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Inverse Sturm-Liouville problems with two supplementary discontinuous conditions on two symmetric disjoint intervals

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Abstract In this paper, we consider Sturm-Liouville problems on two symmetric disjoint intervals with two supplementary discontinuous conditions at an interior point. First, we investigate some spectral properties of boundary value problems, and obtain the asymptotic form of the eigenvalues and the eigenfunctions. Then, the eigenfunction expansion of Green's function is presented and we prove the uniqueness theorems for the solution of the inverse problem, and reconstruct the Sturm-Liouville operator and the coefficients of boundary conditions using the Weyl m-function and spectral data. Also, numerical examples are presented.

Keywords. Inverse Sturm-Liouville problems, discontinuous conditions, Green's function, expansion theorem, Weyl m-function.

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1. Introduction

Consider the boundary value problem (BVP) $\wp := \wp(q(x), \alpha_{ij})$, $i, j = 1, 2$, consisting of the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda y, \quad x \in [-a, 0) \cup (0, a], \tag{1.1}$$

with the boundary conditions

$$\alpha_{11}y(-a, \lambda) + \alpha_{12}y'(-a, \lambda) = 0, \quad \alpha_{21}y(a, \lambda) + \alpha_{22}y'(a, \lambda) = 0, \tag{1.2}$$

and the discontinuous conditions

$$\begin{cases} c_{11}y'(0+, \lambda) + c_{12}y(0+, \lambda) + c_{13}y'(0-, \lambda) + c_{14}y(0-, \lambda) = 0, \\ d_{11}y'(0+, \lambda) + d_{12}y(0+, \lambda) + d_{13}y'(0-, \lambda) + d_{14}y(0-, \lambda) = 0. \end{cases} \tag{1.3}$$

Here, $\lambda = \rho^2$ is a spectral parameter, $0 < a < \infty$, α_{ij}, $i, j = 1, 2$, are real numbers, $\alpha_{12}\alpha_{22} \neq 0$, $c_{1\ell}, d_{1\ell}$, $\ell = 1, 2, 3, 4$, are real constants, $q(x)$ is a real continuous function on $[-a, 0) \cup (0, a]$, and $\lim_{x\to 0^\pm} |q(x)| < \infty$.

Eq. (1.1) often appears in mathematics, physics, chemistry, mathematical physics, mathematical chemistry and other branches of natural sciences (for example, see [6, 7, 14, 16, 17, 18, 23] and references therein). Also, BVPs with discontinuous

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conditions can be applied in such problems in physics, electronics and mechanics (see, for example, [24]).

In [7, 20, 22], the uniqueness theorems for inverse problems with discontinuous conditions together with separated boundary conditions on the interval $(0,1)$ or $(0,\pi)$ were investigated. Further, the asymptotic behavior of eigenvalues and eigenfunctions of Sturm-Liouville BVPs on $(0,\pi)$ or $(-\pi,\pi)$ were studied in [2, 3, 4]. In [8, 10, 11, 12], the authors considered Sturm-Liouville BVPs with a discontinuity inside the interval $(0,\pi)$ and eigen-parameter dependent boundary conditions, and investigated some their spectral properties. We mention that in [19, 25], the authors proved Hochstadt-Lieberman-type theorems for BVPs with a finite number of discontinuities at interior points on finite intervals.

In the present paper, first, we study the spectral properties of $\wp$ on $[-a,0)\cup(0,a]$, and the asymptotic behavior of the fundamental solutions of (1.1), the eigenvalues and the eigenfunctions of $\wp$ are investigated (see section 2). Then, we present the eigenfunction expansion of the Green's function of the problem $\wp$ in section 3. In section 4, we introduce the Weyl m-function of $\wp$, prove the uniqueness theorems for the solution of the inverse problem, and reconstruct the Sturm-Liouville operator and the coefficients of boundary conditions using the spectral mappings method and the spectral data. Finally, numerical examples are presented in section 5.

2. Asymptotic form of the solutions, eigenvalues and eigenfunctions

In this section, we study the asymptotic behavior of some fundamental solutions of (1.1) and obtain the asymptotic form of the eigenvalues and the eigenfunctions of BVP $\wp$.

Denote

$$H_{s\ell}=\det\begin{pmatrix} c_{1s} & c_{1\ell}\\ d_{1s} & d_{1\ell}\end{pmatrix},\quad 1\le s<\ell\le 4.$$

Assume that $H_{12},H_{34}>0$. Let $y_{11}(x,\lambda),y_{12}(x,\lambda)$ and $y_{21}(x,\lambda),y_{22}(x,\lambda)$ be the solutions of (1.1) on the intervals $[-a,0)$ and $(0,a]$ respectively, which satisfy the initial conditions

$$y_{11}(-a,\lambda)=\alpha_{12},\quad y_{11}'(-a,\lambda)=-\alpha_{11}, \tag{2.1}$$

$$\begin{cases} y_{12}(0-,\lambda)=-\frac{H_{14}}{H_{34}}y_{22}(0+,\lambda)-\frac{H_{24}}{H_{34}}y_{22}'(0+,\lambda),\\[2mm] y_{12}'(0-,\lambda)=\frac{H_{13}}{H_{34}}y_{22}(0+,\lambda)+\frac{H_{23}}{H_{34}}y_{22}'(0+,\lambda),\end{cases} \tag{2.2}$$

$$\begin{cases} y_{21}(0+,\lambda)=\frac{H_{23}}{H_{12}}y_{11}(0-,\lambda)+\frac{H_{24}}{H_{12}}y_{11}'(0-,\lambda),\\[2mm] y_{21}'(0+,\lambda)=-\frac{H_{13}}{H_{12}}y_{11}(0-,\lambda)-\frac{H_{14}}{H_{12}}y_{11}'(0-,\lambda),\end{cases} \tag{2.3}$$

$$y_{22}(a,\lambda)=-\alpha_{22},\quad y_{22}'(a,\lambda)=\alpha_{21}. \tag{2.4}$$

It is known that for each fixed x, the solutions $y_{11}(x,\lambda)$ and $y_{22}(x,\lambda)$ are entire in λ. By the method used in [1], it can be shown that for each fixed x, $y_{12}(x,\lambda)$ and $y_{21}(x,\lambda)$ are also entire in λ. From (2.1)-(2.4) and using the method of variation of

parameters (see [5]), we have:
(a) for $x \in [-a, 0)$,

$$\begin{aligned} y_{11}(x,\lambda) =& -\alpha_{11}\frac{\sin(\rho(x+a))}{\rho} + \alpha_{12}\cos(\rho(x+a)) \\ &+ \frac{1}{\rho}\int_{-a}^{x} q(\xi)\sin(\rho(x-\xi))y_{11}(\xi,\lambda)d\xi, \end{aligned} \tag{2.5}$$

$$\begin{aligned} y_{12}(x,\lambda) =& \Big(\frac{H_{13}}{H_{34}}y_{22}(0+,\lambda) + \frac{H_{23}}{H_{34}}y'_{22}(0+,\lambda)\Big)\frac{\sin(\rho x)}{\rho} \\ &- \Big(\frac{H_{14}}{H_{34}}y_{22}(0+,\lambda) + \frac{H_{24}}{H_{34}}y'_{22}(0+,\lambda)\Big)\cos(\rho x) \\ &+ \frac{1}{\rho}\int_{x}^{0} q(\xi)\sin(\rho(x-\xi))y_{12}(\xi,\lambda)d\xi, \end{aligned}$$

(b) for $x \in (0, a]$,

$$\begin{aligned} y_{21}(x,\lambda) =& -\Big(\frac{H_{13}}{H_{12}}y_{11}(0-,\lambda) + \frac{H_{14}}{H_{12}}y'_{11}(0-,\lambda)\Big)\frac{\sin(\rho x)}{\rho} \\ &+ \Big(\frac{H_{23}}{H_{12}}y_{11}(0-,\lambda) + \frac{H_{24}}{H_{12}}y'_{11}(0-,\lambda)\Big)\cos(\rho x) \\ &+ \frac{1}{\rho}\int_{0}^{x} q(\xi)\sin(\rho(x-\xi))y_{21}(\xi,\lambda)d\xi, \end{aligned}$$

$$\begin{aligned} y_{22}(x,\lambda) =& -\alpha_{21}\frac{\sin(\rho(a-x))}{\rho} - \alpha_{22}\cos(\rho(a-x)) \\ &+ \frac{1}{\rho}\int_{x}^{a} q(\xi)\sin(\rho(x-\xi))y_{22}(\xi,\lambda)d\xi. \end{aligned}$$

Multiplying (2.5) by $\exp(-\tau(x+a))$ where $\tau = |Im\rho|$, we get

$$y_{11}(x,\lambda) = O\Big(\exp(\tau(x+a))\Big)$$

as $|\lambda| \to \infty$. Therefore, from (2.5) we obtain

$$y_{11}(x,\lambda) = \alpha_{12}\cos(\rho(x+a)) + O\Big(\frac{1}{|\rho|}\exp(\tau(x+a))\Big).$$

Similarly, the asymptotic form of the solutions $y_{12}(x,\lambda)$, $y_{21}(x,\lambda)$ and $y_{22}(x,\lambda)$ can be obtained and we have the following lemma.

Lemma 2.1. *As $|\lambda| \to \infty$, the following asymptotic formulas hold:*

$$y_{11}(x,\lambda) = \alpha_{12}\cos(\rho(x+a)) + O\Big(\frac{1}{|\rho|}\exp(\tau(x+a))\Big), \tag{2.6}$$

$$y_{12}(x,\lambda) = -\frac{H_{24}\alpha_{22}}{H_{34}}\rho\sin(\rho a)\cos(\rho x) + O\Big(\exp(\tau(a-x))\Big), \tag{2.7}$$

$$y_{21}(x,\lambda) = \frac{H_{24}\alpha_{12}}{H_{12}}\rho\sin(\rho a)\cos(\rho x) + O\Big(\exp(\tau(x+a))\Big), \tag{2.8}$$

$$y_{22}(x,\lambda) = -\alpha_{22}\cos(\rho(a-x)) + O\Big(\frac{1}{|\rho|}\exp(\tau(a-x))\Big). \tag{2.9}$$

Now, we define the characteristic function

$$\Delta(\lambda) := W\Big(y_{11}(x,\lambda), y_{12}(x,\lambda)\Big),$$

where $W(u,v)$ is the wronskian of u and v. Since the wronskian y_{11} and y_{12} is independent of x (see [15]), hence,

$$\Delta(\lambda) = y_{11}(0-,\lambda)y'_{12}(0-,\lambda) - y'_{11}(0-,\lambda)y_{12}(0-,\lambda). \tag{2.10}$$

Now, we obtain the asymptotic forms of the eigenvalues and the eigenfunctions of the boundary value problem $\wp$.

Theorem 2.2. *(a) The eigenvalues of the boundary value problem $\wp$ have the following asymptotic form:*

$$\rho_n = \sqrt{\lambda_n} = \frac{n\pi}{2a} + O\Big(\frac{1}{n}\Big), \tag{2.11}$$

as $n \to \infty$.
(b) The asymptotic form of the eigenfunctions

$$\{y(x,\lambda_n) = (y_{11}(x,\lambda_n), y_{21}(x,\lambda_n))\}_{n\geq 0}$$

, of $\wp$ is as follows:

$$\begin{cases} y_{11}(x,\lambda_n) = \alpha_{12}\cos(\rho_n(x+a)) + O\Big(\frac{1}{n}\Big), & x \in [-a,0), \\ y_{21}(x,\lambda_n) = \frac{H_{24}\alpha_{12}}{H_{12}}\rho_n\sin(\rho_n a)\cos(\rho_n x) + O\Big(1\Big), & x \in (0,a]. \end{cases}$$

Proof. We know from [15] that the eigenvalues of $\wp$ coincide with the zeros of the characteristic function $\Delta(\lambda)$. Substituting (2.6) and (2.7) into (2.10), we have

$$\Delta(\lambda) = -\frac{H_{24}\alpha_{12}\alpha_{22}}{H_{34}}\rho^2\sin^2(\rho a) + O\Big(|\rho|\exp(2\tau a)\Big). \tag{2.12}$$

Moreover, it follows from the Rouche's theorem [21] that $\Delta(\lambda)$ has the same number of zeros inside the appropriate large contours as the term

$$-\frac{H_{24}\alpha_{12}\alpha_{22}}{H_{34}}\rho^2\sin^2(\rho a).$$

So, $\Delta(\lambda)$ has a countable set of zeros which can be numbered as $\lambda_1 < \lambda_2 < \dots$, and $\rho_n = \frac{n\pi}{2a} + \kappa_n$, where $\kappa_n < \frac{\pi^2}{4a}$ for sufficiently large n. This and (2.12) yield that

$\kappa_n = O\Big(\frac{1}{n}\Big)$, and hence we obtain (2.11). Now, substituting (2.11) into (2.6) and (2.8), we arrive at (b). $\square$

3. The eigenfunction expansion of Green's function

Let us consider the Cauchy problem $\wp_0$ consisting of the nonhomogeneous differential equation

$$-z'' + (q(x) - \lambda)z = f(x), \qquad x \in [-a, 0) \cup (0, a], \tag{3.1}$$

together with the same boundary and discontinuous conditions (1.2)-(1.3) when

$$\delta(\lambda) := H_{34}\delta^{-}(\lambda) = H_{12}\delta^{+}(\lambda) \neq 0,$$

where

$$\delta^{-}(\lambda) = W(y_{11}(x,\lambda), y_{12}(x,\lambda)), \quad \delta^{+}(\lambda) = W(y_{21}(x,\lambda), y_{22}(x,\lambda)).$$

We know from [9] that the general solution $z(x,\lambda)$ of (3.1) has the form

$$z(x,\lambda) = \begin{cases} k_{11}y_{11}(x,\lambda) + k_{12}y_{12}(x,\lambda) + \frac{H_{34}}{\delta(\lambda)}y_{12}(x,\lambda)\int_{-a}^{x} y_{11}(t,\lambda)f(t)dt \\ +\frac{H_{34}}{\delta(\lambda)}y_{11}(x,\lambda)\int_{x}^{0} y_{12}(t,\lambda)f(t)dt, \qquad x \in [-a,0), \\ \\ k_{21}y_{21}(x,\lambda) + k_{22}y_{22}(x,\lambda) + \frac{H_{12}}{\delta(\lambda)}y_{22}(x,\lambda)\int_{0}^{x} y_{21}(t,\lambda)f(t)dt \\ +\frac{H_{12}}{\delta(\lambda)}y_{21}(x,\lambda)\int_{x}^{a} y_{22}(t,\lambda)f(t)dt, \qquad x \in (0,a], \end{cases}$$

where k_{ij}, $i,j = 1,2$, are arbitrary constants. From (1.2)-(1.3), we obtain $k_{12} = k_{21} = 0$ and

$$k_{11} = \frac{H_{12}}{\delta(\lambda)}\int_0^a y_{22}(t,\lambda)f(t)dt, \quad k_{22} = \frac{H_{34}}{\delta(\lambda)}\int_{-a}^0 y_{11}(t,\lambda)f(t)dt.$$

Therefore, we obtain the following formula for the solution $z(x,\lambda)$ of $\wp_0$:

$$z(x,\lambda) = \begin{cases} \frac{H_{34}}{\delta(\lambda)}y_{12}(x,\lambda)\int_{-a}^{x} y_{11}(t,\lambda)f(t)dt \\ +\frac{H_{34}}{\delta(\lambda)}y_{11}(x,\lambda)\int_{x}^{0} y_{12}(t,\lambda)f(t)dt \\ +\frac{H_{12}}{\delta(\lambda)}y_{11}(x,\lambda)\int_{0}^{a} y_{22}(t,\lambda)f(t)dt, \quad x \in [-a,0), \\ \\ \frac{H_{34}}{\delta(\lambda)}y_{22}(x,\lambda)\int_{-a}^{0} y_{11}(t,\lambda)f(t)dt \\ +\frac{H_{12}}{\delta(\lambda)}y_{22}(x,\lambda)\int_{0}^{x} y_{21}(t,\lambda)f(t)dt \\ +\frac{H_{12}}{\delta(\lambda)}y_{21}(x,\lambda)\int_{x}^{a} y_{22}(t,\lambda)f(t)dt, \quad x \in (0,a]. \end{cases} \tag{3.2}$$

Hence, the Green's function of the problem $\wp_0$ has the form:

$$G(x,t;\lambda) = \begin{cases} \frac{U(t,\lambda)V(x,\lambda)}{\delta(\lambda)}, \quad -a \le t \le x \le a, \quad x,t \neq 0, \\ \\ \frac{U(x,\lambda)V(t,\lambda)}{\delta(\lambda)}, \quad -a \le x \le t \le a, \quad x,t \neq 0, \end{cases}$$

where

$$U(x,\lambda)=\begin{cases} y_{11}(x,\lambda), & x\in[-a,0),\\ y_{21}(x,\lambda), & x\in(0,a],\end{cases}$$
$$V(x,\lambda)=\begin{cases} y_{12}(x,\lambda), & x\in[-a,0),\\ y_{22}(x,\lambda), & x\in(0,a].\end{cases} \tag{3.3}$$

Therefore, the solution (3.2) can be rewritten in the terms of the Green's function as

$$z(x,\lambda)=H_{34}\int_{-a}^{0}G(x,t;\lambda)f(t)dt+H_{12}\int_{0}^{a}G(x,t;\lambda)f(t)dt.$$

Since $\lambda=0$ is not an eigenvalue of the boundary value problem $\wp_0$, the homogeneous equation $-z''+q(x)z=0$ has only the trivial solution $z\equiv 0$. Now, let $G_0(x,t)=G(x,t;0)$, we see that the function

$$z_0(x,\lambda)=H_{34}\int_{-a}^{0}G_0(x,t)f(t)dt+H_{12}\int_{0}^{a}G_0(x,t)f(t)dt$$

solves the nonhomogeneous equation $-z''+q(x)z=f(x)$ and satisfies all boundary and discontinuous conditions (1.2)-(1.3).

Theorem 3.1. *The Green's function $G_0(x,t)$ can be expanded into an eigenfunction series*

$$G_0(x,t)=-\sum_{n=0}^{\infty}\frac{1}{\lambda_n}U(x,\lambda_n)U(t,\lambda_n), \tag{3.4}$$

which converges absolutely and uniformly on $([-a,0)\cup(0,a])^2$.

Proof. Using the asymptotic behavior of the eigenvalues λ_n and the eigenfunctions $U(x,\lambda_n)$, it is not difficult to show that the series in (3.4) converges absolutely and uniformly, and therefore represents a continuous function. Now, to prove the equality (3.4), suppose on the contrary that

$$\widetilde{G}_0(x,t)=G_0(x,t)+\sum_{n=0}^{\infty}\frac{1}{\lambda_n}U(x,\lambda_n)U(t,\lambda_n)\neq 0.$$

Since $\widetilde{G}_0(x,t)$ is a symmetric function, by the theory of integral equations (for example, see [13]), the kernel $\widetilde{G}_0(x,t)$ has at least one eigenfunction. Therefore, we can prove that there is a real number λ^0 and real-valued function ϕ^0 such that

$$H_{34}\int_{-a}^{0}\widetilde{G}_0(x,t)\phi^0(t)dt+H_{12}\int_{0}^{a}\widetilde{G}_0(x,t)\phi^0(t)dt=\lambda^0\phi^0(x). \tag{3.5}$$

Multiplying (3.5) by $U(x,\lambda_n)$, we obtain by necessary calculations that

$$\begin{aligned}
&H_{34}\int_{-a}^{0}\{H_{34}\int_{-a}^{0}\widetilde{G}_0(x,t)U(x,\lambda_n)dx\\
&\quad+H_{12}\int_{0}^{a}\widetilde{G}_0(x,t)U(x,\lambda_n)dx\}\phi^0(t)dt\\
&\quad+H_{12}\int_{0}^{a}\{H_{34}\int_{-a}^{0}\widetilde{G}_0(x,t)U(x,\lambda_n)dx\\
&\quad+H_{12}\int_{0}^{a}\widetilde{G}_0(x,t)U(x,\lambda_n)dx\}\phi^0(t)dt\\
&=\lambda^0\{H_{34}\int_{-a}^{0}U(x,\lambda_n)\phi^0(x)dx+H_{12}\int_{0}^{a}U(x,\lambda_n)\phi^0(x)dx\}.
\end{aligned}\tag{3.6}$$

Since the eigenfunctions $U(x,\lambda_n)$, $n\geq 0$, form an orthonormal set in the sense of

$$H_{34}\int_{-a}^{0}U(x,\lambda_m)U(x,\lambda_n)dx+H_{12}\int_{0}^{a}U(x,\lambda_m)U(x,\lambda_n)dx=\delta_{mn},$$

(δ_{mn} is the Kronecker delta) it is easy to see that

$$\begin{aligned}
&H_{34}\int_{-a}^{0}\widetilde{G}_0(x,t)U(x,\lambda_n)dx+H_{12}\int_{0}^{a}\widetilde{G}_0(x,t)U(x,\lambda_n)dx\\
&=H_{34}\int_{-a}^{0}G_0(x,t)U(x,\lambda_n)dx+H_{12}\int_{0}^{a}G_0(x,t)U(x,\lambda_n)dx\\
&\quad+\frac{U(x,\lambda_n)}{\lambda_n}.
\end{aligned}\tag{3.7}$$

Substituting (3.7) into (3.6) gives us (for $n=0,1,2,3,\dots$)

$$H_{34}\int_{-a}^{0}U(x,\lambda_n)\phi^0(x)dx+H_{12}\int_{0}^{a}U(x,\lambda_n)\phi^0(x)dx=0.\tag{3.8}$$

Consequently, $\phi^0(x)$ is orthogonal to all eigenfunctions. On the other hand, from (3.5) and (3.8) we derive

$$H_{34}\int_{-a}^{0}G_0(x,t)\phi^0(t)dt+H_{12}\int_{0}^{a}G_0(x,t)\phi^0(t)dt=\lambda^0\phi^0(x).$$

Thus, $\phi^0(x)$ is also an eigenfunction of $\wp$ corresponding to the eigenvalue $-(\lambda^0)^{-1}$. So, it is orthogonal to itself, i.e.

$$H_{34}\int_{-a}^{0}(\phi^0(x))^2dx+H_{12}\int_{0}^{a}(\phi^0(x))^2dx=0.$$

This together with $H_{12},H_{34}>0$ yields $\phi^0(x)\equiv 0$. This is a contradiction. The proof is complete. $\square$

4. Recovering the Sturm-Liouville problem

In this section, we prove the uniqueness theorems for the solution of the inverse problem associated with $\wp$, and we study the inverse problem by using the Weyl m-function and the spectral data.

4.1. **Reconstruction by the Weyl m-function.** In this subsection, we define the Weyl m-function of $\wp$, and present the nodes (zeros of the eigenfunctions). Then, we prove two uniqueness theorems and solve the inverse problem using the spectral mappings method. Denote

$$m(\lambda) = -\frac{\alpha_{12}^{-1} y_{22}(-a,\lambda)}{\Delta(\lambda)}. \tag{4.1}$$

$m(\lambda)$ is called the Weyl m-function of $\wp$. From (2.9) and (2.12), we get

$$m(\lambda) = H_{24}^{-1} H_{34} \alpha_{12}^{-1} \lambda^{-1} + O\Big(|\lambda|^{-1} \exp(2\tau a)\Big), \quad as\ |\rho| \to \infty.$$

Let $S(x,\lambda)$ be the solution of (1.1) satisfies (1.3) and the following initial conditions

$$S(-a,\lambda) = 0, \quad S'(-a,\lambda) = H_{12}^{-1}.$$

We denote

$$\Phi(x,\lambda) := \frac{y_{22}(x,\lambda)}{\Delta(\lambda)}, \tag{4.2}$$

which is called the Weyl-solution of $\wp$. From (2.1), (4.1) and (4.2), we have

$$\Phi(x,\lambda) = -S(x,\lambda) - m(\lambda) y_{11}(x,\lambda). \tag{4.3}$$

We agree that together with $\wp$ we consider a boundary value problem $\widetilde{\wp}$ consisting of (1.1)-(1.3) but with different coefficients $\widetilde{q}(x)$ and $\widetilde{\alpha}_{ij}$, $\widetilde{c}_{1\ell} = c_{1\ell}$, $\widetilde{d}_{1\ell} = d_{1\ell}$, $i,j = 1,2$, $\ell = 1,2,3,4$, $\widetilde{\alpha}_{12}\widetilde{\alpha}_{22} \neq 0$.

In the following theorem, we prove the first uniqueness theorem for the solution of the inverse problem.

Theorem 4.1. *If $m(\lambda) = \widetilde{m}(\lambda)$, then $\wp = \widetilde{\wp}$, i.e. $q(x) = \widetilde{q}(x)$ and $\alpha_{ij} = \widetilde{\alpha}_{ij}$, $i,j = 1,2$.*

Proof. We define the matrix $\mathbf{P}(x,\lambda) = \Big(\mathbf{P}_{ij}(x,\lambda)\Big)_{i,j=1,2}$ by the formula

$$\mathbf{P}(x,\lambda) \begin{pmatrix} \widetilde{y}_{11}(x,\lambda) & \widetilde{\Phi}(x,\lambda) \\ \widetilde{y}'_{11}(x,\lambda) & \widetilde{\Phi}'(x,\lambda) \end{pmatrix} = \begin{pmatrix} y_{11}(x,\lambda) & \Phi(x,\lambda) \\ y'_{11}(x,\lambda) & \Phi'(x,\lambda) \end{pmatrix}.$$

Since $W(\widetilde{y}_{11}(x,\lambda), \widetilde{\Phi}(x,\lambda)) \equiv 1$, we obtain

$$\begin{cases} \mathbf{P}_{11}(x,\lambda) = y_{11}(x,\lambda)\widetilde{\Phi}'(x,\lambda) - \widetilde{y}'_{11}(x,\lambda)\Phi(x,\lambda), \\ \mathbf{P}_{12}(x,\lambda) = \widetilde{y}_{11}(x,\lambda)\Phi(x,\lambda) - y_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda), \end{cases} \tag{4.4}$$

$$\begin{cases} \mathbf{P}_{21}(x,\lambda) = y'_{11}(x,\lambda)\widetilde{\Phi}'(x,\lambda) - \widetilde{y}'_{11}(x,\lambda)\Phi'(x,\lambda), \\ \mathbf{P}_{22}(x,\lambda) = \widetilde{y}_{11}(x,\lambda)\Phi'(x,\lambda) - y'_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda). \end{cases}$$

Using the above equalities and the relations (4.1) and (4.3), we conclude that for fixed x, $\mathbf{P}_{ij}(x,\lambda)$, $i,j=1,2$, are meromorphic in λ, and have simple poles in the eigenvalues $\lambda_n, \widetilde{\lambda}_n$ of $\wp, \widetilde{\wp}$, respectively. Denote

$$\begin{cases} D_\delta := \{\lambda : |\lambda - \lambda_n| > \delta,\ n = 1,2,3,\dots\}, \\ \\ \widetilde{D}_\delta := \{\lambda : |\lambda - \widetilde{\lambda}_n| > \delta,\ n = 1,2,3,\dots\}. \end{cases}$$

Now, it follows from (4.3), (4.4) and $\widetilde{\Phi}(x,\lambda) = -\widetilde{S}(x,\lambda) - \widetilde{m}(\lambda)\widetilde{y}_{11}(x,\lambda)$ that

$$\begin{cases} \mathbf{P}_{11}(x,\lambda) = & -y_{11}(x,\lambda)\widetilde{S}'(x,\lambda) + \widetilde{y}'_{11}(x,\lambda)F(x,\lambda) \\ & +\Big(m(\lambda) - \widetilde{m}(\lambda)\Big) y_{11}(x,\lambda)\widetilde{y}'_{11}(x,\lambda), \\ \\ \mathbf{P}_{12}(x,\lambda) = & -\widetilde{y}_{11}(x,\lambda)S(x,\lambda) + y_{11}(x,\lambda)\widetilde{S}(x,\lambda) \\ & +\Big(\widetilde{m}(\lambda) - m(\lambda)\Big) y_{11}(x,\lambda)\widetilde{y}_{11}(x,\lambda). \end{cases} \tag{4.5}$$

Hence, $m(\lambda) = \widetilde{m}(\lambda)$ yields that $\mathbf{P}_{11}(x,\lambda)$ and $\mathbf{P}_{12}(x,\lambda)$ are entire in λ for fixed x. Moreover, from (2.9), (2.12) and (4.2), for $m = 0,1$ we derive

$$\begin{cases} |\Phi^{(m)}(x,\lambda)| \le r_\delta |\rho|^{m-2} \exp(-\tau(x+a)), \quad \rho \in D_\delta, \\ \\ |\widetilde{\Phi}^{(m)}(x,\lambda)| \le \widetilde{r}_\delta |\rho|^{m-2} \exp(-\tau(x+a)), \quad \rho \in \widetilde{D}_\delta, \end{cases}$$

where r_δ and $\widetilde{r}_\delta$ are positive constants. Thus, for $\rho \in D_\delta \cap \widetilde{D}_\delta$, we derive the following approximations as $|\rho| \to \infty$:

$$|\mathbf{P}_{11}(x,\lambda)| \le r_\delta, \quad |\mathbf{P}_{12}(x,\lambda)| \le r_\delta |\rho|^{-2}. \tag{4.6}$$

From (4.5) and (4.6) we obtain $\mathbf{P}_{11}(x,\lambda) \equiv h(x)$ and $\mathbf{P}_{12}(x,\lambda) \equiv 0$. Thus, we have

$$y_{11}(x,\lambda) = h(x)\widetilde{y}_{11}(x,\lambda), \quad \Phi(x,\lambda) = h(x)\widetilde{\Phi}(x,\lambda).$$

Since $W(y_{11},\Phi) = W(\widetilde{y}_{11},\widetilde{\Phi}) \equiv 1$, $h(x) \equiv 1$. Therefore, $y_{11}(x,\lambda) = \widetilde{y}_{11}(x,\lambda)$ and $\Phi(x,\lambda) = \widetilde{\Phi}(x,\lambda)$. Consequently, $\wp = \widetilde{\wp}$, i.e.

$$q(x) = \widetilde{q}(x), \qquad \alpha_{ij} = \widetilde{\alpha}_{ij}.$$

The proof is complete. □

Remark 4.2. According to the asymptotic form of the eigenfunctions $y(x,\lambda_n)$ of the boundary value problem $\wp$, the set $X := \{x_n^j\}$, $n = 1,2,3,\dots$, $j = 1,2,\dots,n-1$, of the nodes of $\wp$ has the form

$$x_n^j = \begin{cases} -a + \frac{(2j-1)\pi}{n-1} + O(\frac{1}{n^2}), & x \in [-a,0), \\ \\ \frac{(2j-1)\pi}{n-1} + O(\frac{1}{n^2}), & x \in (0,a]. \end{cases}$$

By the method used in the proof of Theorem 3 in [18], the following uniqueness theorem can be proved.

Theorem 4.3. *Let $\{x_n^j\}$ and $\{\widetilde{x}_n^j\}$ be the nodes of $\wp$ and $\widetilde{\wp}$, respectively. If $x_n^j = \widetilde{x}_n^j$, then $q(x) = \widetilde{q}(x)$ a.e. on $(-a,0) \cup (0,a)$.*

4.2. **Reconstruction by the spectral data.** First, we define the norming constants by

$$\beta_n := ||y(x,\lambda_n)||^2 = \int_{-a}^{0} y_{11}^2(x,\lambda_n)dx + \int_{0}^{a} y_{21}^2(x,\lambda_n)dx. \tag{4.7}$$

The numbers $\{\lambda_n, \beta_n\}$ are called the spectral data of the boundary value problem $\wp$.

Now, we consider the inverse problem of recovering $\wp$ from the spectral data $\{\lambda_n, \beta_n\}$. Let us choose a model boundary value problem $\widetilde{\wp} = \wp(\widetilde{q}(x), \widetilde{\alpha}_{ij})$ with the conditions $c_{1j} = \widetilde{c}_{1j}$, $d_{1j} = \widetilde{d}_{1j}$, $j = 1,2,3,4$, $\widetilde{\alpha}_{12}\widetilde{\alpha}_{22} \neq 0$, and $\lim_{x\to 0^{\pm}} |\widetilde{q}(x)| < \infty$.

Theorem 4.4. *If $\lambda_n = \widetilde{\lambda}_n$, $\beta_n = \widetilde{\beta}_n$, $n \geq 0$, then $\wp = \widetilde{\wp}$.*

Proof. Let $\rho_n^0 = \frac{(n-1)\pi}{2a}$. It follows from Lemma 2.1 that

$$|U^{(\nu)}(x,\lambda_n)| \leq C(|\rho_n^0| + 1)^{\nu}, \quad |\widetilde{U}^{(\nu)}(x,\widetilde{\lambda}_n)| \leq C(|\rho_n^0| + 1)^{\nu}, \ \nu = 0,1,$$

where here and below, the symbol C denotes various positive constants in estimates, and $U(x,\lambda), \widetilde{U}(x,\lambda)$ are defined as (3.3). Moreover, according to the hypothesis of the theorem, we get

$$\begin{cases} |U^{(\nu)}(x,\lambda) - \widetilde{U}^{(\nu)}(x,\lambda)| \leq C|\rho|^{\nu-1}\exp(\tau x), \ \nu = 0,1, \\ |V^{(\nu)}(x,\lambda) - \widetilde{V}^{(\nu)}(x,\lambda)| \leq C|\rho|^{\nu-1}\exp(\tau(a-x)), \ \nu = 0,1, \end{cases} \tag{4.8}$$

where $\tau = |Im\rho|$, and $V(x,\lambda)$, $\widetilde{V}(x,\lambda)$ are defined as (3.3). Now, from (4.1) and (31) we have

$$|\Phi^{(\nu)}(x,\lambda) - \widetilde{\Phi}^{(\nu)}(x,\lambda)| \leq r_\delta|\rho|^{\nu-2}\exp(-\tau x), \ \nu = 0,1, \ \rho \in D_\delta \cap \widetilde{D}_\delta,$$

where r_δ, D_δ and $\widetilde{D}_\delta$ is defined as in the proof of Theorem 4.1. Therefore, as $|\rho| \to \infty$, we arrive at (4.6) and hence the proof is similar to the proof of Theorem 4.1. □

Now, we denote

$$\begin{aligned} D_i(x,\lambda,\eta) :=& \frac{W(y_{11}(x,\lambda), y_{11}(x,\eta))}{\gamma_{in}(\lambda-\eta)} \\ =& \frac{1}{\gamma_{in}}\int_{-a}^{x} y_{11}(t,\lambda)y_{11}(t,\eta)dt, \quad i = 1,2, \end{aligned} \tag{4.9}$$

where $\gamma_{1n} = \beta_n$ and $\gamma_{2n} = \widetilde{\beta}_n$. By the method used in the proof of Lemma 4.4.1 in [7], we can obtain the following relation:

$$\begin{aligned} y_{11}(x,\lambda) =& \widetilde{y}_{11}(x,\lambda) \\ &- \sum_{n=0}^{\infty}\Big(y_{11}(x,\lambda_n)\widetilde{D}_1(x,\lambda,\lambda_n) - y_{11}(x,\widetilde{\lambda}_n)\widetilde{D}_2(x,\lambda,\widetilde{\lambda}_n)\Big). \end{aligned} \tag{4.10}$$

In the next theorem, we construct the potential function $q(x)$ and the coefficients α_{ij} of $\wp$.

Theorem 4.5. *The following relations hold:*

$$q(x) = \widetilde{q}(x) - 2\varepsilon'(x), \tag{4.11}$$

where

$$\varepsilon(x) = \sum_{n=0}^{\infty} \Big(\frac{1}{\beta_n} y_{11}(x,\lambda_n)\widetilde{y}_{11}(x,\lambda_n) - \frac{1}{\widetilde{\beta}_n} y_{11}(x,\widetilde{\lambda}_n)\widetilde{y}_{11}(x,\widetilde{\lambda}_n) \Big). \tag{4.12}$$

Moreover,

$$\begin{cases} \alpha_{11} = \widetilde{\alpha}_{11} + \varepsilon(-a)\widetilde{\alpha}_{12}, \quad \alpha_{12} = \widetilde{\alpha}_{12} - \varepsilon(-a), \\ \alpha_{21} = \widetilde{\alpha}_{21} + \varepsilon(a)\widetilde{\alpha}_{22}, \quad \alpha_{22} = \widetilde{\alpha}_{22}. \end{cases} \tag{4.13}$$

Proof. It follows from (4.9), (4.10) and (4.12) that

$$\begin{aligned} \widetilde{y}'_{11}(x,\lambda) =& y'_{11}(x,\lambda) + \widetilde{y}_{11}(x,\lambda)\varepsilon(x) \\ &+ \sum_{n=0}^{\infty} \Big(y'_{11}(x,\lambda_n)\widetilde{D}_1(x,\lambda,\lambda_n) - y'_{11}(x,\widetilde{\lambda}_n)\widetilde{D}_2(x,\lambda,\widetilde{\lambda}_n) \Big), \end{aligned} \tag{4.14}$$

$$\begin{aligned} &\widetilde{y}''_{11}(x,\lambda) = y''_{11}(x,\lambda) \\ &+ \sum_{n=0}^{\infty} \Big(y''_{11}(x,\lambda_n)\widetilde{D}_1(x,\lambda,\lambda_n) - y''_{11}(x,\widetilde{\lambda}_n)\widetilde{D}_2(x,\lambda,\widetilde{\lambda}_n) \Big) \\ &+ 2\widetilde{y}_{11}(x,\lambda) \sum_{n=0}^{\infty} \Big(\frac{1}{\beta_n} y'_{11}(x,\lambda_n)\widetilde{y}_{11}(x,\lambda_n) - \frac{1}{\widetilde{\beta}_n} y'_{11}(x,\widetilde{\lambda}_n)\widetilde{y}_{11}(x,\widetilde{\lambda}_n) \Big) \\ &+ \sum_{n=0}^{\infty} \Big\{ \frac{1}{\beta_n} \Big(\widetilde{y}_{11}(x,\lambda)\widetilde{y}_{11}(x,\lambda_n) \Big)' y_{11}(x,\lambda_n) \\ &- \frac{1}{\widetilde{\beta}_n} \Big(\widetilde{y}_{11}(x,\lambda)\widetilde{y}_{11}(x,\widetilde{\lambda}_n) \Big)' y_{11}(x,\widetilde{\lambda}_n) \Big\}. \end{aligned} \tag{4.15}$$

Since

$$y''_{11}(x,\lambda) = (q(x) - \lambda)y_{11}(x,\lambda), \ \ \widetilde{y}''_{11}(x,\lambda) = (\widetilde{q}(x) - \lambda)\widetilde{y}_{11}(x,\lambda), \tag{4.16}$$

applying (4.10), (4.15), (4.16), and then after cancelling the terms with $\widetilde{y}'_{11}(x,\lambda)$, we arrive at (4.11). Taking $x = -a$ and $x = a$ in (4.10) and (4.14), we have (4.13). ☐

Now, Theorem 4.5 gives us an algorithm for the solution of the inverse problem as well as necessary and sufficient conditions for its solvability. The boundary value problem $\wp$ can be constructed by the following algorithm.

Algorithm 4.6. Let the numbers $\{\lambda_n, \beta_n\}$ be given. Then

i) Choose $\widetilde{\wp}$ such that $\widetilde{q} \in C([-a,0) \cup (0,a])$, $\lim_{x\to 0^{\pm}} |\widetilde{q}(x)| < \infty$ and $\widetilde{\alpha}_{12}\widetilde{\alpha}_{22} \neq 0$;

ii) Find $y_{11}(x,\lambda)$ by (4.10);

iii) Construct $q(x)$ and α_{ij} by (4.11) and (4.13).

5. Numerical examples

In this section, we present two illustrative examples to demonstrate the applicability of our scheme. In the first example, we consider a boundary value problem of the form (1.1)-(1.3), and obtain its Green's function. In the second example, we choose a model of discontinuous boundary value problem $\widetilde{\wp}$, and show the efficiency of Algorithm 4.6 to construct the problem $\wp$.

Although the Green's function looks as simple as that of standard Sturm-Liouville problems, it is rather complicated because of the transmission conditions. To illustrate this situation, let us give the following example.

Example 5.1. Consider the following boundary value problem:

$$\begin{aligned} -y''(x,\lambda) &= \lambda y(x,\lambda), \quad x \in [-2,0)\cup(0,2], \\ y(-2,\lambda)+y'(-2,\lambda) &= 0, \quad y(2,\lambda)-y'(2,\lambda)=0, \\ y'(0-,\lambda)-y'(0+,\lambda) &= 0, \quad y(0-,\lambda)-2y(0+,\lambda)=0, \end{aligned}$$

where λ is a complex spectral parameter. Therefore, $H_{12}=H_{23}=2$, $H_{13}=H_{24}=0$, $H_{34}=-H_{14}=1$. Putting $\lambda=\rho^2$, we find that

$$\begin{aligned} y_{11}(x,\lambda) &= \cos(\rho(x+2)) - \frac{1}{\rho}\sin(\rho(x+2)), \\ y_{21}(x,\lambda) &= \Big(\cos 2\rho - \frac{1}{\rho}\sin 2\rho\Big)\cos(\rho x) - \frac{1}{2}\Big(\sin 2\rho + \frac{1}{\rho}\cos 2\rho\Big)\sin(\rho x), \\ y_{12}(x,\lambda) &= \Big(1+\sin^2 2\rho + \frac{1}{2\rho}\sin 4\rho\Big)\cos(\rho(2-x)) \\ &\qquad -\Big(\frac{1}{2}\sin 4\rho + \frac{1}{\rho}(1+\cos^2 2\rho)\Big)\sin(\rho(2-x)), \\ y_{22}(x,\lambda) &= \cos(\rho(2-x)) - \frac{1}{\rho}\sin(\rho(2-x)). \end{aligned}$$

The Green's function has the following form:

$$G(x,t;\lambda) = \begin{cases} (\delta^-(\lambda))^{-1} y_{11}(t,\lambda) y_{12}(x,\lambda), & -2\le t\le x<0, \\ (\delta^-(\lambda))^{-1} y_{11}(t,\lambda) y_{22}(x,\lambda), & -2\le t<0,\ 0<x\le 2, \\ (2\delta^+(\lambda))^{-1} y_{21}(t,\lambda) y_{22}(x,\lambda), & 0<t\le x\le 2, \\ (\delta^-(\lambda))^{-1} y_{11}(x,\lambda) y_{12}(t,\lambda), & -2\le x\le t<0, \\ (2\delta^+(\lambda))^{-1} y_{11}(x,\lambda) y_{22}(t,\lambda), & -2\le x<0,\ 0<t\le 2, \\ (2\delta^+(\lambda))^{-1} y_{21}(x,\lambda) y_{22}(t,\lambda), & 0<x\le t\le 2, \end{cases}$$

where

$$\delta^-(\lambda) = \frac{3}{2}(\rho-\frac{1}{\rho})\sin 4\rho - 6\sin^2 2\rho + 3, \quad \delta^+(\lambda) = \frac{H_{34}}{H_{12}}\delta^-(\lambda) = \frac{1}{2}\delta^-(\lambda).$$

Now, we show the efficiency of Algorithm 4.6 in the following example.

Example 5.2. Take the boundary value problem $\widetilde{\wp}$ on $[-1,0)\cup(0,1]$ such that $\widetilde{q}(x)\equiv 0$, $\widetilde{\alpha}_{11}=\widetilde{\alpha}_{21}=0$, $\widetilde{\alpha}_{12}=\widetilde{\alpha}_{22}=1$, and discontinuous conditions

$$\gamma y'(0-,\lambda)-y'(0+,\lambda)=0, \quad y(0-,\lambda)-\gamma y(0+,\lambda)=0,$$

where $\gamma>0$ is an arbitrary constant. Hence $H_{12}=H_{34}=2$, $H_{13}=H_{24}=0$, $H_{14}=-1$, $H_{23}=4$. Let $\{\widetilde{\lambda}_n,\widetilde{\beta}_n\}$ be the spectral data of $\widetilde{\wp}$. Clearly, $\widetilde{\lambda}_0=0$, $\widetilde{\beta}_0=1+\gamma^{-2}$ and

$$\widetilde{y}_{11}(x,\widetilde{\lambda}_0)=\begin{cases}1, & -1\le x<0,\\ \gamma^{-1}, & 0<x\le 1.\end{cases}$$

Let $\lambda_n=\widetilde{\lambda}_n$ $(n\ge 0)$, $\beta_n=\widetilde{\beta}_n$ $(n\ge 1)$, $\beta_0>0$ be an arbitrary positive number, and let $\vartheta:=\beta_0^{-1}-\widetilde{\beta}_0^{-1}$. Then, it follows from (4.10) and (4.12) that

$$\begin{aligned} y_{11}(x,\lambda_0)&=\widetilde{y}_{11}(x,\lambda_0)-\vartheta y_{11}(x,\lambda_0)\int_{-1}^{x}\widetilde{y}_{11}^2(t,\lambda_0)dt,\\ \varepsilon(x)&=\vartheta y_{11}(x,\lambda_0)\widetilde{y}_{11}(x,\lambda_0).\end{aligned}$$

Consequently,

$$y_{11}(x,\lambda_0)=\begin{cases}(1+\vartheta x)^{-1}, & -1\le x<0,\\ \gamma^{-1}(1+\vartheta+\vartheta\gamma^{-2}x)^{-1}, & 0<x\le 1,\end{cases}$$

$$\varepsilon(x)=\begin{cases}\vartheta(1+\vartheta x)^{-1}, & -1\le x<0,\\ \vartheta\gamma^{-2}(1+\vartheta+\vartheta\gamma^{-2}x)^{-1}, & 0<x\le 1.\end{cases}$$

Therefore, according to (4.11) and (4.13), we calculate

$$q(x)=\begin{cases}2\vartheta^2(1+\vartheta x)^{-2}, & -1\le x<0,\\ 2\vartheta^2\gamma^{-4}(1+\vartheta+\vartheta\gamma^{-2}x)^{-2}, & 0<x\le 1,\end{cases}$$

$$\begin{aligned}\alpha_{11}&=\vartheta(1-\vartheta)^{-1},\ \alpha_{12}=1-\vartheta(1-\vartheta)^{-1},\\ \alpha_{21}&=\vartheta\gamma^{-2}(1+\vartheta+\vartheta\gamma^{-2})^{-1},\ \alpha_{22}=1.\end{aligned}$$

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