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Matrix inverse eigenvalue problem for stabilization of fractional descriptor discrete-time linear systems by forward and propositional output feedback

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Abstract In this paper, stabilization of unstable fractional descriptor discrete-time linear system via forward and propositional output feedback is done to obtain satisfactory responses. To gain forward and propositional output feedback matrices, two standard linear systems need to exist. Assigning nonzero arbitrary eigenvalues to the first standard system and inverted the desired eigenvalues for standard descriptor system to the second one, desired eigenvalues are assigned by matrix inverse eigenvalue problem. Numerical examples are also presented to illustrate our method.

Keywords. Fractional descriptor, Matrix inverse eigenvalue problem, Discrete-time, Output feedback, Moore-Penrose inverse.

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1. Introduction

Non-integer derivatives (or fractional derivatives) have become nowadays a precious tool, currently used in the study of the behaviour of real systems and provides an excellent instrument for the description of memory and hereditary properties of various materials and processes such as viscoelastic systems, chaotic synchronization, electromagnetic systems, electrical circuits theory, fractances, mechatronics systems, signal processing, chemical mixing, and Diabetes control [1, 3, 7, 9, 17, 18, 21, 22].

Fractional descriptor systems describe a more complete class of dynamical models than the fractional state-space systems, which are not only theoretical interest but also have a great importance in practice like using Kirchhoff's laws for the electrical circuits.

A modification of the finite integration method with the radial basis function method is combined to solve a time-fractional convection diffusion equation with variable coefficients [2]. Also we may find several methods for stability of standard,

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fractional, and descriptor systems. Some of them were derived by the use of Drazin inverse, Shuffle algorithm, and dynamic compensators. However, we do not deal with some initial conditions like having full row rank matrices in every performed algorithm and finding index of Shuffle and Drazin [6, 10] and we may not devote our method for only positive fractional systems. Also just assigning distinct eigenvalues is considered in [23] by dynamic compensators. Stability and stabilization of descriptor systems by state feedback are investigated in [15, 16]. However, stabilization of descriptor and fractional systems by output feedback controller has attracted considerable attention because it is usually not possible or practical to sense all the states and feed them back.

In this article, a method based on matrix inverse eigenvalue problem for stabilization of fractional descriptor discrete-time linear systems via forward and propositional output feedback is investigated. Definition of fractional order is given to convert the fractional descriptor systems to standard descriptor systems but with unlimited delay in state. Having decreasing sequence of coefficients of delay and defining a new state vector may help us obtain standard descriptor systems without any delays. Using forward output feedback is not suitable for standard descriptor linear systems, because the open-loop matrix in this system is not full rank. To gain forward and propositional output feedback matrices, two standard linear systems need to exist. Finding closed-loop matrix is the aim of matrix inverse eigenvalue problem such that improve the dynamic response of linear systems by output feedback. Moore-Penrose inverse is also useful to calculate output feedback matrices. Assigning nonzero arbitrary eigenvalues to the first standard system and inverted the desired eigenvalues for standard descriptor system to the second one, desired eigenvalues are assigned by matrix inverse eigenvalue problem.

Practical stability of the fractional descriptor discrete-time linear systems for not necessarily positive by matrix inverse eigenvalue problem and output feedback has not been considered yet.

This paper is organized as follows. In next section, some definitions of fractional order are recalled and the fractional descriptor discrete-time linear systems are introduced. Stabilization of these systems, converting to standard descriptor systems, and comparing forward output feedback and forward and propositional output feedback are given in section 3. In section 4, the matrix inverse eigenvalue problem and obtaining propositional and forward output feedbacks will be displayed. In section 5, numerical examples are proposed for more intuitive results. Conclusions are also given in final section.

The following notation will be used: $\mathbb{R}$ – the set of real numbers, $\mathbb{C}$ – the set of complex numbers, $\mathbb{R}^{n\times m}$ – the set of $n \times m$ real matrices, $\mathbb{R}^m = \mathbb{R}^{m\times 1}$, and A^t – the transposed matrix of A.

2. Preliminaries and definitions

2.1. **Fractional order derivatives.** Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator $D^{\alpha}_{a,t}$, where a and t are the bounds of the operation and $\alpha \in \mathbb{R}$. The general integro-differential operator is defined as

$$D_{a,t}^{\alpha} = \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}} & : \alpha > 0, \\ 1 & : \alpha = 0, \\ \int_a^t (d\tau)^{\alpha} & : \alpha < 0. \end{cases} \tag{2.1}$$

The three most frequently used definitions for the continues fractional derivative are: the Grunwald-Letnikov (GL), the Riemann-Liouville (RL), and the Caputo definitions. This consideration is based on the fact that for a wide class of functions, the three best-known definitions GL, RL, and Caputo are equivalent under some conditions [13, 19].

Definition 2.1. The Grunwald-Letnikov fractional derivative with fractional order $\alpha \in \mathbb{R}^+$ is defined by

$${}_{GL}D_{a,t}^{\alpha} x(t) = \lim_{h \to 0} h^{-\alpha} \sum_{i=0}^{[\frac{t-a}{h}]} (-1)^i \binom{\alpha}{i} x(t - ih), \tag{2.2}$$

where [.] means the integer part and

$$\binom{\alpha}{i} = \begin{cases} 1 & for \quad i = 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(i+1)\Gamma(\alpha-i+1)} & for \quad i = 1, 2, ... \end{cases} \tag{2.3}$$

Definition 2.2. If A is a $m \times n$ matrix, then there exists a unique Moore-Penrose inverse matrix $A^{\dagger}_{n\times m}$ that satisfies the four following conditions [20]:

$$\begin{cases} AA^{\dagger}A = A \\ A^{\dagger}AA^{\dagger} = A^{\dagger} \\ (AA^{\dagger})^H = AA^{\dagger} \\ (A^{\dagger}A)^H = A^{\dagger}A \end{cases} \tag{2.4}$$

where A^H is the conjugate transposed matrix of A.

Theorem 2.3. *For $n \in \mathbb{N}$ and $0 < \alpha < 1$ we have* [13]

$$D^{n+\alpha} x(t) = D^n D^{\alpha} x(t). \tag{2.5}$$

We can easily assume $0 < \alpha < 1$ by this theorem.

Definition 2.4. The discrete fractional derivative of the order $\alpha \in \mathbb{R}^+$ with zero initial point is defined by [5]

$$\Delta^{\alpha} x(t_k) = \Delta^{\alpha} x_k = \sum_{i=0}^{k} (-1)^i \binom{\alpha}{i} x_{k-i}, \quad 0 < \alpha < 1. \tag{2.6}$$

2.2. **Fractional descriptor discrete-time linear systems.** Consider the fractional descriptor discrete-time linear system described by

$$\begin{cases} E\Delta^{\alpha} x_{k+1} = Ax_k + Bu_k \\ \quad y_k = Cx_k \end{cases} \tag{2.7}$$

where α is fractional order difference of the state vector and $0 < \alpha < 1$, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and $y_k \in \mathbb{R}^r$ are state, input, and output vectors, the matrices $E, A \in \mathbb{R}^{n\times n}$, $B \in$

$\mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{r\times n}$ are known constant matrices which $1 \le m \le n$, $rank(B) = m$, and $rank(C) = r$, $rank(E) < n$, and also $k \in \mathbb{Z}^+ = \{0, 1, 2, \cdots\}$.

Definition 2.5. The fractional descriptor system (2.7) is called asymptotically stable if and only if $\lim_{k\to\infty} x_k = 0$ for any $x_0 \in \mathbb{R}^n$ and $u_k = 0$.

Using the definition 2.4 we may write the equation (2.7) in the form

$$\begin{cases} Ex_{k+1} = A_\alpha x_k + \sum_{i=1}^{k} c_i E x_{k-i} + Bu_k \\ \quad y_k = Cx_k \end{cases} \tag{2.8}$$

which

$$c_i = c_i(\alpha) = (-1)^i \binom{\alpha}{i+1}, \quad i = 1, 2, \cdots, k \tag{2.9}$$

and $A_\alpha = A + \alpha E$. Also $\binom{\alpha}{i+1}$ is defined by (2.3).
Note that the equation (2.8) describes a linear descriptor discrete-time system with unlimited delay in the state. To make the control of this system possible, we should change it to standard descriptor linear system. Although the converted standard descriptor linear systems may have large matrices, but stability of them is proved [4].

3. Stability of fractional descriptor discrete-time linear systems

The coefficients c_i in (2.9) strongly decrease for increasing i when $0 < \alpha < 1$. Assuming $c_i = 0$ for $i > L$ the system (2.8) is converted to the linear descriptor system with L delays [5]

$$\begin{cases} Ex_{k+1} = A_\alpha x_k + \sum_{i=1}^{L} c_i E x_{k-i} + Bu_k \\ \quad y_k = Cx_k \end{cases} \tag{3.1}$$

We may convert the time delay system (3.1) to the standard descriptor system

$$\begin{cases} \bar{E}X_{k+1} = \bar{A}X_k + \bar{B}U_k \\ \quad Y_k = \bar{C}X_k \end{cases} \tag{3.2}$$

where

$$X_k = \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \\ x_{k-L} \end{bmatrix} \in \mathbb{R}^{\bar{n}}, \quad Y_k = \begin{bmatrix} y_k \\ y_{k-1} \\ y_{k-2} \\ \vdots \\ y_{k-L} \end{bmatrix} \in \mathbb{R}^{r}, \tag{3.3}$$

$U_k = u_k \in \mathbb{R}^m$ are state, output, and input vectors, $\bar{n} = n(L+1)$, and

$$\bar{A} = \begin{bmatrix} A_\alpha & c_1E & \cdots & c_{L-1}E & c_LE \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \in \mathbb{R}^{\bar{n}\times\bar{n}},$$

$$\bar{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{\bar{n}\times m}, \bar{E} = \begin{bmatrix} E & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix} \in \mathbb{R}^{\bar{n}\times\bar{n}},$$

$$\bar{C} = \begin{bmatrix} C & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{r\times\bar{n}}. \tag{3.4}$$

Definition 3.1. The fractional descriptor system (2.7) is called practically stable if and only if the time delay descriptor system (3.1) or equivalently the system (3.2) is asymptotically stable [5].

3.1. **Eigenvalue assignment with forward output feedback law.** Consider system (3.2) by forward output feedback law

$$\begin{aligned} U_k &= K'_f Y_{k+1} \\ &= K'_f \bar{C} X_{k+1}. \end{aligned} \tag{3.5}$$

The aim is to design the forward output feedback (3.5) which produces a closed-loop system of (3.2) with the satisfactory response by assigning desirable eigenvalues $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$, where $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$, and are self-conjugate complex numbers for $i = 1, 2, \cdots, \bar{n}$.
To establish the proposed results, consider the following assumptions

$$I) rank[\bar{E}|\bar{B}] = \bar{n}, \quad II) rank[\bar{A}] = \bar{n}, \quad III) rank[\bar{B}] = m. \tag{3.6}$$

If assumption (I) holds, then there exists K'_f such that [4]

$$rank[\bar{E} - \bar{B}K'_f\bar{C}] = \bar{n}. \tag{3.7}$$

Substituting feedback (3.5) into the equation (3.2) one can write

$$\bar{E}X_{k+1} = \bar{A}X_k + \bar{B}K'_f\bar{C}X_{k+1},$$

therefore

$$X_{k+1} = (\bar{E} - \bar{B}K'_f\bar{C})^{-1}\bar{A}X_k \tag{3.8}$$

is a standard linear system which is well-defined by (3.7).

Theorem 3.2. *The standard discrete-time linear system (3.8) is asymptotically stable if and only if eigenvalues of $(\bar{E} + \bar{B}K'_f\bar{C})^{-1}\bar{A}$ lie in the unit disk* [8].

Lemma 3.3. *Consider a matrix $M \in \mathbb{R}^{n\times n}$ with $rank(M) = n$ and the eigenvalues equal to $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then, the eigenvalues of M^{-1} are $\lambda_1^{-1}, \lambda_2^{-1}, \cdots,$ λ_n^{-1}* [11, 12].

Theorem 3.4. *Define the matrices N, M as*

$$N = \bar{A}^{-1}\bar{E}, \quad M = -\bar{A}^{-1}\bar{B}, \tag{3.9}$$

such that the pair of (M,N) *be controllable. Also, let* K'_f *be output feedback matrix, such that* $\{\lambda_1^{-1},\lambda_2^{-1},\cdots,\lambda_{\bar{n}}^{-1}\}$ *is the set of eigenvalues of the closed-loop system*

$$\begin{cases} z_{k+1} = Nz_k + Mw_k \\ g_k = \bar{C}z_k \end{cases} \tag{3.10}$$

via output feedback law

$$w_k = K'_f g_k, \tag{3.11}$$

where $\lambda_i \in \mathbb{C}$ *and* $\lambda_i \neq 0$, $i = 1,2,\cdots,\bar{n}$, *are arbitrarily assigned, then for this gained* K'_f, *the desired spectrum* $\Lambda = \{\lambda_1,\lambda_2,\cdots,\lambda_{\bar{n}}\}$ *is the eigenvalues of the controlled system (3.2) with forward feedback (3.5) and the condition (3.7) holds.*

Proof. Considering that (M,N) is controlled, then one can find the output feedback matrix K'_f such that the controlled system (3.10) via output feedback (3.11) given by

$$z_{k+1} = (N + MK'_f\bar{C})z_k \tag{3.12}$$

has eigenvalues equal to $\lambda_1^{-1},\lambda_2^{-1},\cdots,\lambda_{\bar{n}}^{-1}$. Now by (3.9) note that:

$$N + MK'_f\bar{C} = \bar{A}^{-1}(\bar{E} - \bar{B}K'_f\bar{C}) \tag{3.13}$$

so

$$(N + MK'_f\bar{C})^{-1} = (\bar{E} - \bar{B}K'_f\bar{C})^{-1}\bar{A}. \tag{3.14}$$

The closed-loop matrices of systems (3.10) and (3.2) via output feedback laws (3.5) and (3.11) are in the inverse of each other by (3.8), (3.12), (3.13), and (3.14). Therefore (3.7) holds and the set of eigenvalue of closed-loop system (3.2) with feedback law (3.5) is equal to $\Lambda = \{\lambda_1,\lambda_2,\cdots,\lambda_{\bar{n}}\}$ by Lemma 3.3. □

Remark 3.5. By the equation (3.4), the matrices $\bar{E}$ and $\bar{A}$ in the system (3.2) are singular because $rank(E) < n$ is the necessary condition in the fractional descriptor discrete-time linear system (2.7) and the matrix including last n columns and first n rows of $\bar{A}$ is not full rank. So the method in subsection 3.1 cannot help us stabilize the system (2.7).

The method based on using forward output feedback when $\bar{A}$ is singular, i.e. the condition (II) in (3.6) is not satisfied, does not work. The problem is removed in next subsection.

3.2. **Eigenvalue assignment with forward and propositional output feedback law.** When we use the forward and propositional output feedback instead of the forward output feedback, we do not need the condition of being full rank of matrix $\bar{A}$ in the system (3.2). It is excellent for using this feedback.

Consider system (3.2) by forward and propositional output feedback law

$$\begin{aligned} U_k &= K_f Y_{k+1} + K_p Y_k \\ &= K_f\bar{C}X_{k+1} + K_p\bar{C}X_k. \end{aligned} \tag{3.15}$$

The aim is to design the forward and propositional output feedback (3.15) which produces a closed-loop system of (3.2) with the satisfactory response by assigning desirable eigenvalues $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$, where $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$, and are self-conjugate complex numbers for $i = 1, 2, \cdots, \bar{n}$.
To establish the proposed results, consider the following assumptions

$$I) rank[\bar{E}|\bar{B}] = \bar{n}, \quad II) rank[\bar{B}] = m.$$

If assumption (I) holds, then there exists K_f such that [4]

$$rank[\bar{E} - \bar{B}K_f\bar{C}] = \bar{n}. \tag{3.16}$$

Substituting feedback (3.15) into the equation (3.2) one can write

$$\bar{E}X_{k+1} = \bar{A}X_k + \bar{B}K_f\bar{C}X_{k+1} + \bar{B}K_p\bar{C}X_k,$$

therefore

$$X_{k+1} = (\bar{E} - \bar{B}K_f\bar{C})^{-1}(\bar{A} + \bar{B}K_p\bar{C})X_k \tag{3.17}$$

is a standard linear system which is well-defined by (3.16).

Theorem 3.6. *The standard descriptor discrete-time linear system (3.17) is asymptotically stable if and only if eigenvalues of $(\bar{E} - \bar{B}K_f\bar{C})^{-1}(\bar{A} + \bar{B}K_p\bar{C})$ lie in the unit disk* [8].

Obtaining propositional and forward output feedbacks K_p and K_f, first, the propositional feedback matrix K_p is obtained by assigning non-zero arbitrary eigenvalues to the closed-loop of system

$$\begin{cases} q_{k+1} = \bar{A}q_k + \bar{B}v_k \\ p_k = \bar{C}q_k \end{cases} \tag{3.18}$$

via output feedback law

$$v_k = K_p p_k, \tag{3.19}$$

then we obtain the forward output feedback matrix K_f by assigning $\{\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_{\bar{n}}^{-1}\}$ to the system (3.21), where $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$, are self-conjugate complex numbers for $i = 1, 2, \cdots, \bar{n}$, and $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$ is the set of desired eigenvalues for the standard descriptor system (3.2) via output feedback (3.15).

Theorem 3.7. *Define the matrices N, M as*

$$N = (\bar{A} + \bar{B}K_p\bar{C})^{-1}\bar{E}, \quad M = -(\bar{A} + \bar{B}K_p\bar{C})^{-1}\bar{B}, \tag{3.20}$$

such that the pair of (M, N) be controllable. Also, let K_f be the output feedback matrix, such that $\{\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_{\bar{n}}^{-1}\}$ is the set of eigenvalues of the closed-loop system

$$\begin{cases} z_{k+1} = Nz_k + Mw_k \\ g_k = \bar{C}z_k \end{cases} \tag{3.21}$$

via output feedback

$$w_k = K_f g_k, \tag{3.22}$$

where $\lambda_i \in \mathbb{C}$ and $\lambda_i \neq 0$, $i = 1, 2, \cdots, \bar{n}$, are arbitrarily assigned. Then for this gained K_f, the desired spectrum $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$ includes the eigenvalues of the *controlled system (3.2) with forward and propositional feedback (3.15) and also, the condition (3.16) holds.*

Proof. Considering that (M, N) is controlled, then one can find an output feedback matrix K_f such that the controlled system (3.21) via output feedback (3.22) given by

$$z_{k+1} = (N + MK_f\bar{C})z_k \tag{3.23}$$

has eigenvalues equal to $\{\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_{\bar{n}}^{-1}\}$. Now by (3.20) note that:

$$N + MK_f\bar{C} = (\bar{A} + \bar{B}K_p\bar{C})^{-1}(\bar{E} - \bar{B}K_f\bar{C}) \tag{3.24}$$

so

$$(N + MK_f\bar{C})^{-1} = (\bar{E} - \bar{B}K_f\bar{C})^{-1}(\bar{A} + \bar{B}K_p\bar{C}). \tag{3.25}$$

The closed-loop matrices of systems (3.21) and (3.2) via output feedback laws (3.15) and (3.22) are in inverse of each other by (3.8), (3.23), (3.24), and (3.25). Therefore (3.16) holds and the set of eigenvalues of the closed-loop system (3.2) with feedback law (3.15) is equal to $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$ by Lemma 3.3. $\square$

4. The matrix inverse eigenvalue problem

In this section, we describe a method for finding output feedback K for system (3.2) via control law (3.5).

Definition 4.1. The matrix inverse eigenvalue problem is that given four linearly independent sets of real n-vectors

$$\{z_1, z_2, ..., z_p\}, \{z_{p+1}, z_{p+2}, ..., z_{p+q}\},$$

$$\{w_1, w_2, ..., w_p\}, \{w_{p+1}, w_{p+2}, ..., w_{p+q}\}$$

with $p + q \leq n$ and a set of complex numbers $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ to find a real matrix $\Omega_{n\times n}$ such that

$$\Omega z_i = w_i, \quad i = 1, 2, ..., p,$$

$$\Omega z_j = w_j, \quad j = p+1, p+2, ..., p+q,$$

and the spectrum of Ω be $\{\lambda_1, \lambda_2, ..., \lambda_n\}$, where $\lambda_i, i = 1, \cdots, n$ are closed under complex conjugation.

Let $X_r = [z_1, z_2, ..., z_p]$, $X_l = [z_{p+1}, z_{p+2}, ..., z_{p+q}]$, $Y_r = [w_1, w_2, ..., w_p]$, and $Y_l = [w_{p+1}, w_{p+2}, ..., w_{p+q}]$. If the matrix Ω of the problem exists, the following consistency condition must be satisfied [14]

$$X_l^t Y_r = Y_l^t X_r. \tag{4.1}$$

Theorem 4.2. *If the matrix inverse eigenvalue problem satisfies the consistency condition (4.1), then the necessary and sufficient condition for the existence of the matrix Ω is that there are vectors $u_i \in \int_u^i$ and $v_i \in \int_v^i$, $i = 1, 2, ..., n$ such that*

$$u_i v_j = \delta_{ij}, \quad i, j = 1, 2, ..., n, \tag{4.2}$$

where $\int_u^i$ *and* $\int_v^i$ *are the null spaces of* $(\lambda_i X_l^t - Y_l^t)$ *and* $(\lambda_i X_r^t - Y_r^t)$ *respectively and* δ_{ij} *is the Kronecker delta function. If such* u_i *exists, then* Ω *can be obtained using the equation*

$$\Omega = T diag(\lambda_1, \lambda_2, ..., \lambda_n) T^{-1}, \tag{4.3}$$

which $T = \begin{bmatrix} u_1 & u_2 & ... & u_n \end{bmatrix}$ [14].

Let the base vectors of $\int_u^i$ and $\int_v^i$ be the form of matrices S_u^i and S_v^i respectively, then vectors u_i and v_i, $i = 1, 2, ..., n$ can be expressed as

$$u_i = S_u^i z_i, \quad v_i = S_v^i w_i.$$

Thus from equation (4.2) we have

$$z_i^t (S_u^i)^t S_v^j w_j = \delta_{ij}, \quad i, j = 1, 2, ..., n, \tag{4.4}$$

with n^2 nonlinear equations and $2n^2 - n(p+q)$ unknowns. The number of unknowns may be greater than n^2 because $p + q \leq n$. Thus we can solve this system using an iterative method by converting to n^2 linear equations as the following algorithm.

Step 1. Form

$$S_{i,j} = S_u^i S_v^j.$$

Step 2. Consider some initial values randomly to all $w_i, i = 1, \cdots, n$. So converted system with n linear equations and single unknown from system (4.4) is obtained as

$$z_i^t [S_{i,1} w_1, S_{i,2} w_2, \cdots, S_{i,n} w_n] = e_i^t, \quad i = 1, \cdots, n$$

or

$$z_i^t \Phi_i = e_i^t, \quad i = 1, \cdots, n,$$

which e_i^t is ith column of identity matrix I_n. The best approximate answer by Moore-Penrose inverse is given by

$$z_i^t = e_i^t \Phi_i^\dagger, \quad i = 1, \cdots, n.$$

Step 3. Substitute gained values of $z_i, i = 1, \cdots, n$ into converted system with n linear equations and single unknown from system (4.4) as

$$[z_1^t S_{1,j}, z_2^t S_{2,j}, \cdots, z_n^t S_{n,j}] w_j = e_j, \quad j = 1, \cdots, n$$

or

$$\Psi_j w_j = e_j, \quad j = 1, \cdots, n.$$

The best approximate answer by Moore-Penrose inverse is given by

$$w_j = \Psi_j^\dagger e_j, \quad j = 1, \cdots, n.$$

Step 4. Repeat steps 2 and 3 until error

$$[\sum_i \sum_j (z_i^t S_{i,j} w_j - \delta_{i,j})^2]^{\frac{1}{2}}$$

be less than desired error.

4.1. **Obtaining forward and propositional output feedback matrices by matrix inverse eigenvalue problem.** In this subsection, more details of matrix inverse eigenvalue problem to gain the closed-loop, propositional output feedback, and forward output feedback matrices are given.

To obtain the propositional output feedback matrix, consider $\Omega_{\bar{n}\times\bar{n}} = \bar{A} + \bar{B}K_p\bar{C}$ as the closed-loop matrix of system (3.18) via output feedback law (3.19) and also U_1 and V_1 as the matrices formed by the base vectors of the null spaces of $\bar{B}^t$ and $\bar{C}$ respectively. So we have

$$\Omega V_1 = (\bar{A} + \bar{B}K_p\bar{C})V_1 = \bar{A}V_1,$$
$$U_1^t\Omega = U_1^t(\bar{A} + \bar{B}K_p\bar{C}) = U_1^t\bar{A}.$$

Let

$$X_l = U_1, \quad X_r = V_1, \quad Y_l = \bar{A}^t U_1, \quad Y_r = \bar{A}V_1. \tag{4.5}$$

By Theorem 4.2 we can find Ω as (4.3). If such Ω exists, the propositional feedback matrix K_p can be computed through the equation

$$K_p = \bar{B}^\dagger(\Omega - \bar{A})\bar{C}^\dagger, \tag{4.6}$$

where $\bar{B}^\dagger$ and $\bar{C}^\dagger$ are the Moore-Penrose generalized inverse of $\bar{B}$ and $\bar{C}$ respectively. This method is generally solved when $\bar{B}$ and $\bar{C}$ are full rank and $rank(\bar{B}) + rank(\bar{C}) \geq rank(\bar{A})$, so we can expect a solution with probability 1 for a given set $\{\lambda_1, \lambda_2, \cdots, \lambda_{\bar{n}}\}$, where $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$, and are arbitrary and self-conjugate complex numbers for $i = 1, 2, \cdots, \bar{n}$.

Remark that we can use repeated eigenvalues, but in the method in [23] using dynamic compensators just distinct eigenvalues are applicable.

It is enough to substitute N, M, and K_f instead of $\bar{A}, \bar{B}$, and K_p to gain the forward output feedback matrix K_f by having the closed-loop matrix $\Omega = N + MK_f\bar{C}$ as the closed-loop matrix of system (3.21) via output feedback law (3.22). It is remarkable that we should consider the inverse of desired eigenvalues for the standard descriptor linear system (3.2) via output feedback law (3.15) to the system (3.21) via output feedback law (3.22).

5. Numerical examples

In this section, we present two examples to show the simplicity of our method.

Example 5.1. Consider the system (2.7) with $\alpha = 0.3, L = 2$, and following matrices:

$$E = \begin{bmatrix} 3 & 1 & -4 \\ 2 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 1 & 0 \\ 4 & -1 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix}.$$

Case (a). Consider this example by the method based on subsection 3.1. $rank(\bar{A}) =$

8 in spite of A is full rank. Therefore the method is not applicable.

Case (b). Consider this example by the method based on subsection 3.2. The results for system (3.18) via output feedback law (3.19) by assigning eigenvalues $\{\pm 0.1, \pm 0.2, \pm 0.3, \pm 0.4, 0.5\}$ are as follow.

$$\bar{A} = \begin{bmatrix} 1.9 & 5.3 & 1.8 & 0.31 & 0.1 & -0.42 & -0.17 & -0.05 & 0.23 \\ -1.4 & 0.4 & 0.3 & 0.21 & -0.21 & 0.1 & -0.11 & 0.11 & -0.05 \\ 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{E} = \begin{bmatrix} 3 & 1 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_p = \begin{bmatrix} -6.64 & 0.53 & 4.2 \\ 29.61 & -6.87 & -18.72 \\ 15.57 & -3.43 & -9.48 \end{bmatrix}.$$

The eigenvalues of closed-loop matrix $\Omega = \bar{A} + \bar{B}K_p\bar{C}$ given by

$$\Omega = \begin{bmatrix} 1.04 & -0.03 & -0.08 & -0.32 & -0.03 & 0.13 & 0.03 & 0.01 & -0.03 \\ 0.08 & 0.99 & -0.24 & -0.02 & -0.39 & 0.27 & 0 & 0.05 & -0.05 \\ 0.16 & 0.01 & 0.66 & -0.08 & -0.05 & -0.03 & 0.01 & 0.01 & -0.01 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

are desired eigenvalues as $\{\pm 0.1, \pm 0.2, \pm 0.3, \pm 0.4, 0.5\}$.

The results for system (3.21) via output feedback law (3.22) by assigning inverse of eigenvalues $\{\pm 0.1, \pm 0.2, \pm 0.3, \pm 0.4, -0.5\}$ are as follow.

$$N = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -117.55 & 60.4 & 19.8 & -5.72 & 34.61 & -105.54 & 6.94 & -6.77 & 17.38 \\ -304.34 & -19.05 & 292.51 & 46.55 & 24.36 & -176.04 & -5.75 & -0.6 & 28.43 \\ -329.21 & 16.83 & 264.91 & 46.36 & 40.98 & -174.56 & -6.03 & -7.46 & 32.24 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -127.75 & 165.1 & -472.25 \\ -160.12 & 191.02 & -623.01 \\ -154.9 & 202.36 & -637.72 \end{bmatrix},$$

$$K_f = \begin{bmatrix} 98.9 & -71.74 & -47.46 \\ 37.87 & -21.07 & -24.84 \\ -12.11 & 11.24 & 3.33 \end{bmatrix}.$$

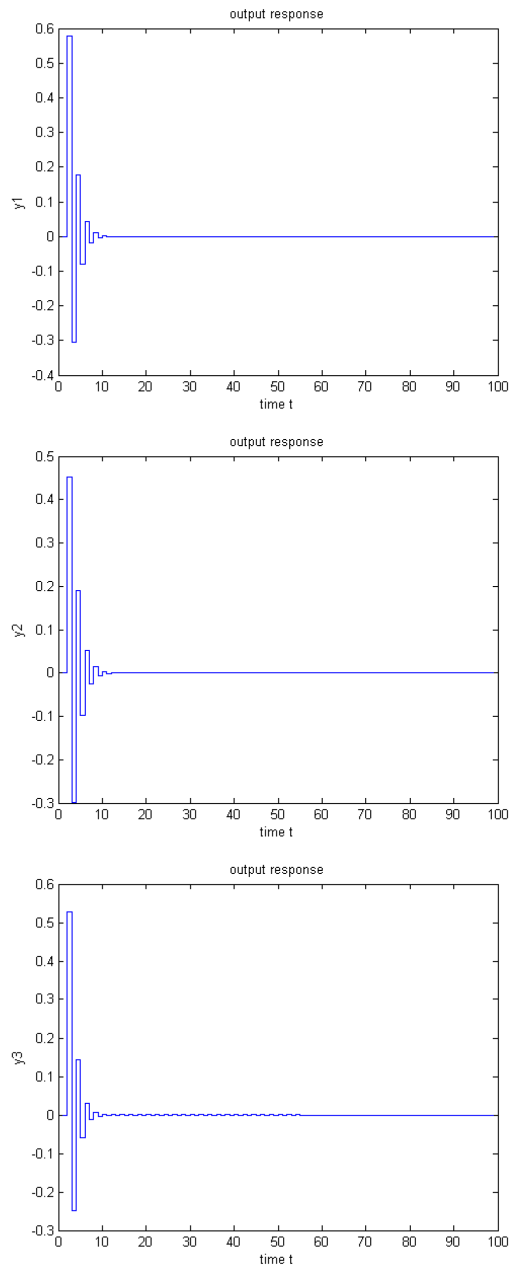
The eigenvalues of closed-loop matrix $(\bar{E} - \bar{B}K_f\bar{C})^{-1}(\bar{A} + \bar{B}K_p\bar{C})$ of the system (3.2) are desired eigenvalues as $\{\pm 0.1, \pm 0.2, \pm 0.3, \pm 0.4, -0.5\}$.

Figure 1 show simulation results.

Example 5.2. Consider the system (2.7) with $\alpha = 0.7, L = 2$, and following matrices:

$$E = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -3 & 4 \\ 2 & -3 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & -1 & 1 \\ 2 & -4 & 1 \end{bmatrix},$$

FIGURE 1. The output vector $y_k(t)$ in example 5.1



$$B = \begin{bmatrix} 1 & -4 \\ 3 & 3 \\ -2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}.$$

Case (a). Consider this example by the method based on subsection 3.1. $rank(\bar{A}) = 8$ and $rank(A) = 2$. Therefore the method is not applicable.

Case (b). Consider this example by the method based on subsection 3.2. The results for system (3.18) via output feedback law (3.19) by assigning eigenvalues $\{0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5\}$ are as follow.

$$\bar{A} = \begin{bmatrix} 1.6 & 1.1 & 1.3 & -0.21 & 0.31 & -0.1 & 0.09 & -0.13 & 0.04 \\ 1.7 & -3.1 & 3.8 & 0.1 & -0.31 & 0.42 & -0.04 & 0.13 & 0.18 \\ 3.4 & -6.1 & 1.7 & 0.21 & -0.31 & 0.1 & -0.09 & 0.13 & -0.04 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{E} = \begin{bmatrix} -2 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 1 & -4 \\ 3 & 3 \\ -2 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} -2 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_p = \begin{bmatrix} -0.27 & -0.03 \\ 0.38 & 0.14 \end{bmatrix}.$$

The eigenvalues of closed-loop matrix $\Omega = \bar{A} + \bar{B}K_p\bar{C}$ given by

$$\Omega = \begin{bmatrix} 3.09 & -3.6 & 0.9 & 1.68 & -2.73 & 2.19 & -0.18 & 0.34 & -0.3 \\ 2.62 & -3.29 & 1.73 & 0.65 & -1.07 & 0.81 & -0.08 & 0.16 & -0.13 \\ 2.07 & -3.18 & 2.9 & -1.18 & 1.87 & -1.45 & 0.09 & -0.16 & 0.14 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

are as $\{0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5\}$.

The results for system (3.21) via output feedback law (3.22) by assigning inverse of eigenvalues $\{-0.1, -0.15, -0.2, -0.25, -0.3, -0.35, -0.4, -0.45, -0.5\}$ are as follow.

$$N = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 513.04 & -942.67 & 660.45 & -1124.52 & 1551.91 & -1078.97 & 119.59 & -179.27 & 137.38 \\ 597.23 & -1230.24 & 1078.88 & -855.46 & 1176.37 & -842.73 & 86.18 & -130.43 & 108 \\ 386.07 & -859.16 & 846.48 & -302.89 & 415.34 & -322.46 & 33.32 & -52.08 & 49.42 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -130.64 & -307.32 \\ -317.29 & -924.34 \\ -436.96 & -888.53 \end{bmatrix},$$

$$K_f = \begin{bmatrix} 1.23 & -0.09 \\ -1.09 & -0.03 \end{bmatrix}.$$

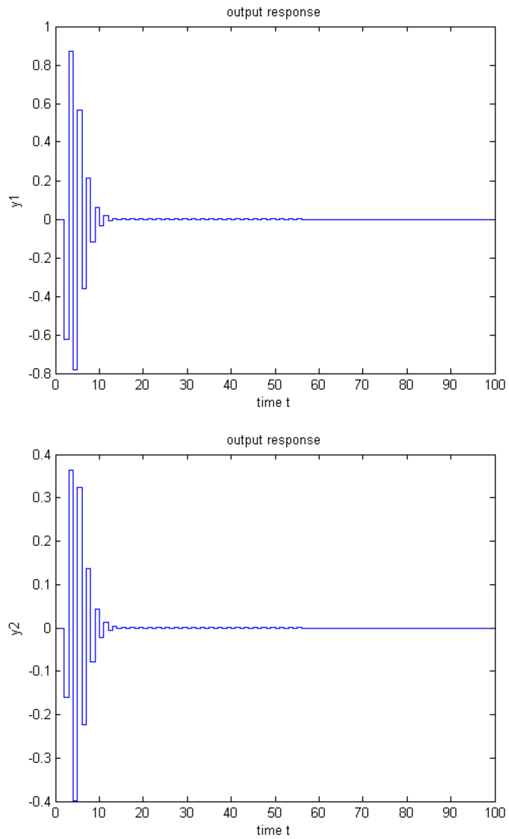
The eigenvalues of closed-loop matrix $(\bar{E} - \bar{B}K_f\bar{C})^{-1}(\bar{A} + \bar{B}K_p\bar{C})$ of the system (3.2) are desired eigenvalues as $\{-0.1, -0.15, -0.2, -0.25, -0.3, -0.35, -0.4, -0.45, -0.5\}$.

Figure 2 show simulation results.

6. Conclusions

A method based on matrix inverse eigenvalue problem for stabilization of the fractional descriptor discrete-time linear system in form of (2.7) has been considered. First, by the definition of fractional order (2.6) and a new state vector (3.3) displayed system has been converted to a standard discrete-time linear system (3.2). It is clear that working with the standard systems is much easier than the fractional mode. Second, the matrix inverse eigenvalue problem in section 3 has been used to

FIGURE 2. The output vector $y_k(t)$ in example 5.2



assign eigenvalues to the system (3.2) and obtain the output feedback K. Finally the state and input vectors in (3.2) has been obtained and illustrated by a numerical example and shown the input and state vectors converge to zero. The results presented in this article are also applicable in stabilization of descriptor, time-delayed, and two-dimensional (2D) systems. An extension of these considerations for fractional descriptor continuous-time linear systems is still an open problem.

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