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Study of new monotone iterative technique for a class of arbitrary order differential equations

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Abstract In this paper, we apply new type monotone iterative technique which is very rarely used to find iterative solutions for boundary value problem (BVPs) of nonlinear fractional order differential equations (NFODEs). With the help of the aforesaid technique, we establish two sequences of upper and lower solutions for the considered BVP. Further the procedure is testified by providing suitable examples.

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1. Introduction

This article is related with the investigation of multiple solutions to the following BVP of NFODEs by using monotone iterative technique

$$\begin{cases} {}^{c}\mathscr{D}^{\varepsilon}u(t) = \theta(t, u(t));\ t \in \mathscr{I} = [0,1];\ \varepsilon \in (1,2], \\ u(0) = \gamma u^{'}(0) = 0,\ u(1) = \delta u^{'}(1), \quad \gamma > 0, \delta > 0, \end{cases} \tag{1.1}$$

where $\theta : \mathscr{I} \times R \longrightarrow R$ is continuous function, while ${}^{c}\mathscr{D}$ stands for Caputo fractional derivative of order ε.

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Arbitrary order differential equations and their systems are powerful tools to describe many biological, physical, psychological phenomena more accurately as compared to classical differential equations. Furthermore, by using NFODEs many related problems to applied sciences and engineering can also be presented more accurately. Number of applications of NFODEs are also available in the field of chemistry, computer networking, control theory, viscoelasticity, complex medium with electrodynamics, aerodynamics, polymer rheology and signal and image processing phenomenon, etc., (see[1, 8, 23, 33]). Therefore, considerable attention is paid to study the area devoted to differential equations of fractional order. In last few years, many researchers studied BVPs of NFODEs with deep interest for the solutions of existence (see; [4, 7, 11, 12, 14, 17, 19, 21, 29] and the references therein). Because BVPs have many valuable applications in the field of applied sciences. The area of fractional order differential equations is well explored using different techniques. Many numerical methods have been implemented to solve fractional order differential equations appears in the literature. However, only in few articles the method of lower and upper solutions are available for the problem of fractional order, see; [15, 16, 25, 27, 28, 34]. The monotone iterative technique combined with the method of extremal (lower and upper) solutions is one of the strong tools being used to find multiple solutions to NFODEs as well as integer order differential equations and their systems. The monotone iterative techniques were used in some papers to develop conditions for existence of iterative solutions for ordinary and NFODES (see[5, 6, 9, 13, 16, 18, 24, 30, 31, 35]). By using the aforesaid technique to establish some adequate conditions for existence of iterative solutions to NFDEs, one need a proper differential inequalities as a comparison results. For instance, Al-Refai [2], described the basic theory of the boundary value problem of fractional order involving in the Caputo derivative. On applying maximum principle he obtained the necessary conditions for the existence of eigenfunctions and also find the upper and lower bound estimates of the eigenvalue. Al-Refai et al. [3] extended the method of upper and lower solutions and the maximum principle to BVPs using the Caputo fractional derivatives. Further, they also showed the uniqueness and positivity results for the considered problem.

Wardowski [32], studied the method of upper and lower solution, extension of the comparison result and monotone iterative method for case of NFODEs.

On the basis of the above mentioned research works, in this article, we extend the maximum principle and the method of upper and lower solutions with Caputo fractional derivatives for a boundary value problem (1.1). Moreover, by using standard technique of functional analysis, also we develop the conditions for uniqueness of positive solution by considering linear BVP of FDEs. Thank to the aforesaid techniques, we develop conditions for extremal solutions for the considered BVP (1.1). For the justification of our main results we illustrate some numerical examples.

2. Preliminaries

This section provide some results of fractional calculus and nonlinear functional analysis which can be traced in [10, 20, 26].

Definition 2.1. A function $w(t),\ t>0$ is said to be in the space $C_v,\ v\in R$ if there exists a real number $q>v$, such that $w(t)=t^q w_t$, where $w_t\in C[0,\infty)$.

Definition 2.2. Let $\varepsilon>0$ and $w:[a,+\infty)\to R$. Then the Riemann-Liouville integral of arbitrary order of $h(t)$ is given by

$$\mathcal{I}^{\varepsilon}_{a+}w(t)=\frac{1}{\Gamma(\varepsilon)}\int_a^t (t-\tau)^{\varepsilon-1}w(\tau)d\tau,\ \text{where}\ \varepsilon\in R_+,$$

show that at the right side the integral is pointwise defined on $(0,\infty)$.

Definition 2.3. In Caputo sense, the fractional order derivative of a function w on the interval $[a,b]$ is given by

$${}^{c}\mathscr{D}^{\varepsilon}_{a+}w(t)=\frac{1}{\Gamma(n-\varepsilon)}\int_a^t (t-\tau)^{n-\varepsilon-1}w^{(n)}(\tau)\,d\tau,\ n=[\varepsilon]+1,$$

show that at the right side the integral is pointwise defined on $(0,\infty)$.

Lemma 2.4. *The unique solution of FDE* ${}^{c}\mathscr{D}^{\varepsilon}w(t)=0$, *for* $w\in C(\mathscr{I})\cap L(\mathscr{I})$ *is given by*

$$\mathcal{I}^{\varepsilon}[{}^{c}\mathscr{D}^{\varepsilon}w(t)]=w(t)+\sum_{k=0}^{n-1}C_k t^k, \tag{2.1}$$

where $C_k\in\mathbb{R}$, *for* $k=0,1,2,\dots,n-1$.

Definition 2.5. A function $v(t)\in C^2(\mathscr{I})$, is called a lower solution of the problem (1.1), if

$$\begin{aligned}&{}^{c}\mathscr{D}^{\varepsilon}v(t)+\theta(t,v(t))\geq 0, t\in\mathscr{I},\ \varepsilon\in(1,2],\\ &v(0)\leq\gamma v^{'}(0),\ \ v(1)\leq\delta v^{'}(1).\end{aligned}$$

Similarly $w\in C^2(\mathscr{I})$ is called upper solution of BVP (1.1), if

$$\begin{aligned}&{}^{c}\mathscr{D}^{\varepsilon}w(t)+\theta(t,w(t))\leq 0, t\in\mathscr{I},\ \varepsilon\in(1,2],\\ &w(0)\geq\gamma w^{'}(0),\ \ w(1)\geq\delta w^{'}(1).\end{aligned}$$

Theorem 2.6. [2] *Assume that* $w\in C^2(\mathscr{I})$ *attains its minimum at* $t_0\in\mathscr{I}$, *then*

$$({}^{c}\mathscr{D}^{\varepsilon}w)(t_0)\geq\frac{1}{\Gamma(2-\varepsilon)}\left[(\varepsilon-1)t_0^{-\varepsilon}(w(0)-w(t_0))-t_0^{1-\varepsilon}w^{'}(0)\right],\ \ 1<\varepsilon<2.$$

Corollary 2.7. *Assume that* $w\in C^2(\mathscr{I})$ *attains its minimum at* $t_0\in\mathscr{I}$, *and* $w^{'}(0)\leq 0$. *Then* $({}^{c}\mathscr{D}^{\varepsilon}w)(t_0)\geq 0,\ \ 1<\varepsilon<2$.

Lemma 2.8. [2] *Let* $w(t)\in C^2(\mathscr{I}),\mu(t,w)\in C(\mathscr{I}\times R)$ *and* $\mu(t,w)<0,\ \ \forall\ t\in\mathscr{I}$. *If* $w(t)$ *satisfies the inequalities*

$$\begin{aligned}&{}^{c}\mathscr{D}^{\varepsilon}w(t)+a(t)w^{'}(t)+\mu(t,w)w\leq 0, t\in\mathscr{I},\\ &w(0)-\gamma w^{'}(0)\geq 0,\ \ w(1)-\delta w^{'}(1)\geq 0,\end{aligned}$$

where $a(t)\in C(\mathscr{I})$ *and* $\gamma,\delta\geq 0$, *then* $w(t)\geq 0$, *for all* $t\in\mathscr{I}$.

Lemma 2.9. [2] *Let β and α be any upper and lower solutions, respectively of BVP (1.1). If $\theta(t,u)$ with respect to u is decreasing, then α,β are ordered, i.e $\alpha(t)\leq\beta(t)$, for $t\in\mathscr{I}$.*

Proof. Consider the lower and upper solution α,β, then (1.1) yields

$$^{c}\mathscr{D}^{\varepsilon}\alpha(t)+\theta(t,\alpha(t))\geq 0,\ t\in\mathscr{I},$$
$$\alpha(0)\leq\gamma\alpha^{'}(0),\ \ \alpha(1)\leq\delta\alpha^{'}(1)$$

and

$$^{c}\mathscr{D}^{\varepsilon}\beta(t)+\theta(t,\beta(t))\leq 0, t\in\mathscr{I},$$
$$\beta(0)\geq\gamma\beta^{'}(0),\ \ \beta(1)\geq\delta\beta^{'}(1).$$

Upon subtracting, we get

$$^{c}\mathscr{D}^{\varepsilon}(\beta-\alpha)+\theta(t,\beta)-\theta(t,\alpha)\leq 0, \tag{2.2}$$

using Mean value theorem from (2.2), we have

$$^{c}\mathscr{D}^{\varepsilon}(\beta-\alpha)+\frac{\partial\theta}{\partial u}(\eta)(\beta-\alpha)\leq 0, \tag{2.3}$$

where $\eta=a\alpha+(1-a)\beta,\ \ a\in\mathscr{I}$. If we put $\beta-\alpha=z$,
(2.3) yields

$$^{c}\mathscr{D}^{\varepsilon}z(t)+\frac{\partial\theta}{\partial u}(\eta)z\leq 0,$$

with $z(0)\leq\gamma z^{'}(0),\ \ z(1)\leq\delta z^{'}(1)$. As θ with respect to u, $\frac{\partial\theta}{\partial u}<0$ is strictly decreasing. Using result in Lemma 2.8, we have $z\geq 0$. □

Lemma 2.10. *If $\theta(t,u)$ with respect to u is strictly decreasing, then BVP (1.1) posses at most one solution.*

Proof. Let u,v be two solutions of BVP (1.1), then

$$^{c}\mathscr{D}^{\varepsilon}u+\theta(t,u)=0 \quad\text{with}\quad u(0)=\gamma u^{'}(0),\ \ u(1)=\delta u^{'}(1),$$
$$^{c}\mathscr{D}^{\varepsilon}v+\theta(t,v)=0 \quad\text{with}\quad v(0)=\gamma v^{'}(0),\ \ v(1)=\delta v^{'}(1),$$

then on subtraction, we get

$$\begin{aligned}&^{c}\mathscr{D}^{\varepsilon}(u-v)+\theta(t,u)-\theta(t,v)=0,\\&u(0)-v(0)=\gamma(u^{'}(0)-v^{'}(0)),\ \ u(1)-v(1)=\delta(u^{'}(1)-v^{'}(1)),\end{aligned} \tag{2.4}$$

applying Mean value theorem to (2.4), we have

$$^{c}\mathscr{D}^{\varepsilon}(u-v)+\frac{\partial\theta}{\partial u}(u-v)=0, \tag{2.5}$$

where $\eta=au+(1-a)v,\ \ a\in[0,1]$. If we put $z=u-v$, then (2.5) yields

$$^{c}\mathscr{D}^{\varepsilon}z(t)+\frac{\partial\theta}{\partial u}(\eta)z=0,$$

with $z(0) = \gamma z^{'}(0), \quad z(1) = \delta z^{'}(1)$. Using result in Lemma 2.8, we have $z \geq 0$. But $-z$ also satisfied (2.5), so $-z \geq 0$. Therefore, $z = 0 \Rightarrow u = v$. Hence, the solution is unique. □

3. Existence of upper and lower solution by using monotone sequences

In this section, we construct monotone iterative sequences and their convergence to obtain upper and lower solution of BVP (1.1). Consider ordered appears lower and upper solution α and β, respectively, then define a set as

$$S = [\alpha, \beta] = \left\{ \mu \in C^2(\mathscr{I}), \alpha \leq \mu \leq \beta \right\},$$

as $\theta(t, u)$ with respect to u is strictly decreasing, so $\frac{\partial \theta}{\partial u}$ is bounded below in S, i.e., there exists a positive constant d such that

$$-d \leq \frac{\partial \theta}{\partial u}(t, \eta) < 0, \quad \forall \ \eta \in S. \tag{3.1}$$

Theorem 3.1. *Consider the BVP (1.1) with $\theta(t, u)$ satisfies (3.1). Let $u^{(0)}$ and $v^{(0)}$ be initial ordered lower and upper solutions of (1.1). Let $u^{(i)}, v^{(i)}, \ i \geq 1$ be respectively, the solution of*

$$\begin{cases} -{}^c\mathscr{D}^{\varepsilon} u^{(i)} + cu^{(i)} = cu^{(i-1)} + g(t, u^{(i-1)}), \quad t \in \mathscr{I}, \\ u^{(i)}(0) = u_0^{(i)} \geq u^{(i-1)}(0), \quad u^{(i)}(1) = u_1^{(i)} \geq u^{(i-1)}(1), \end{cases} \tag{3.2}$$

and

$$\begin{cases} -{}^c\mathscr{D}^{\varepsilon} v^{(i)} + cv^{(i)} = cv^{(i-1)} + g(t, v^{(i-1)}), \quad t \in \mathscr{I}, \\ v^{(i)}(0) = v_0^{(i)} \geq v^{(i-1)}(0), \quad v^{(i)}(1) = v_1^{(i)} \geq v^{(i-1)}(1), \end{cases} \tag{3.3}$$

then

(1) *The sequence $u^{(i)}, i > 0$, for BVP (1.1) is increasing sequence of lower solution.*

(2) *The sequence $v^{(i)}, i > 0$, for BVP (1.1) is increasing sequence of upper solution. Moreover,*
(3) $u^{(i)} \leq v^{(i)}, \quad \forall \ i \geq 1.$

Proof. To prove (1), we need to prove that

(i) $u^{(i)}, u^{(i-1)} > 0$ for each $i > 1$, and

(ii) $u^{(i)}$ is a lower solution for each $i \geq 1$.

To prove (i), we use induction procedure. It follows from (3.2) with $i = 1$ that

$$-{}^c\mathscr{D}^{\varepsilon} u^{(1)} + cu^{(1)} = cu^{(0)} + g(t, u^{(0)}). \tag{3.4}$$

Since $u^{(0)}$ is a lower solution,

$${}^c\mathscr{D}^{\varepsilon} u^{(0)} + cu^{(i)} + g(t, u^{(0)}) \geq 0. \tag{3.5}$$

Adding (3.4) and (3.5), we obtain

$$^c\mathscr{D}^\varepsilon(u^{(1)} - u^{(0)}) - c(u^{(1)} - u^{(0)}) \leq 0.$$

Let $u^{(1)} - u^{(0)} = z$, then z satisfies

$$^c\mathscr{D}^\varepsilon(z) - cz \leq 0, \quad z(0) \geq \gamma z^{'}(0), \quad z(1) \geq \delta z^{'}(1).$$

Since $-c < 0$, by positivity result in Lemma 2.8, we have $z \geq 0$ or $u^{(0)} < u^{(1)}$, and the result is true for $i = 1$. Now suppose that the result is true for $n \leq i$ and prove for $n = i + 1$.
From (3.2), we have

$$-^c\mathscr{D}^\varepsilon(u^{(i+1)} - u^{(i)}) + c(u^{(i+1)} - u^{(i)}) = c(u^(i) - u^{(i-1)}) + g(t, u^i) - g(t, u^{(i-1)}).$$

Let $z = u^{(i+1)} - u^{(i)}$, applying mean value theorem and and using induction hypothesis $u^{(i+1)} < u^{(i)}$, we obtain

$$\begin{aligned} ^c\mathscr{D}^\varepsilon(z) + cz &= c(u^{(i)} - u^{(i-1)}) + \frac{\partial g}{\partial u}(\zeta)(u^{(i)} - u^{(i-1)}) \\ &\geq c(u^{(i)} - u^{(i-1)}) - c(u^{(i)} - u^{(i-1)}) = 0. \end{aligned}$$

Or $^c\mathscr{D}^\varepsilon(z) - cz \leq 0$, which gives $z \geq 0$, by Lemma 2.8. Hence, $u^{(i)} \leq u^{(i+1)}$ and the result is proved for $n = i + 1$. Therefore $u^{(i+1)} \leq u^{(i)}$ for all $i \geq 1$, which proves (i).
To prove (ii) subtracting $g(t, u^{(i)})$ from both sides of (3.2) and using mean value theorem, we get

$$\begin{aligned} &^c\mathscr{D}^\varepsilon u^{(i)} + g(t, u^{(i)}) = c(u^{(i)} - u^{(i-1)}) + g(t, u^{(i)}) - g(t, u^{(i-1)}) \\ &\left(c + \frac{\partial g}{\partial u}(\zeta)\right)(u^{(i)} - u^{(i-1)}) \geq 0. \end{aligned}$$

So $u^{(i)}, \;\; i > 1$ is a lower solution of BVP (1.1). Hence, we proved (ii). Clearly the proof of (2) and the proof of (1) are similar, so we omit it.
The proof of (3) is clear from Lemma 2.8, since by (1) and (2), $u^{(i)}$ and $v^{(i)}$ are lower and upper solutions, respectively. □

For the convergence results, we provide the following theorems.

Theorem 3.2. *Consider the BVP presented in (1.1), with $\theta(t, u)$ condition (3.1). Let $u^{(i)}$ and $v^{(i)}$, $i \geq 0$ be same as stated in Theorem 3.1. Then the sequences $\{u^{(i)}\}$ and $\{v^{(i)}\}, i \geq 0$, converge uniformly to u^* and v^*, respectively with $u^{(*)} \leq v^{(*)}$.*

Proof. Since $u^{(i)}$ is bounded above by $v^{(0)}$ and monotonically increasing sequence, it converges to say u^*. Similarly, the sequence $w^{(i)}$ is bounded below by $u^{(0)}$ and monotonically decreasing, and it converges to say v^*. The sequences $u^{(i)}$ and $v^{(i)}$ are sequences of continuous functions defined on the compact $\mathscr{I}$, therefore by Dini's theorem [22], the convergence is uniform. Since by Theorem 3.1, $u^{(i)} \leq v^{(i)}, \;\; \forall \;\; i \geq 0$, we have

$$u^* = \lim_{i\to\infty} u^{(i)} \leq \lim_{i\to\infty} v^{(i)} = v^*.$$

□

Theorem 3.3. *If the boundary conditions in* (3.2) *and* (3.3) *are the same as in BVP* (1.1)*, i.e.,* $u^{(i)}(0) = v^{(i)}(0)$ *and* $u^{(i)}(1) = v^{(i)}(1), \quad i \geq 1$*, then* $u^* = v^* = w$*, where* z *is the local solution to BVP* (1.1)*.*

Proof. We shall prove that $u^* = v^*$ by showing that u^* and v^* are solution to (1.1) and by Lemma 2.10, we get $u^* = v^*$. It follows from (3.2) that

$$-{}^c\mathscr{D}^\varepsilon u^{(i)} + cu^{(i)} = cu^{(i-1)} + g(t, u^{(i-1)}). \tag{3.6}$$

Applying the operator j^β for $1 \leq \beta \leq 2$, (3.6) yields

$$-u^{(i)} + c_0^{(i)} + c_1^{(i)}t + cj^\beta u^{(i)} = cj^\beta u^{(i-1)} + j^\beta g(t, u^{(i-1)}).$$

Taking the limit and $u^{(i)} \to u^*$ and using the continuity of g, we have

$$-u^* + c_0^\infty + c_1^\infty t + cj^\beta u^* = cj^\beta u^* + j^\beta g(t, u^*), \tag{3.7}$$

where $c_0^\infty = \lim_{i\to\infty} u^{(i)}(0) = a$ and $c_1^\infty = \lim_{i\to\infty} (u^i)^{'}(0) = a$. Applying ${}^c\mathscr{D}^\varepsilon$ to (3.7), using (2.1), and denoting ${}^c\mathscr{D}^\varepsilon t^i = 0$, for $i = 0, 1$, (3.7) reduces to

$$-{}^c\mathscr{D}^\varepsilon u^* + cu^* = cu^* + g(t, u^*),$$

or ${}^c\mathscr{D}^\varepsilon u^* + g(t, u^*) = 0$ which means that u^* is the solution of the problem (1.1). Since $u^{(i)}(0) = v_i^{'}(0)$ and $v^{(i)}(1) = \delta v_i^{'}(1), \; i \geq 1, u^*(0) = \gamma u_i^{'}(0)$ and $u^*(1) = \delta u_i^{'}(1)$, and it shows that u^* is a solution of problem (1.1). The same result applied to $v^{(i)}$ shows that v^* is a solution of (1.1). Conclusion is that $u^* = v^* = w$, the uniqueness of the solution of the problem. □

4. Illustrative examples

In this section, we present the following numerical examples to demonstrate our existence results.

Example 4.1. Consider the following fractional order boundary values problem

$$\begin{cases} {}^c\mathscr{D}^{1.75} = (u^3 - 5); \; t \in \mathscr{I}, \\ u(0) = 0.5u^{'}(0), \; u(1) = 0.5u^{'}(1). \end{cases} \tag{4.1}$$

From the above system (4.1), one can verify that $\theta(t, u) = -u^3 + 5$ and let lower and upper solution be $\alpha^{(0)}(t) = 0, \quad \beta^{(0)}(t) = 1$. Then $\theta(t, u)$ is decreasing as

$$-3 \leq \frac{\partial\theta}{\partial u} = -3u^2 < 0, \quad \forall \; u \in [0, 1], \text{ with } d = 3.$$

These extremal solutions can be computed from taking limit of monotone iterative sequences which can be developed .

Example 4.2. Consider the following fractional order boundary values problem

$$\begin{cases} {}^c\mathscr{D}^{1.5}u = u\exp(u) - 6; \; t \in \mathscr{I}, \\ u(0) = 0.333u^{'}(0), \; u(1) = 0.333u^{'}(1). \end{cases} \tag{4.2}$$

From the above system (4.2), we see that

$$\theta(t, u) \quad = \quad -u\exp(u) + 6$$

and let $\alpha^{(0)}(t) = 0, \quad \beta^{(0)}(t) = 1$ be lower and upper solution respectively, then $\theta(t, u)$ is decreasing with

$$-3e^2 \leq \frac{\partial \theta}{\partial u} \quad = \quad \exp(u)(-u+1) < 0, \quad \forall \ u \in \mathscr{I}.$$

Thus the boundary value problem (4.2) has an extremal solution.

5. Conclusion

In this paper, we apply a new type monotone iterative technique to find iterative solutions for boundary value problem (BVPs) of nonlinear fractional order differential equations (NFODEs). With the help of the aforesaid technique, we establish two sequences of upper and lower solutions for the considered BVP. Further the procedure is testified by providing suitable examples.

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