2 > \odot y(t)+ < \frac{t}{2}, t, \frac{3t}{2} >, \\ y(0) =< 0,1,2 > . \end{cases} \tag{5.10}$$

It is obvious that for all $t>0$, $a(t)\in\mathbb{R}^{+}_{\mathcal{F}}$ and also

$$\begin{aligned} y_0+\int_0^t b(s)e^{-\int_0^s a_c(r)dr}ds &=< 0,1,2 > \\ &+ < \int_0^t \frac{s}{2}e^{-\frac{s^3}{3}-s}ds, \int_0^t se^{-\frac{s^3}{3}-s}ds, \int_0^t \frac{3s}{2}e^{-\frac{s^3}{3}-s}ds > \\ &=< \frac{1}{2}g(t), 1+g(t), 2+\frac{3}{2}g(t) >\in\mathbb{R}^{+}_{\mathcal{F}}, \end{aligned}$$

where $g(t)=\int_0^t se^{-\frac{s^3}{3}-s}ds$.
On the other hand, utilizing Definition 4.1, we have

$$\begin{aligned} &\int_0^t (a(s)-a_c(s))\odot(y_0+\int_0^s b(r)e^{-\int_0^r a_c(u)du})ds = \\ &\int_0^t \Big(< s^2, s^2+1, s^2+2 > \odot < \frac{1}{2}g(s), 1+g(s), 2+\frac{3}{2}g(s) > \\ &\ominus(s^2+1) < \frac{1}{2}g(s), 1+g(s), 2+\frac{3}{2}g(s) >\Big)\, ds \\ &= \int_0^t < -(1+g(s)), 0, 1+g(s) > ds \\ &=< -\int_0^t (1+g(s))ds, 0, \int_0^t (1+g(s))ds > . \end{aligned}$$

We apply Case 1 of Theorem 4.2 in order to obtain the following solution for Problem (5.10) which is (i)-differentiable for $t>0$

$$y(t)=e^{\frac{t^3}{3}+t} < \frac{1}{2}g(t)-t-\int_0^t g(s)ds, 1+g(t), 2+\frac{3}{2}g(t)+t+\int_0^t g(s)ds > .$$

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