



Lyapunov exponents for discontinuous dynamical systems of Filippov type

Zahra Monfared*

Department of Applied Mathematics,
Ferdowsi University of Mashhad(FUM), Mashhad, Iran.
E-mail: zahra.monfared@mail.um.ac.ir

Zohreh Dadi

Department of Mathematics, Faculty of
Basic Sciences, University of Bojnord, Bojnord, Iran.
E-mail: z.dadi@ub.ac.ir

Zahra Afsharnezhad

Department of Applied Mathematics,
Ferdowsi University of Mashhad(FUM), Mashhad, Iran.
E-mail: afsharnezhad@math.um.ac.ir

Abstract

The area of discontinuous dynamical systems is almost a young research area, and the enthusiasm and necessity for analyzing these systems have been growing. On the other hand, chaos appears in a rather wide class of discontinuous systems. One of the most important properties of chaos is the sensitive dependence on initial conditions. Also, one of the most effective ways to diagnosis chaotic systems is defining Lyapunov exponents of these systems. Also, defining and calculating Lyapunov exponents for all discontinuous systems are real challenges. This paper is devoted to define Lyapunov exponents for discontinuous dynamical systems of Filippov type in order to investigate chaos for these systems.

Keywords. Chaos, Lyapunov exponents, Filippov systems.

2010 Mathematics Subject Classification. 34H10 , 34D08.

1. INTRODUCTION

Nonlinear dynamical systems can have either continuous or discontinuous nonlinearities or both. Indeed, nonlinear systems dealing with impact, friction, freeplay switching, and so on are discontinuous. Furthermore, discontinuous dynamical systems have many applications in various fields like mechanical and electrical engineering, biomechanics, control theory, economics, ecology, etc. By using this kind of differential equations, dynamical phenomena can be modeled more realistically and reasonably.

One important type of discontinuous systems is Filippov systems. The vector field of these systems is discontinuous but the trajectories of these systems are continuous with respect to the time t . Filippov systems consist of at least one discontinuity

Received: 4 November 2018 ; Accepted: 21 April 2019.

* corresponding.

boundary which divides the phase space into disjoint regions with different dynamics. Such systems appear in applications such as control systems with switching control laws, or population dynamics; see [6]. Moreover, Filippov systems can be used in modelling of mechanical systems exhibiting dry friction [6]. Furthermore, Filippov in [8, 9] has shown that many results in the classical theory of differential equation is valid also for differential equations with discontinuous right-hand sides. In recent years, remarkable researches and studies have performed in investigating the discontinuous dynamical system and developing the theory of these systems. For more information see [2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18].

Chaos is a generic property of many dynamical systems such as in electronic circuits, chemical reactions, lasers, mechanical devices, and a lot of models of biology, economics and physics; see [7]. There are papers concluding chaos in a rather wide class of discontinuous systems analytically and using Melnikov method; for example see [3, 7, 13]. On the other hand, one of the most important properties of chaos is the sensitive dependence on initial conditions. To measure this property, we use Lyapunov exponent concept. In fact, the most useful way to diagnosis chaotic systems is defining Lyapunov exponents of these systems. The calculation of Lyapunov exponents for continuous systems is well developed, while it is not done yet for all kinds of discontinuous systems. Furthermore, the enthusiasm and necessity for defining Lyapunov exponents of discontinuous dynamical systems as a tool of investigating chaos in these systems have been growing. To the best of our knowledge, defining and calculating Lyapunov exponents for all discontinuous systems represent real challenges. However, the calculation of Lyapunov exponents for some discontinuous systems is performed, for instance, see [5, 10, 12, 14, 15, 16, 18]. The present work attempts to define Lyapunov exponents for discontinuous dynamical systems of Filippov type. This can help us to investigate chaos for these systems.

This paper has organized in two sections. In the first section, we give those definitions and theorems that are going to use in the next section. The second section is devoted to study chaos for Filippov systems by the aid of defining Lyapunov exponents for these systems.

2. PRELIMINARIES

2.1. Discontinuous dynamical systems. Here, we give the definition of Filippov systems and also state some concepts and basic results which are important for our purposes in this paper.

In [8], Filippov systems are introduced as discontinuous dynamical systems consist of two or more smooth vector fields that are separated by discontinuity boundaries. For simplicity, we assume that there is just one discontinuity boundary. The formal definition of such systems given in [9] is as follows.

Let us consider the system

$$\begin{aligned} \dot{x} &= f(t, x) \\ x(t_0) &= x_0, \end{aligned} \tag{2.1}$$



in which $f : U \rightarrow \mathbb{R}^n$, and $U = J \times U' \subseteq \mathbb{R} \times \mathbb{R}^n$ is a domain. Moreover, suppose that f satisfies the following assumptions:

(I) f has the following form

$$f(t, x) = \begin{cases} f^-(t, x); & x \in S^-, t \in J \\ f^+(t, x); & x \in S^+, t \in J \end{cases}, \quad (2.2)$$

such that U' is divided into two open and disjoint sets S^- and S^+ by a hypersurface Σ . The discontinuity boundary Σ and sets S^+ and S^- can be defined by a scalar function $h : U' \rightarrow \mathbb{R}, h \in C^r(U', \mathbb{R}), r \geq 1$ as

$$S^- = \{x \in U' \mid h(x) < 0\}, \quad (2.3)$$

$$S^+ = \{x \in U' \mid h(x) > 0\}, \quad (2.4)$$

and

$$\Sigma = \{x \in U' \mid h(x) = 0\}. \quad (2.5)$$

(II) The normal of the hypersurface Σ , given by $n(x) = [Dh(x)]^T$ is chosen such that it is always nonzero, i.e., $n(x) \neq 0$ for each $x \in \Sigma$.

(III) There are functions $g^\pm : J \times V^\pm \rightarrow \mathbb{R}^n$ with the features

(1) $S^\pm \cup \Sigma \in V^\pm$, where V^\pm are domain in \mathbb{R}^n ,

(2) $g^\pm \in C^r(J \times V^\pm, \mathbb{R}^n), r \geq 1$,

(3) $g^\pm(t, x) = f^\pm(t, x), \quad \forall t \in J, x \in S^\pm$.

This means the existence of C^r -extensions of f^\pm . Therefore f^\pm must be C^r , too.

Now, let us define the set-valued extension of system (2.2) for each $t \in J$ as follows

$$F(t, x) = \begin{cases} f^-(t, x); & x \in S^- \\ \{(1 - \lambda)g^-(t, x) + \lambda g^+(t, x), \forall \lambda \in [0, 1]\}; & x \in \Sigma \\ f^+(t, x); & x \in S^+. \end{cases} \quad (2.6)$$

Then, the system (2.1) can be considered as a differential inclusion:

$$x' \in F(t, x), \quad (2.7)$$

which is known as Filippov's convex method.

Definition 2.1 (Fillipov solution). Function $x : I \rightarrow \mathbb{R}^n$, at which $I \subseteq \mathbb{R}$ is an interval, is called a solution of differential inclusion(2.7) if x is almost every where continuous and $x'(t) \in F(t, x(t))$, for almost all $t \in I$,



The assumptions (I) – (III) assure that differential equations (2.7) and (2.1) have a solution in the sense of Filippov solution; for more details see [2, 9, 11].

Definition 2.2. Suppose that $x : I \rightarrow \mathbb{R}^n$ is a solution of (2.1) which reaches to the discontinuity boundary Σ at the point $x_\Sigma \in \Sigma$ and time $t_\Sigma \in I$, i.e., $x(t_\Sigma) = x_\Sigma$. We say that solution $x(t)$ crosses the hypersurface Σ transversally at x_Σ if

$$n^T(x_\Sigma) g^-(t_\Sigma, x_\Sigma) n^T(x_\Sigma) g^+(t_\Sigma, x_\Sigma) > 0, \tag{2.8}$$

where $n(x) = [Dh(x)]^T$; see [9, 11].

2.2. Lyapunov exponents for continuous systems.

Definition 2.3. Consider the following dynamical system

$$\dot{x} = f(x(t)); \quad x(t_0) = x_0, \quad f \in C^1. \tag{2.9}$$

The time evaluation of a tangent vector $\delta x(t)$ at $x(t)$ is represented by linearizing equation (2.9) as follows

$$\delta \dot{x} = F(x(t)) \delta x(t), \tag{2.10}$$

where

$$F(x(t)) = \left. \frac{\partial f(x)}{\partial x^T} \right|_{x=x(t)}. \tag{2.11}$$

Then, the spectrum of the Lyapunov exponents λ_i is given for some different initial conditions $\delta x_i(t)$ as

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|\delta x(t)\|}{\|\delta x_i(t_0)\|}. \tag{2.12}$$

For more information in this direction see [1, 4, 12, 17].

3. LYAPUNOV EXPONENTS FOR FILIPPOV SYSTEMS

Here we are going to define Lyapunov exponents for discontinuous dynamical systems of Filippov type.

Consider the following Filippov system

$$\dot{x} = \begin{cases} f^-(x); & x \in S^-, \quad f^- \in C^1 \\ f^+(x); & x \in S^+, \quad f^+ \in C^1 \end{cases}, \tag{3.1}$$

such that

$$S^+ = \{x \in U \subseteq \mathbb{R}^n; h(x) > 0\}, \tag{3.2}$$

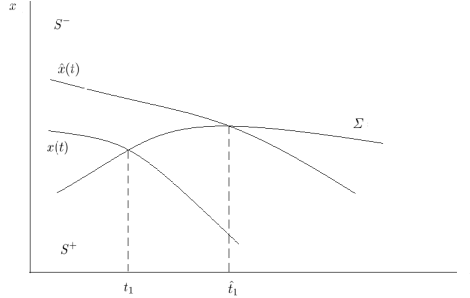
$$S^- = \{x \in U \subseteq \mathbb{R}^n; h(x) < 0\}, \tag{3.3}$$

where $h : U' \rightarrow \mathbb{R}$ satisfies in the hypothesis of Section 2.1. Moreover, the discontinuity boundary Σ which separates two regions S^+ and S^- is

$$\Sigma = \{x \in U \subseteq \mathbb{R}^n; h(x) = 0\}, \tag{3.4}$$



FIGURE 1. Nearby trajectories $x(t)$ and $\hat{x}(t)$ of system (3.1)



for more details see Section 2.1.

Here we suppose that orbits of (3.1) transversally cross the discontinuity boundary Σ , i.e., we assume that the orbits of (3.1) satisfy in relation (2.8) of Definition 2.2.

Now for computing the complete spectrum of Lyapunov exponents of the system (3.1), we have to linearize equation (3.1). Indeed our aim is doing a linearization similar to equation (2.10) for the Filippov system (3.1) by using the properties of discontinuous systems. For this purpose, we do the following process:

Let $x(t)$ be a solution of system (3.1) with $x(t_0) = x_0 \in S^-$. Moreover, it is assumed that $x(t)$ reaches the boundary Σ at time $t_1 > t_0$. For this solution, we can write

$$\begin{cases} \dot{x}(t) = f^-(x(t)); & t_0 \leq t < t_1 \\ h(x(t_1)) = 0; & t = t_1 \\ \dot{x}(t) = f^+(x(t)); & t > t_1 \end{cases} . \tag{3.5}$$

Note that the existence of transversality conditions for all solutions of system (3.1) plays an important role in our process. Here transversality condition for $x(t)$ results

$$\exists t_1^+ > t_1 \quad s.t. \quad h(x(t_1^+)) > 0. \tag{3.6}$$

We put

$$t_1^{inf} := inf\{t_1^+ | h(x(t_1^+)) > 0\}. \tag{3.7}$$

To check the chaotic behaviors of the system (3.1), we have to investigate the behavior of trajectories that are close to the solutions of the system (3.1). So, discuss the behavior of nearby trajectories, we compare $x(t)$ with a trajectory $\hat{x}(t)$ such that $\hat{x}(t_0) = x_0 + \delta x_0 \in S^-$. As well as two trajectories $x(t)$ and $\hat{x}(t)$, satisfies in the following relation

$$\hat{x}(t) - x(t) = \delta x(t), \quad \forall t \tag{3.8}$$

Let $\hat{x}(t)$ touches the discontinuity boundary at time $\hat{t}_1 > t_0$, where $\hat{t}_1 = t_1 + kt$. Without loss of generality, we can assume that $kt > 0$. By this assumption, first the



trajectory $x(t)$ reaches to the discontinuity boundary Σ .
 Now for the trajectory $\hat{x}(t)$ we can write

$$\begin{cases} \dot{\hat{x}}(t) = f^-(\hat{x}(t)); & t_0 \leq t < \hat{t}_1 \\ h(\hat{x}(\hat{t}_1)) = 0; & t = \hat{t}_1 \\ \dot{\hat{x}}(t) = f^+(\hat{x}(t)); & t > \hat{t}_1 \end{cases} \quad (3.9)$$

For $t_1^- < t_1$ and $\hat{t}_1^+ > \hat{t}_1$ we define

$$\begin{aligned} \delta x_- &:= \delta x(t_1^-) = \hat{x}(t_1^-) - x(t_1^-), \\ \delta x_+ &:= \delta x(\hat{t}_1^+) = \hat{x}(\hat{t}_1^+) - x(\hat{t}_1^+). \end{aligned} \quad (3.10)$$

Applying a Taylor series expansion up to the first order terms to $h(\hat{x}(\hat{t}_1))$ yields

$$\begin{aligned} 0 &= h(\hat{x}(\hat{t}_1)) = h(\hat{x}(t_1 + kt)) \approx h(\hat{x}(t_1) + kt \cdot \dot{\hat{x}}(t_1)) \\ &= h(\hat{x}(t_1) + kt \cdot f^-(\hat{x}(t_1))) \end{aligned} \quad (3.11)$$

$x(t)$ is continuous for $t_0 \leq t < \hat{t}_1$ and also by the first equation of system (3.5), we can obtain $x(t_1^-)$, for some $t_1^- < t_1$ sufficiently close to t_1 . This implies that the last line of (3.11) becomes

$$\begin{aligned} &\approx h(\hat{x}(t_1^-) + kt \cdot f^-(\hat{x}(t_1^-))) = h(x(t_1^-) + \delta x_- + kt \cdot f^-(x(t_1^-) + \delta x_-)) \\ &\approx h\left(x(t_1^-) + \delta x_- + kt \left[f^-(x(t_1^-)) + \delta x_- \cdot \frac{\partial f(x)}{\partial x^T} \Big|_{x=x(t_1^-)} \right]\right) \\ &\approx h(x(t_1^-)) + \frac{\partial h(x)}{\partial x^T} \Big|_{x=x(t_1^-)} \cdot [\delta x_- + kt \cdot f^-(x(t_1^-))] \\ &\approx H(x(t_1^-)) \cdot [\delta x_- + kt \cdot f^-(x(t_1^-))], \end{aligned}$$

where

$$H(x(t_1^-)) = \frac{\partial h(x)}{\partial x^T} \Big|_{x=x(t_1^-)}. \quad (3.12)$$

Therefore,

$$H(x(t_1^-)) \cdot [\delta x_- + kt \cdot f^-(x(t_1^-))] = 0. \quad (3.13)$$

Hence we have

$$kt = -\frac{H(x(t_1^-)) \cdot \delta x_-}{H(x(t_1^-)) \cdot f^-(x(t_1^-))}. \quad (3.14)$$



On the other hand, for \hat{t}_1^+ sufficiently close to \hat{t}_1

$$\begin{aligned} \hat{x}(\hat{t}_1^+) &= \hat{x}(t_1^+ + kt) \approx \hat{x}(t_1^+) + kt.\dot{\hat{x}}(t_1^+) = \hat{x}(t_1^+) + kt.f^-(\hat{x}(t_1^+)) \\ &= \hat{x}(t_1^+) + kt.f^-(x(t_1^+) + \delta x(t_1^+)) \approx \hat{x}(t_1^+) + kt.f^-(x(t_1^+)), \end{aligned} \tag{3.15}$$

where $t_1 < t_1^+ < \hat{t}_1^+$. Also,

$$x(\hat{t}_1^+) = x(t_1^+ + kt) \approx x(t_1^+) + kt.\dot{x}(t_1^+) = x(t_1^+) + kt.f^+(x(t_1^+)), \tag{3.16}$$

where $t_1 < t_1^+$. Thus,

$$\delta x_+ = \hat{x}(\hat{t}_1^+) - x(\hat{t}_1^+) = \delta x(t_1^+) + kt [f^-(x(t_1^+)) - f^+(x(t_1^+))]. \tag{3.17}$$

By substituting kt from (3.14) we have

$$\delta x_+ = \delta x(t_1^+) - \frac{H(x(t_1^-)).\delta x_-}{H(x(t_1^-)).f^-(x(t_1^-))} \cdot [f^-(x(t_1^+)) - f^+(x(t_1^+))]. \tag{3.18}$$

The above computations which led to calculate δx_+ and kt , will help us to perform our desired linearization for Filippov system (3.1). In fact, by the aid of these calculations, we can define a linearization for the system (3.1) in the form of the following set

$$\begin{cases} \delta \dot{x} = F^-(x(t))\delta x; & t_0 \leq t < t_1 \\ \delta \dot{x} = F^+(x(t))\delta x; & t > t_1 \end{cases}, \tag{3.19}$$

in which F^- and F^+ and also initial conditions for both equations must be determined. Notice that the first equation of (3.19) is the linearization of the system (3.1) before the discontinuity, and the second one is the linearization of (3.1) after the discontinuity. We also need a suitable transition condition of the linearized equations at the instant of discontinuity. Putting

$$F^-(x(t)) = \left. \frac{\partial f^-(x)}{\partial x^T} \right|_{x=x(t)}, \tag{3.20}$$

and considering $\delta x(t_0) = \delta x_0$ as an initial condition for the first equation of (3.19), will complete the first part of linearization. To accomplish the second part of linearization, if we set

$$F^+(x(t)) = \left. \frac{\partial f^+(x)}{\partial x^T} \right|_{x=x(t)}, \tag{3.21}$$

just defining a reasonable initial condition for the second equation of (3.19) will be needed. The second equation of (3.19) is defined for $t > t_1$, and the time starts from t_1^{inf} . Therefore by choosing $\delta x(t_1^{inf}) = \delta x_+$, we can find a suitable initial condition for the second part of linearization. Moreover, the relation (3.18) is the transition condition of the linearized equations at the instant of discontinuity.

Due to the mentioned explanations, the following set together with the transition condition (3.18) can describe a linearization for the Filippov system (3.1)

$$\begin{cases} \delta \dot{x} = F^-(x(t))\delta x; & \delta x(t_0) = \delta x_0, \quad t_0 \leq t < t_1, \\ \delta \dot{x} = F^+(x(t))\delta x; & \delta x(t_1^{inf}) = \delta x_+, \quad t > t_1 \end{cases}, \tag{3.22}$$

where F^- , F^+ and δx_+ are stated by the relations (3.20), (3.21) and (3.18).



Using this linearization, we can calculate the spectrum of Lyapunov exponents in the case of Filippov systems. It should be mentioned that here we supposed $kt > 0$. Notice that the same results can be obtained for $kt < 0$.

4. CONCLUSION

Due to the wide applications of Filippov systems, here we defined Lyapunov exponents for these systems in order to investigate chaos for them. In fact to compute the complete spectrum of Lyapunov exponents of the discontinuous system (3.1), we performed a linearization for that by the aid of the properties of discontinuous systems.

REFERENCES

- [1] K. T. Alligood, D. T. Sauer, and J. A. Yorke, *Chaos: An Introduction to Dynamical Systems*, Springer, 1996.
- [2] Z. Afsharnezhad and M. Karimi Amaleh, *Continuation of the Periodic Orbits for the Differential Equation with Discontinuous Right Hand Side*, Journal of Dynamics and Differential Equations, Springer, 23(1) (2011), 71–92. doi:10.1007/s10884-010-9194-x
- [3] F. Battelli and M. Fečkan, *Nonsmooth homoclinic orbits, Melnikov functions and chaos in discontinuous systems*, Physica D: Nonlinear Phenomena, 241, (2012), 1962–1975. doi:10.1016/j.physd.2011.05.018
- [4] C. Chicone, *Ordinary differential equations with applications*, Springer, 2006.
- [5] M. F. Danca, *Lyapunov exponents of a class of piecewise continuous systems of fractional order*, Nonlinear Dynamics, 81 (2015), 227–237. doi:10.1007/s11071-015-1984-6
- [6] K. Efsthathiou, X. Liu, and H. W. Broer, *The Boundary-Hopf-Fold Bifurcation in Filippov Systems*, SIAM Journal on Applied Dynamical Systems, 14(2) (2015), 914–941.
- [7] M. Fečkan, *Bifurcation and chaos in discontinuous and continuous systems*, Springer, 2011.
- [8] A. F. Filippov, *Differential Equations with Discontinuous Right hand Sides*, Kluwer Academic Publishers, 1988.
- [9] P. Kukučka, *Jumps of the fundamental solution matrix in discontinuous systems and applications*, Nonlinear analysis, 66 (2007), 2529–2546. doi:10.1016/j.na.2006.03.037
- [10] G. Licskó and G. Csernák, *On the chaotic behaviour of a simple dry-friction oscillator*, Mathematics and Computers in Simulation, 96 (2014), 55–62. doi:10.1016/j.matcom.2013.03.002
- [11] Z. Monfared, Z. Afsharnezhad, and J. Abolfazli Esfahani, *Flutter, limit cycle oscillation, bifurcation and stability regions of an airfoil with discontinuous freeplay nonlinearity*, Nonlinear dynamics, 90 (2017), 1965–1986. doi:10.1007/s11071-017-3773-x
- [12] P. C. Müller, *Calculation of Lyapunov exponents for dynamic systems with discontinuities*, Chaos, Solitons & Fractals, 5 (1995), 1671–1681. doi:10.1016/0960-0779(94)00170-U
- [13] L. S. Shi, Y. K. Zou, and T. Küpper, *Melnikov method and detection of chaos for non-smooth systems*, Acta Mathematicae Applicatae Sinica, English Series, 29 (2013), 881–896.
- [14] S. L. T. De Souza and I. L. Caldas, *Calculation of Lyapunov exponents in systems with impacts*, Chaos, Solitons & Fractals, 19 (2004), 569–579. doi:10.1016/S0960-0779(03)00130-9
- [15] A. Stefanski and T. Kapitaniak, *Using chaos synchronization to estimate the largest Lyapunov exponent of nonsmooth systems*, Discrete Dynamics in Nature and Society, 4 (2000), 207–215. doi:10.1155/S1026022600000200
- [16] A. Stefanski, *Estimation of the largest Lyapunov exponent in systems with impacts*, Chaos, Solitons & Fractals, 11 (2000), 2443–2451. doi:10.1016/S0960-0779(00)00029-1
- [17] S. Wiggins, *Introduction to applied dynamical systems and chaos*, Springer-Verlag, 1990.
- [18] J. Wojewoda, A. Stefański, M. Wiercigroch, T. Kapitaniak, *Estimation of Lyapunov exponents for a system with sensitive friction model*, Archive of Applied Mechanics, 79 (2009), 667–677. doi:10.1007/s00419-008-0291-9

