Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 3, 2020, pp. 408-423
DOI:10.22034/cmde.2020.32623.1515

A new operational matrix of Müntz-Legendre polynomials and Petrov-Galerkin method for solving fractional Volterra-Fredholm integro-differential equations

Sedigheh Sabermahani
Department of Mathematics, Faculty of Mathematical Sciences,
Alzahra University, Tehran, Iran.
E-mail: s.saber@alzahra.ac.ir

Yadollah Ordokhani*
Department of Mathematics, Faculty of Mathematical Sciences,
Alzahra University, Tehran, Iran.
E-mail: ordokhani@alzahra.ac.ir

Abstract This manuscript is devoted to present an efficient numerical method for finding numerical solution of Volterra-Fredholm integro-differential equations of fractional-order. This technique is based on applying Müntz-Legendre polynomials and Petrov-Galerkin method. A new Riemann-Liouville operational matrix for Müntz-Legendre polynomials is proposed using Laplace transform. Employing this operational matrix and Petrov-Galerkin method, transforms the problem into a system of algebraic equations. Next, we solve this system by applying any iterative method. An estimation of the error is proposed. Moreover, some numerical examples are implemented in order to show the validity and accuracy of the suggested method.

Keywords. Müntz-Legendre polynomials, Petrov-Galerkin method, Laplace transform.
2010 Mathematics Subject Classification. 65R20, 65L60.

1. Introduction

The origin of fractional calculus is in the question whether the meaning of a integer order derivative can be valid for an inappropriate order derivative. Regard to letter of L'Hopital to Libeniz on 30 September 1695, this question was raised. In response to this question, Leibniz said that it's an apparent paradox that one day, the consequences will be beneficial [13].

Recently, fractional calculus is a concern for many researchers and mathematicians, since the fractional equations emerge in the modeling of many phenomena in physics, chemistry, engineering, and so on. Fractional order integro-differential equation is a type of fractional equations which has various applications in various sciences such as earthquake engineering, biomedical engineering, fluid mechanics, thermal systems, turbulence, fluid flow, mechanics, etc [2, 3, 10].

Received: 20 March 2019 ; Accepted: 4 May 2019.
* Corresponding author.

Moreover, due to the importance of these equations, several analytical and numerical methods are presented to solve this class of fractional equations, such as collocation method [4], CAS wavelet [17], Chebyshev wavelet [21], Bernoulli wavelets [7], hybrid functions [9], method based on the block backward differential formula [6] and so on.

On the other hand, a set of basic functions is the Mütz-Legendre polynomials, which is a set of generalized orthogonal polynomials. Badalyan introduced Mütz-Legendre polynomials in 1955 [1]. Also, scholars like McCarthy et. al (1993) [12], Stefánsson (2010) [19] have been studying this kind of polynomials. Recently, these polynomials have been used as the basis polynomials for numerical solution of various types of fractional problems. Here are some these research papers: Solving delay Fredholm integro-differential equations by Müntz-Legendre matrix method [20], Müntz-Legendre polynomials and numerical solution of gas solution in a fluid [5], Müntz-Legendre wavelet for finding numerical solution of the fractional pantograph differential equations [15], etc.

Employing the operational matrices of derivative or integral of orthogonal functions, transforms the dynamical system problems into a solution for a system of algebraic equations. This is the main advantage of using their. These operational matrices can be derive based on the specific orthogonal functions, uniquely [9].

Due to the presence of λ_k factor, Müntz-Legendre polynomials have the advantages of fractional functions in addition to the advantages of the orthogonal functions. The existence of parameter λ_k makes these functions suitable for solving the fractional equations. We employ these polynomials together Petrov-Galerkin method for solving one kind of fractional equations.

The structure of this study is as follows. Section 2 recalls some basic definitions which are needed for this manuscript. In Section 3, we derive Müntz-Legendre polynomials operational matrix of Riemann-Liouville integration using Laplace transform. In Section 4, we present our numerical method and an estimation of the error is given. Finally, in Section 5 the achieved validity and efficiency numerical results are verified by various examples.

2. Preliminaries

Definition 2.1. Assume that $g : [a,\ b] \to R,\ \nu \in R, \nu > 0$ and $n = \lceil \nu \rceil$, the Riemann-Liouville integral is defined as [16]

$$I^{\nu} g(x) = \begin{cases} \frac{1}{\Gamma(\nu)} x^{\nu-1} * g(x), & \nu > 0, \\ g(x), & \nu = 0, \end{cases} \tag{2.1}$$

in which $*$ denotes the convolution product. For this fractional integral, we have

$$(I^{\nu} D^{\nu} g)(x) = g(x) - \sum_{i=0}^{\lceil \nu \rceil - 1} \frac{x^i}{i!} g^{(i)}(0), \tag{2.2}$$

where D^{ν} is Caputo's derivative, which is defined in Ref. [16].

Definition 2.2. [16] Assume that $D^{k\gamma}g(x) \in C(0,1]$ for $k = 0, 1, \dots, n$. Then, we have:

$$g(x) = \sum_{k=0}^{n-1} \frac{x^{k\gamma}}{\Gamma(k\gamma+1)} D^{k\gamma} g(0^+) + \frac{x^{n\gamma}}{\Gamma(n\gamma+1)} D^{n\gamma} g(\xi),$$

with $0 < \xi \le x, \ \forall x \in (0,\ 1]$. In addition, one has:

$$|g(x) - \sum_{k=0}^{n-1} \frac{x^{k\gamma}}{\Gamma(k\gamma+1)} D^{k\gamma} g(0^+)| \le M_\gamma \frac{x^{n\gamma}}{\Gamma(n\gamma+1)}, \tag{2.3}$$

in which $M_\gamma \ge \sup_{\xi \in (0,\ 1]} |D^{n\gamma} g(\xi)|$.

2.1. **Müntz-Legendre polynomials.** The Müntz-Legendre polynomials are orthogonal with the weight function $\omega(x) = 1$ on the interval $(0,\ 1]$ which are defined in [12] as follows:

$$P_n(x) := P_n(x,\ \Lambda_n) = \sum_{k=0}^{n} c_{n,k} x^{\lambda_k}, \quad c_{n,k} = \frac{\prod_{j=0}^{n-1}(\lambda k - \lambda_j + 1)}{\prod_{j=0,\ j\ne k}(\lambda_k - \lambda_j)}, \tag{2.4}$$

The following orthogonality condition is satisfied for the polynomials

$$\langle P_n,\ P_m \rangle = \int_0^1 P_n(x) P_m(x) dx = \frac{\delta_{nm}}{2\lambda_n + 1},\ (n \ge m). \tag{2.5}$$

In this study, we suppose that $\lambda_k = k\gamma, \gamma > 0$.

2.2. **Function approximation.** Suppose that $H = L^2[0,1]$ is a Hilbert space, $\{P_0(x), P_1(x), \dots, P_N(x)\} \subset H$ is the set of Müntz-Legendre polynomials and $Y = span\{P_0(x), P_1(x), \dots, P_N(x)\}$. Let g be an arbitrary function in H. Since Y is a finite dimensional vector space, g has the best unique approximation such as $\tilde{g}$ out of Y, that is

$$\exists \tilde{g} \in Y, \quad s.t \quad \forall y \in Y, \quad \|g - \tilde{g}\| \le \|g - y\|. \tag{2.6}$$

Then

$$\forall y \in Y, \quad \langle g - \tilde{g},\ y \rangle = 0, \tag{2.7}$$

where, $\langle , \rangle$ denotes inner product.

On the other hand, $\tilde{g} \in Y$ then the unique coefficients $a_0,\ a_1, \dots, a_N$ exist such that

$$g(x) \simeq \tilde{g}(x) = \sum_{n=0}^{N} a_n P_n(x) = A^T \Phi(x), \tag{2.8}$$

where

$$A = [a_0, a_1, \dots, a_N]^T, \quad \Phi(x) = [P_0(x), P_1(x), \dots, P_N(x)]^T, \tag{2.9}$$

and T indicates transposition. Using Eq. (2.7), we get

$$\langle g(x) - A^T\Phi(x),\ P_n(x)\rangle = 0, \quad n = 0, 1, \dots, N. \tag{2.10}$$

Then the coefficient vector A can be obtained as follows:

$$A^T = G^T D^{-1}, \tag{2.11}$$

where

$$G = [g_0, g_1, \dots, g_N]^T, \quad g_n = \int_0^1 g(x)P_n(x)dx, \ \ n = 0, 1, \dots, N,$$

and

$$D = \int_0^1 \Phi^T(x)\Phi(x)dx = \begin{bmatrix} \frac{1}{2\lambda_0+1} & 0 & \cdots & 0 \\ 0 & \frac{1}{2\lambda_1+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2\lambda_N+1} \end{bmatrix}. \tag{2.12}$$

Remark 2.3. Assume that $D^{k\gamma}y \in C[0,1], (k = 0, 1, \dots, N)$, $Y = span\{P_0(x), P_1(x), \dots, P_N(x)\}$. Since, $\tilde{g}(x)$ is the best approximation of g out of Y, $\tilde{g}(x) \in Y$, therefore, the error bound of the numerical solution $\tilde{g}(x)$ utilizing Müntz-Legendre polynomials series can be achieved as:

$$\|g - \tilde{g}\|_2 \leq \frac{M_\gamma}{\Gamma((N+1)\gamma+1)\sqrt{((2N+2)\gamma+1)}}, \quad M_\gamma = sup_{x\in[0,1]}|D^{(N+1)\gamma}g(x)|. \tag{2.13}$$

$$\|g - \tilde{g}\|_\infty \leq \frac{M_\gamma}{\Gamma((N+1)\gamma+1)}, \quad M_\gamma = sup_{x\in[0,1]}|D^{(N+1)\gamma}g(x)|. \tag{2.14}$$

Proof. Consider

$$\widehat{g}(x) = \sum_{k=0}^{N} \frac{x^{k\gamma}}{\Gamma(k\gamma+1)} D^{k\gamma}g(0^+).$$

Considering Definition 2.2, we have

$$g(x) = \sum_{k=0}^{N} \frac{x^{k\gamma}}{\Gamma(k\gamma+1)} D^{k\gamma}g(0^+) + \frac{x^{(N+1)\gamma}}{\Gamma((N+1)\gamma+1)} D^{(N+1)\gamma}g(\xi), \quad \xi \in [0,1],$$

and

$$|g(x) - \widehat{g}(x)| \leq \frac{x^{(N+1)\gamma}}{\Gamma((N+1)\gamma+1)} \sup_{x\in[0,1]} |D^{(N+1)\gamma}g(x)|.$$

Using the above equation, we get

$$\begin{aligned}\|g-\tilde{g}\|_2^2 &\leq \|g-\widehat{g}\|_2^2=\int_0^1|g(x)-\widehat{g}(x)|^2dx\leq\int_0^1\frac{x^{(2N+2)\gamma}}{\Gamma((N+1)\gamma+1)^2}M_\gamma^2dx\\ &= \frac{M_\gamma^2}{\Gamma((N+1)\gamma+1)^2}\int_0^1x^{(2N+2)\gamma}dx=\frac{M_\gamma^2}{\Gamma((N+1)\gamma+1)^2((2N+2)\gamma+1)}.\end{aligned}$$

where $M_\gamma = sup_{x\in[0,1]}|D^{(N+1)\gamma}g(x)|$. By taking the square roots, Eq. (2.13) is derived. Similar to the proof of Eq. (2.13), we prove Eq. (2.14). For this approach, using Definition 2.2, we have

$$\begin{aligned}\|g-\tilde{g}\|_\infty &\leq \|g-\widehat{g}\|_\infty=\sup_{x\in[0,1]}|g(x)-\widehat{g}(x)|\leq\frac{x^{(N+1)\gamma}}{\Gamma((N+1)\gamma+1)}M_\gamma\\ &\leq \frac{M_\gamma}{\Gamma((N+1)\gamma+1)}.\end{aligned}$$

Then, the proof is complete. □

3. Riemann-Liouville operational matrix of Müntz-Legendre polynomials

Suppose $\Phi(x)$ is Müntz-Legendre polynomials vector proposed in Eq. (2.9), we have

$$I^\nu\Phi(x)\simeq F^{(\nu,\,\gamma)}\Phi(x), \tag{3.1}$$

where $F^{(\nu,\,\gamma)}$ is operational of the Riemann-Liouville integration of order ν. Using definition of the operator I^ν, we have

$$I^\nu P_n(x)=\frac{1}{\Gamma(\nu)}x^{\nu-1}*P_n(x),\quad n=0,1,\ldots,N. \tag{3.2}$$

Now, by taking Laplace transform of Eq. (3.2), we get

$$\mathcal{L}[I^\nu P_n(x)]=\mathcal{L}[\frac{1}{\Gamma(\nu)}x^{\nu-1}]\mathcal{L}[P_n(x)],\quad n=0,1,\ldots,N, \tag{3.3}$$

where

$$\mathcal{L}[\frac{1}{\Gamma(\nu)}x^{\nu-1}]=r^{-v}. \tag{3.4}$$

Also, for $P_n(x)$, we have

$$\mathcal{L}[P_n(x)]=\mathcal{L}[\sum_{k=0}^n c_{n,k}x^{k\gamma}]=\sum_{k=0}^n c_{n,k}\mathcal{L}[x^{k\gamma}]=\sum_{k=0}^n c_{n,k}\Gamma(1+k\gamma)r^{-1-k\gamma}. \tag{3.5}$$

Using Eqs (3.4) and (3.5), we obtain

$$\mathcal{L}(I^{\nu}P_n(x)) = \sum_{k=0}^{n} c_{n,k}\Gamma(1+k\gamma)r^{-1-k\gamma-\nu}.$$

Now, taking the inverse Laplace transform of the above equation, obtains

$$I^{\nu}P_n(x) = \sum_{k=0}^{n} c_{n,k}\Gamma(1+k\gamma)\frac{x^{k\gamma+\nu}}{\Gamma(1+k\gamma+\nu)}. \tag{3.6}$$

Then, we expand $x^{k\gamma+\nu}$ as

$$x^{k\gamma+\nu} \simeq \sum_{l=0}^{N} \tilde{a}_l P_l(x) = \tilde{A}_n^T\Phi(x). \tag{3.7}$$

Therefore, we get

$$F^{(\nu,\,\gamma)} = [\widehat{A}_n], \quad n = 0,1,\ldots,N, \tag{3.8}$$

where

$$\widehat{A}_n = \sum_{k=0}^{n} \frac{c_{n,k}\tilde{A}_n^T\Gamma(1+k\gamma)}{\Gamma(1+k\gamma+\nu)}. \tag{3.9}$$

Remark 3.1. As in the discussion above, the error bound of $F^{(\nu,\gamma)}$ is as follows:

$$E^{(\nu,\gamma)}(x) = I^{\nu}\Phi(x) - F^{(\nu,\gamma)}\Phi(x), \quad E^{(\nu,\gamma)} = \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_N \end{bmatrix}. \tag{3.10}$$

Only the approximation occurs in Eq. (3.7), using Eq. (2.14), we get

$$\|e_n\|_{\infty} \le \sum_{k=0}^{n} \frac{M_{k,\gamma}}{\Gamma((N+1)\gamma+1)}, \quad n = 0,1,\ldots,N, \tag{3.11}$$

where $M_{k,\gamma} = \sup_{x\in[0,1]} |(x^{k\gamma+\nu})^{(N+1)\gamma}(x)|$.

4. Numerical method

This work is motivated by the desire to find approximate solution of fractional order Fredholm-Volterra integro-differential equation

$$\begin{aligned} D^{\nu}y(x) &= g(x)y(x) + f(x) + \lambda\int_0^1 k_1(x,t)N_1(t,\ y(t), D^{\nu}y(t))dt \\ &+ \mu\int_0^x k_2(x,t)N_2(t,\ y(t), D^{\nu}y(t))dt, \quad x\in[0,1], \end{aligned} \tag{4.1}$$

subject to

$$y^{(i)}(0) = y_{i0}, \quad i = 0,\ 1, \ldots,\ s,$$

where $m-1 < \nu \leq m,\ m \in N,\ f(x), g(x), N_1(t, y(t), D^{\nu}y(t)), N_2(t, y(t), D^{\nu}y(t))$ are known functions, $k_1(x,t), k_2(x,t) \in L^2([0,1]\times[0,1])$ are continuous and known functions, $y(x)$ is an unknown function, $\lambda,\ \mu$ and y_{i0} are real constants, $s = 0, 1, \ldots, \lceil \nu \rceil - 1$.

In our method, we expand $D^{\nu}y(x), f(x)$ and $g(x)$ with respect to the Müntz-Legendre polynomials as follows:

$$D^{\nu}y(x) \simeq y^T\Phi(x), \quad f(x) \simeq f^T\Phi(x), \quad g(x) \simeq g^T\Phi(x). \tag{4.2}$$

Also, by the initial condition and using property of fractional integration, we obtain

$$y(x) \simeq y^T F^{(\nu,\gamma)}\Phi(x) + E(x), \quad E(x) = \sum_{i=0}^{s} \frac{x^i}{i!} y_{i0},$$

then, we get

$$y(x) \simeq y^T F^{(\nu,\gamma)}\Phi(x) + E^T\Phi(x), \tag{4.3}$$

where

$$E(x) \simeq E^T\Phi(x).$$

Now, we substitute these approximations in Eq. (4.1), we have:

$$\begin{aligned} y^T\Phi(x) &= g^T\Phi(x)(\Phi^T(x)(F^{(\nu,\gamma)})^T y + \Phi^T(x)E) + f^T\Phi(x) \\ &+ \lambda \int_0^1 k_1(x,t) N_1\Big(t, y^T F^{(\nu,\gamma)}\Phi(t) + E^T\Phi(t), y^T\Phi(t))\Big) dt \\ &+ \mu \int_0^x k_2(x,t) N_2\Big(t, y^T F^{(\nu,\gamma)}\Phi(t) + E^T\Phi(t), y^T\Phi(t))\Big) dt. \end{aligned} \tag{4.4}$$

Applying the Gauss-Legendre numerical integration for Eq. (4.4), we achieve

$$\begin{aligned} y^T\Phi(x) &= g^T\Phi(x)\Phi^T(x)(F^{(\nu,\gamma)})^T y + g^T\Phi(x)\Phi^T(x)E + f^T\Phi(x) \\ &+ \lambda \sum_{j=0}^{\tilde{n}} \Big[\frac{\omega_j}{2} k_1(x, \frac{1}{2} + \frac{\tau_j}{2}) N_1\Big(\frac{1}{2} + \frac{\tau_j}{2}, y^T F^{(\nu,\gamma)}\Phi(\frac{1}{2} + \frac{\tau_j}{2}) \\ &+ E^T\Phi(\frac{1}{2} + \frac{\tau_j}{2}), y^T\Phi(\frac{1}{2} + \frac{\tau_j}{2}))\Big)\Big] + \mu \sum_{j=0}^{\tilde{n}} \Big[\frac{x}{2}\omega_j k_2(x, \frac{x}{2} + \frac{x}{2}\tau_j) N_2 \\ &\times \Big(\frac{x}{2} + \frac{x}{2}\tau_j, y^T F^{(\nu,\gamma)}\Phi(\frac{x}{2} + \frac{x}{2}\tau_j) + E^T\Phi(\frac{x}{2} + \frac{x}{2}\tau_j), y^T\Phi(\frac{x}{2} + \frac{x}{2}\tau_j))\Big)\Big] \end{aligned} \tag{4.5}$$

where ω_j and τ_j are weights and nodes of Gauss-Legendre. The resulting equation is solved using Petrov-Galerkin method which is convergence [8] as

$$\langle H, \phi \rangle = 0.$$

For this approach, we consider $\phi(x) = \Phi(x)$, where

$$\begin{aligned} H(x) &= y^T\Phi(x) - \Bigg[g^T\Phi(x)\Phi^T(x)(F^{(\nu,\gamma)})^T y + g^T\Phi(x)\Phi^T(x)E + f^T\Phi(x) \\ &+ \lambda\sum_{j=0}^{\tilde{n}}\Bigg[\frac{\omega_j}{2}k_1(x,\frac{1}{2}+\frac{\tau_j}{2})N_1\Bigg(\frac{1}{2}+\frac{\tau_j}{2}, y^T F^{(\nu,\gamma)}\Phi(\frac{1}{2}+\frac{\tau_j}{2}) \\ &+ E^T\Phi(\frac{1}{2}+\frac{\tau_j}{2}), y^T\Phi(\frac{1}{2}+\frac{\tau_j}{2})\Bigg)\Bigg] + \mu\sum_{j=0}^{\tilde{n}}\Bigg[\frac{x}{2}\omega_j k_2(x,\frac{x}{2}+\frac{x}{2}\tau_j) \\ N_2 &\Bigg(\frac{x}{2}+\frac{x}{2}\tau_j, y^T F^{(\nu,\gamma)}\Phi(\frac{x}{2}+\frac{x}{2}\tau_j) + E^T\Phi(\frac{x}{2}+\frac{x}{2}\tau_j), y^T\Phi(\frac{x}{2}+\frac{x}{2}\tau_j)\Bigg)\Bigg]\Bigg]. \end{aligned}$$

Now, we can solve the obtained system of algebraic equations by any iterative method.

5. Error estimate

Theorem 5.1. *Suppose that the function $y \in C^{N+1}[0,1]$ and $\tilde{y}$ is approximate of y using Müntz-Legendre polynomials. Moreover, let*

1) $|N_i(t, y, D^\nu y)| \le \rho_i, \quad i = 1, 2, \quad \forall x \in [0,1]$
2) *Nonlinear terms N_i, $i = 1,\ 2$, are satisfied in the following conditions*

$$|N_i(t, y_1, D^\nu y_1) - N_i(t, y_2, D^\nu y_2)| \le L|y_1 - y_2|,$$

where $L > 0$.

3) $|k_i(x,t)| \le \tilde{k}_i,\ i = 1, 2, \quad \forall (x,t) \in [0,1] \times [0,1]$.

Consider

$$\begin{aligned} \widehat{e}_N(x) &= \|\lambda\int_0^1 k_1(x,t)\Big(N_1(t, y, D^\nu y) - N_1(t, \tilde{y}, D^\nu \tilde{y})\Big)dt \\ &+ \mu\int_0^x k_2(x,t)\Big(N_2(t, y, D^\nu y) - N_2(t, \tilde{y}, D^\nu \tilde{y})\Big)dt\|_\infty. \end{aligned}$$

Then, we have:

$$\widehat{e}_N(x) \le \frac{LM_\gamma(|\lambda|\tilde{k}_1 + |\mu|\tilde{k}_2)}{\Gamma((N+1)\gamma+1)}, \tag{5.1}$$

where $M_\gamma = sup_{x\in[0,1]}|D^{(N+1)\gamma}g(x)|$.

Proof. By using Eq. (2.14) and $0 \le x \le 1$, we have

$$\begin{aligned}
\widehat{e}_N(x) &= \|\lambda \int_0^1 k_1(x,t)\Big(N_1(t,y,D^\nu y) - N_1(t,\tilde{y},D^\nu \tilde{y})\Big)dt \\
&+ \mu \int_0^x k_2(x,t)\Big(N_2(t,y,D^\nu y) - N_2(t,\tilde{y},D^\nu \tilde{y})\Big)dt\|_\infty \\
&\leq |\lambda|\| \int_0^1 k_1(x,t)\Big(N_1(t,y,D^\nu y) - N_1(t,\tilde{y},D^\nu \tilde{y})\Big)dt\|_\infty \\
&+ |\mu|\| \int_0^x k_2(x,t)\Big(N_2(t,y,D^\nu y) - N_2(t,\tilde{y},D^\nu \tilde{y})\Big)dt\|_\infty \\
&\leq |\lambda|\tilde{k}_1 \int_0^1 \|N_1(t,y,D^\nu y) - N_1(t,\tilde{y},D^\nu \tilde{y})\|_\infty dt \\
&+ |\mu|\tilde{k}_2 \int_0^x \|N_2(t,y,D^\nu y) - N_2(t,\tilde{y},D^\nu \tilde{y})\|_\infty dt \\
&\leq |\lambda|\tilde{k}_1 L \int_0^1 \|y-\tilde{y}\|_\infty dt + |\mu|\tilde{k}_2 L \int_0^x \|y-\tilde{y}\|_\infty dt \\
&\leq |\lambda|\tilde{k}_1 L \int_0^1 \left(\frac{M_\gamma}{\Gamma((N+1)\gamma+1)}\right)dt + |\mu|\tilde{k}_2 L \int_0^x \left(\frac{M_\gamma}{\Gamma((N+1)\gamma+1)}\right)dt \\
&\leq \frac{LM_\gamma(|\lambda|\tilde{k}_1 + |\mu|\tilde{k}_2)}{\Gamma((N+1)\gamma+1)}.
\end{aligned}$$

□

Theorem 5.2. *Let $y(x)$, $\tilde{y}(x)$ be the analytical solution and approximate solution of Eq. (4.1), respectively. Moreover assume that*

1) $|N_i(t,y,D^\nu y)| \leq \rho_i, \quad i=1,2, \quad \forall x \in [0,1]$
2) *Nonlinear terms N_i, $i=1,\ 2$, are satisfied in the following conditions*

$$|N_i(t,y_1,D^\nu y_1) - N_i(t,y_2,D^\nu y_2)| \leq L|y_1-y_2|,$$

where $L>0$.

3) $|k_i(x,t)| \leq \tilde{k}_i,\ i=1,2, \quad \forall (x,t) \in [0,1]\times[0,1]$.

Consider

$$\begin{aligned}
G(y(x)) &= D^\nu y(x) - \Big(g(x)y(x) + f(x) + \lambda \int_0^1 k_1(x,\ t)N_1(t,y(t),D^\nu y(t))dt \\
&+ \mu \int_0^x k_2(x,\ t)N_2(t,y(t),D^\nu y(t))dt\Big).
\end{aligned} \tag{5.2}$$

Then, we have

$$\widehat{E}(x) \leq \frac{M_\gamma}{\Gamma((N+1)\gamma+1)}(2 + (E^{(\nu,\gamma)}(x) + \frac{M_\gamma}{\Gamma((N+1)\gamma+1)})) + \widehat{e}_N(x). \tag{5.3}$$

where $\widehat{E}(x) = \|G(y) - G(\tilde{y})\|_\infty$, $M_\gamma = sup_{x\in[0,1]}|D^{(N+1)\gamma}g(x)|$.

Proof. We consider Eq. (5.2) and $0 \le x \le 1$. then, we have

$$\begin{aligned}
\widehat{E} &\le \|D^\nu y(x) - D^\nu \tilde{y}(x) - (g(x) - \tilde{g}(x))(y(x) - \tilde{y}(x)) - (f(x) - \tilde{f}(x)) \\
&- \lambda \int_0^1 k_1(x,t)[N_1(t,y,D^\nu y) - N_1(t,\tilde{y},D^\nu \tilde{y})]dt \\
&- \mu \int_0^x k_2(x,t)[N_2(t,y,D^\nu y) - N_2(t,\ \tilde{y},D^\nu \tilde{y})]dt\|_\infty \\
&\le \|D^\nu y(x) - D^\nu \tilde{y}(x)\|_\infty + \|g - \tilde{g}\|_\infty \|y - \tilde{y}\|_\infty + \|f - \tilde{f}\|_\infty \\
&+ \|\lambda \int_0^1 k_1(x,t)\Big(N_1(t,y,D^\nu y) - N_1(t,\tilde{y},D^\nu \tilde{y})\Big)dt \\
&+ \mu \int_0^x k_2(x,t)\Big(N_2(t,y,D^\nu y) - N_2(t,\tilde{y},D^\nu \tilde{y})\Big)dt\|_\infty.
\end{aligned}$$

Now, using Eqs. (2.14), (3.11) and Theorem 5.1, we derive

$$\begin{aligned}
\widehat{E} &\le \frac{M_\gamma}{\Gamma((N+1)\gamma+1)} + (\frac{M_\gamma}{\Gamma((N+1)\gamma+1)})(E^{(\nu,\gamma)}(x) + \frac{M_\gamma}{\Gamma((N+1)\gamma+1)}) \\
&+ \frac{M_\gamma}{\Gamma((N+1)\gamma+1)} + \widehat{e}_N(x) \qquad (5.4) \\
&= \frac{M_\gamma}{\Gamma((N+1)\gamma+1)}(2 + (E^{(\nu,\gamma)}(x) + \frac{M_\gamma}{\Gamma((N+1)\gamma+1)})) + \widehat{e}_N(x).
\end{aligned}$$

Then, the proof is complete. □

6. Illustrative test problems

Here, we employ the proposed method with $\lambda_k = k\gamma$, for finding the approximate solution of some examples in three subsections and show that the numerical technique is both accurate and efficient for solving this class of fractional equations.

6.1. Fractional-order Volterra integro-differential equation (FVIDE).

Example 6.1.1. Let us consider the following problem

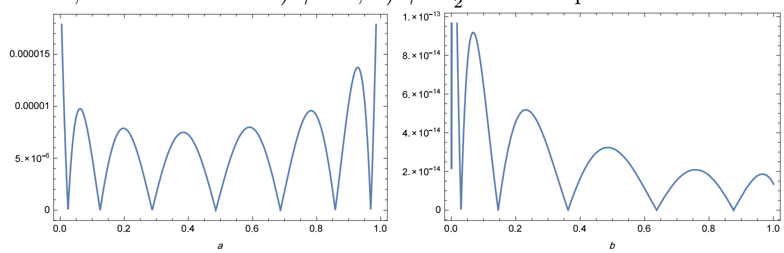
$$\begin{cases} D^{0.5}y(x) = f(x) + \int_0^x \sqrt{x+t}y(t)dt, \quad 0 \le x \le 1, \\ y(0) = 0, \end{cases}$$

$f(x)$ is a function that the analytical solution of this example is $y(x) = x^2$. We apply our method to find numerical solution of this problem for $N = 3$ and various values of γ. For this approach, we consider

$$D^{0.5}y(x) \simeq A^T\Phi(x), \quad A = [a_0, a_1, a_2, a_3]^T, \quad \Phi(x) = [P_0(x), P_1(x), P_2(x), P_3(x)]^T,$$

$$f(x) \simeq f^T\Phi(x),$$

FIGURE 1. Absolute errors between the analytical and approximate solutions, with $N = 3$ and a) $\gamma = 1$, b) $\gamma = \frac{1}{2}$ for Example 6.1.1.



$$F^{(0.5,1)} = \begin{bmatrix} \frac{4}{3\sqrt{\pi}} & \frac{8}{315\sqrt{\pi}} & \frac{-4}{35\sqrt{\pi}} & \frac{16}{315\sqrt{\pi}} \\ \frac{-4}{15\sqrt{\pi}} & \frac{8}{21\sqrt{\pi}} & \frac{4}{15\sqrt{\pi}} & \frac{-16}{231\sqrt{\pi}} \\ \frac{-4}{105\sqrt{\pi}} & \frac{-8}{45\sqrt{\pi}} & \frac{20}{77\sqrt{\pi}} & \frac{112}{585\sqrt{\pi}} \\ \frac{-4}{315\sqrt{\pi}} & \frac{-8}{231\sqrt{\pi}} & \frac{-28}{195\sqrt{\pi}} & \frac{16}{77\sqrt{\pi}} \end{bmatrix},$$

$$F^{(0.5,0.5)} = \begin{bmatrix} \frac{4}{3\sqrt{\pi}} & \frac{2}{3\sqrt{\pi}} & 0 & 0 \\ \frac{-32+9\pi}{12\sqrt{\pi}} & \frac{-20+9\pi}{15\sqrt{\pi}} & \frac{3}{20\sqrt{\pi}} & 0 \\ \frac{28-9\pi}{3\sqrt{\pi}} & \frac{110-36\pi}{15\sqrt{\pi}} & \frac{80-21\pi}{35\sqrt{\pi}} & \frac{8}{21\sqrt{\pi}} \\ \frac{-896+285}{24\sqrt{\pi}} & \frac{-104+33\pi}{3\sqrt{\pi}} & \frac{3(-512+161\pi)}{112\sqrt{\pi}} & \frac{-96+35\pi}{42\sqrt{\pi}} \end{bmatrix},$$

substituting these approximation in the problem and applying Petrov-Galerkin method, we achieve the approximate solution of this problem.

Figure 1 displays the absolute errors achieved by the proposed technique. In this figure, we can see the effectiveness of λ_k, for this problem.

Example 6.1.2. We consider the nonlinear FVIDE given in [4]

$$D^{0.5}y(x) = g(x)y(x) + f(x) + \sqrt{x}\int_0^x y^2(t)dt, \quad y(0) = 0,$$

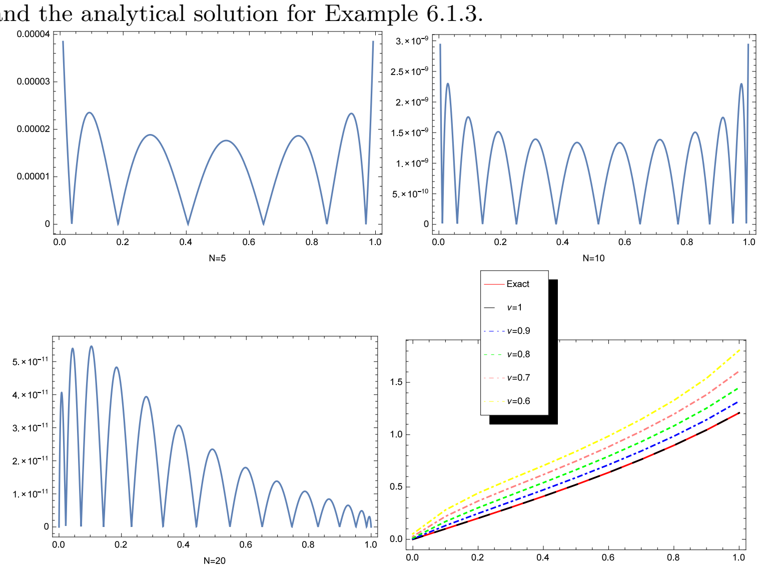
such that

$$g(x) = 2\sqrt{x}+2x^{0.75}-(\sqrt{x}+x^{0.75})Ln(1+x), \quad f(x) = \frac{2Arcsinh(\sqrt{x})}{\sqrt{\pi}\sqrt{1+x}}-2x^{0.75}.$$

The analytical solution of this problem is $y(x) = Ln(1 + x)$. The absolute error derived by using the proposed method with $\gamma = \frac{1}{2}, N = 4, 6$ are 1.568337×10^{-4} and 2.472309×10^{-5}, respectively. However, these values for collocation method with $M = 4, 6$ are 1×10^{-2} [4].

Example 6.1.3. Let us consider the following nonlinear FVIDE [14]

FIGURE 2. Comparison of $y(x)$ for $N = 5, \gamma = \nu$, $\nu = 0.6, 0.7, 0.8, 0.9, 1$ and the analytical solution for Example 6.1.3.



$$D^{\nu}y(x) = 1 + \int_0^x y(t)D^{\nu}y(t)dt, \qquad 0 \leq x \leq 1,\ 0 < \nu \leq 1,$$

with the initial condition $y(0) = 0$. The analytical solution of this example for $\nu = 1$ is $y(x) = \sqrt{2}tan(\frac{\sqrt{2}}{2}x)$.

Figure2 displays the absolute errors for $\nu = \gamma = 1$ for various values of N. Also, this figure shows the obtained results for $N = 5, \nu = \gamma$ with different choices of ν and the analytical solution. The comparisons display that as $\nu \to 1$, the obtained solutions tend to the analytical solution. We can see that the presented scheme is in high agreement with the analytical solution, in this figure.

6.2. **Fractional Fredholm integro-differential equation (FFIDE).**

Example 6.2.1. Consider the following linear FFIDE [18]

$$D^{\nu}y(x) = f(x) - \int_0^1 y(t)dt,$$

where $y(0) = 0$ and $f(x)$ is a function that the analytical solution of this example is $y(x) = x^{\frac{3}{2}} + x$. TABLE 1 displays the comparison of approximate solution for various values of $\nu, \gamma, N = 3$. The method is based on the Chebyshev wavelet of second kind for $M = 4, k = 4$ and the exact solution.

Example 6.2.2. Consider the following nonlinear FFIDE [17]

TABLE 1. Comparison of approximate solution for various values of $\nu, \gamma, N = 3$. Method based on Chebyshev wavelet of second kind and analytic solution for Example 6.2.1.

x	Exact solution	Chebyshev wavelet k=4, M=4	Present method $N = 3, \gamma = 1$	Present method $N = 3, \gamma = \frac{1}{2}$
0.1	0.131622777	0.1333	0.1333	0.131622776
0.3	0.464316767	0.4661	0.4628	0.464316765
0.5	0.853553388	0.8556	0.8524	0.853553390
0.7	1.285662019	1.2880	1.2874	1.285662016
0.9	1.753814968	1.7564	1.7531	1.753814965

TABLE 2. Comparison of the root-mean-square error in Example 6.2.2.

Method	$\|e\|_2$
CAS wavelet (k=4, M=1; 48 number basis)	5.3445×10^{-6}
Chebyshev wavelet (k=4, M=2; 16 number basis)	1.4484×10^{-6}
Chebyshev wavelet (k=5, M=2; 32 number basis)	2.3374×10^{-7}
Present method ($\gamma = \frac{1}{2}$, N=4; 5 number basis)	2.5018×10^{-15}

$$D^{0.5}y(x) = f(x) + \int_0^1 xty^4(t)dt, \quad 0 \le x \le 1,$$

where $y(0) = 0$ and $f(x) = \frac{1}{\Gamma(0.5)}(\frac{8}{3}\sqrt{x^3} - 2\sqrt{x}) - \frac{x}{1260}$. The analytic solution of the problem is $y(x) = x^2 - x$. The comparison results in TABLE 2 displays the comparison results between the root-mean-square error for this problem using 48 ($k = 4, M = 1$) CAS wavelet basis [17], 16 ($k = 4, M = 2$) the second kind Chebyshev wavelet basis [21] and present method for $\gamma = 0.5$ by only 5 ($N = 4$) Müntz-Legendre basis. In this table, we can see that the proposed method is effective.

6.3. Fractional Volterra-Fredholm integro-differential equations (FVFIDE).

Example 6.3.1. Consider the following linear FVFIDE

$$D^{\nu}y(x) = f(x) - \int_0^x \sqrt{x-t}y(t)dt - \int_0^1 (x+t)^2 y(t)dt,$$

where $y(0) = 0$, $f(x)$ is a function that the analytical solution of this problem for $\nu = \frac{1}{2}$ is $y(x) = x^2$. FIGURE 3 shows the comparison of $y(x)$ for $N = 7, \gamma = \nu$, and $\nu = 0.5, 0.7, 0.9, 1$. Also, this figure displays the absolute error of $\nu = \gamma = \frac{1}{2}$. In this figure, we can see that the technique is convergence to the analytic solution of this equation.

Example 6.3.2. Consider the non-linear equation [11]

FIGURE 3. The comparison of $y(x)$ for $N = 7$, $\gamma = \nu$ with various values of ν and the analytic solution for $\nu = \frac{1}{2}$, in Example 6.3.1.

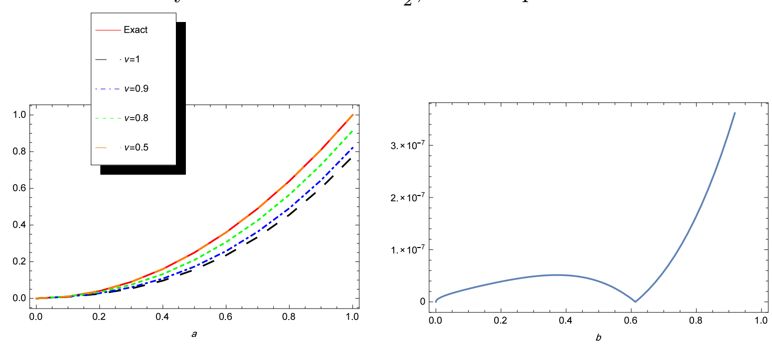
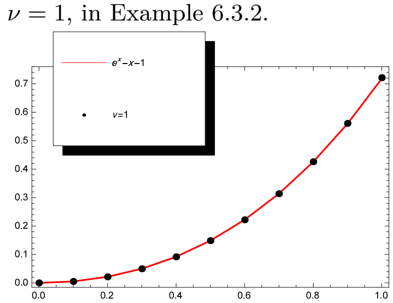


FIGURE 4. The comparison of $y(x)$ for $N = 4$, $\gamma = \nu$ with $\nu = 1$ and the exact solution for $\nu = 1$, in Example 6.3.2.



$$D^{\nu+1}y(x) = \int_0^x (e^t + 1)y^2(t)dt + \int_0^1 xty^2(t)dt + g(x),$$

where $y(0) = y^{'}(0) = 0$, and $g(x)$ is given that the analytical solution is $y(x) = e^x - x - 1$ in $\nu = 1$.

We apply the present technique for $N = 4, 6$ and various values of $\gamma = \nu$. FIGURE 4 shows the comparison of numerical results for $\gamma = \nu = 1$ and the analytical solution for $\nu = 1$. In comparing this figure with Figure. 10 in [11] which is the obtained results using the Legendre wavelets method, we can see that we have found a good approximation than the Legendre wavelets method by using a small number of bases. For more investigation, we present the absolute errors for $N = 4, 6$ and $\gamma = \nu = 1$.

FIGURE 4 and TABLE 3 show the advantages and the accuracy of the proposed method to solve nonlinear FVFIDEs.

7. Conclusion

In this study, we presented an efficient technique to find numerical solution of fractional order Volterra-Fredholm integro-differential equations. This technique is

TABLE 3. The absolute errors for $\nu = \gamma = 1$ and $N = 4, 6$ for Example 6.3.2.

x	N=4	N=6
0.1	7.4286×10^{-5}	1.11801×10^{-6}
0.3	4.9982×10^{-5}	3.11964×10^{-7}
0.5	1.3212×10^{-4}	2.85482×10^{-5}
0.7	2.4035×10^{-5}	2.86379×10^{-6}
0.9	1.4888×10^{-3}	1.74588×10^{-4}

based on Müntz-Legendre polynomials and Petrove-Galerkin method. We derived a new operational matrix of fractional integration using Laplace transform for Müntz-Legendre polynomials. This operational matrix and Petrove-Galerkin method are employed to approximate the problem. We proposed an estimation of the error. Numerical examples are proposed to test the efficiency, effectiveness and accuracy of the present scheme.

Acknowledgments

The authors are grateful to editor and anonymous referees for their constructive comments, which are very helpful to improve the presentation of the paper.

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