Lyapunov exponents for discontinuous dynamical systems of Filippov type

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Abstract
The area of discontinuous dynamical systems is almost a young research area, and the enthusiasm and necessity for analysing these systems have been growing. On the other hand, chaos appears in a rather wide class of discontinuous systems. One of the most important properties of chaos is sensitive dependence on initial conditions. Also, the most effective way to diagnosis chaotic systems is defining Lyapunov exponents of these systems. In addition, defining and calculating Lyapunov exponents for all discontinuous systems are real challenges. This paper is devoted to define Lyapunov exponents for discontinuous dynamical systems of Filippov type in order to investigate chaos for these systems.

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1. INTRODUCTION
Nonlinear dynamical systems can have either continuous or discontinuous nonlinearities or both. Indeed, nonlinear systems dealing with impact, friction, freeplay, switching and so on are discontinuous. Furthermore, discontinuous dynamical systems have many applications in various fields like mechanical and electrical engineering, biomechanics, control theory, economics, ecology and etc. By using this kind of differential equations, dynamical phenomena can be modeled in a more realistic and reasonable way.

One important type of discontinuous systems is Filippov systems. The vector field of these systems is discontinuous but the trajectories of these systems are continuous with respect to the time $t$. Filippov systems consist of at least one discontinuity...
boundary which divides the phase space into disjoint regions with different dynamics. Such systems appear in applications such as control systems with switching control laws, or population dynamics; see [6]. Moreover, Filippov systems can be used in modelling of mechanical systems exhibiting dry friction [6]. Furthermore, Filippov in [8, 9] has shown that many results in the classical theory of differential equation are valid also for differential equations with discontinuous right hand sides. In recent years, remarkable researches and studies have performed in investigating discontinuous dynamical system and developing the theory of these systems. For more information see [2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18].

Chaos is a generic property of many dynamical systems such as in electronic circuits, chemical reactions, lasers, mechanical devices, and a lot of models of biology, economics and physics; see [7]. There are papers concluding chaos in a rather wide class of discontinuous systems analytically and by means of Melnikov method; for example see [3, 7, 13]. On the other hand, one of the most important properties of chaos is sensitive dependence on initial conditions. To measure this property, we use Lyapunov exponent concept. In fact, the most useful way to diagnosis chaotic systems is defining Lyapunov exponents of these systems. The calculation of Lyapunov exponents for continuous systems is well developed, while it is not done yet for all kinds of discontinuous systems. Furthermore, the enthusiasm and necessity for defining Lyapunov exponents of discontinuous dynamical systems as a tool of investigating chaos in these systems have been growing. To the best of our knowledge, defining and calculating Lyapunov exponents for all discontinuous systems represent real challenges. However, calculation of Lyapunov exponents for some discontinuous systems is performed, for instance see [5, 10, 12, 15, 16, 14, 18]. The present work attempts to define Lyapunov exponents for discontinuous dynamical systems of Filippov type. This can help us to investigate chaos for these systems.

This paper has organized in two sections. In the first section, we give those definitions and theorems which are going to use in the next section. The second section is devoted to study chaos for Filippov systems by the aid of defining Lyapunov exponents for these systems.

2. Preliminaries

2.1. Discontinuous dynamical systems. Here, we give the definition of Filippov systems and also state some concepts and basic results which are important for our purposes in this paper.

In [8], Filippov systems are introduced as discontinuous dynamical systems consist of two or more smooth vector fields that are separated by discontinuity boundaries. For simplicity, we assume that there is just one discontinuity boundary. The formal definition of such systems given in [9] is as follows.

Let us consider the system

\[ \dot{x} = f(t, x) \]
\[ x(t_0) = x_0, \]

(2.1)
in which \( f : U \rightarrow \mathbb{R}^n \), and \( U = J \times U' \subseteq \mathbb{R} \times \mathbb{R}^n \) is a domain. Moreover, suppose that \( f \) satisfies the following assumptions:

(I) \( f \) has the following form

\[
\begin{align*}
  f(t,x) = \begin{cases} 
    f^-(t,x); & x \in S^-, t \in J \\
    f^+(t,x); & x \in S^+, t \in J 
  \end{cases}
\end{align*}
\]  

(2.2)

such that \( U' \) is divided into two open and disjoint sets \( S^- \) and \( S^+ \) by a hypersurface \( \Sigma \). The discontinuity boundary \( \Sigma \) and sets \( S^+ \) and \( S^- \) can be defined by a scalar function \( h : U' \rightarrow \mathbb{R}, h \in C^r(U', \mathbb{R}), r \geq 1 \) as

\[
\begin{align*}
  S^- &= \{ x \in U' \mid h(x) < 0 \}, \\
  S^+ &= \{ x \in U' \mid h(x) > 0 \}, \\
  \Sigma &= \{ x \in U' \mid h(x) = 0 \}. 
\end{align*}
\]  

(2.3, 2.4, 2.5)

(II) The normal of the hypersurface \( \Sigma \), given by \( n(x) = [Dh(x)]^T \) is chosen such that it is always nonzero, i.e., \( n(x) \neq 0 \) for each \( x \in \Sigma \).

(III) There are functions \( g^\pm : J \times V^\pm \rightarrow \mathbb{R}^n \) with the features

(1) \( S^\pm \cup \Sigma \in V^\pm \), where \( V^\pm \) are domain in \( \mathbb{R}^n \),

(2) \( g^\pm \in C^r(J \times V^\pm, \mathbb{R}^n), r \geq 1 \),

(3) \( g^\pm(t,x) = f^\pm(t,x), \forall t \in J, x \in S^\pm \).

This means the existence of \( C^r \)-extensions of \( f^\pm \). Therefore \( f^\pm \) must be \( C^r \), too.

Now, let us define the set-valued extension of system (2.2) for each \( t \in J \) as follows

\[
F(t,x) = \begin{cases} 
    f^-(t,x); & x \in S^- \\
    \{(1-\lambda)g^-(t,x) + \lambda g^+(t,x), \forall \lambda \in [0,1]\}; & x \in \Sigma \\
    f^+(t,x); & x \in S^+. 
  \end{cases}
\]  

(2.6)

Then, the system (2.1) can be considered as a differential inclusion:

\[
x' \in F(t,x),
\]  

(2.7)

which is known as Filippov’s convex method.

**Definition 2.1 (Fillipov solution).** Function \( x : I \rightarrow \mathbb{R}^n \), at which \( I \subseteq \mathbb{R} \) is an interval, is called a solution of differential inclusion (2.7) if \( x \) is almost everywhere continuous and \( x'(t) \in F(t,x(t)) \), for almost all \( t \in I \),
The assumptions \((I) - (III)\) assure that differential equations (2.7) and (2.1) have a solution in the sense of Filippov solution; for more details see [2, 9, 11].

**Definition 2.2.** Suppose that \(x : I \rightarrow \mathbb{R}^n\) is a solution of (2.1) which reaches to the discontinuity boundary \(\Sigma\) at the point \(x_\Sigma \in \Sigma\) and time \(t_\Sigma \in I\), i.e., \(x(t_\Sigma) = x_\Sigma\). We say that solution \(x(t)\) crosses the hypersurface \(\Sigma\) transversally at \(x_\Sigma\) if
\[
    n^T(x_\Sigma) g^-(t_\Sigma, x_\Sigma) n^T(x_\Sigma) g^+(t_\Sigma, x_\Sigma) > 0,
\]
where \(n(x) = [Dh(x)]^T\); see [9, 11].

**2.2. Lyapunov exponents for continuous systems.**

**Definition 2.3.** Consider the following dynamical system
\[
    \dot{x} = f(x(t)); \quad x(t_0) = x_0, \quad f \in C^1. \tag{2.9}
\]
The time evaluation of a tangent vector \(\delta x(t)\) at \(x(t)\) is represented by linearizing equation (2.9) as follows
\[
    \delta \dot{x} = F(x(t))\delta x(t), \tag{2.10}
\]
where
\[
    F(x(t)) = \left. \frac{\partial f(x)}{\partial x^T} \right|_{x=x(t)}. \tag{2.11}
\]
Then, the spectrum of the Lyapunov exponents \(\lambda_i\) is given for some different initial conditions \(\delta x_i(t)\) as
\[
    \lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|\delta x(t)\|}{\|\delta x_i(t_0)\|}. \tag{2.12}
\]
For more information in this direction see [1, 4, 12, 17].

**3. Lyapunov exponents for Filippov systems**

Here we are going to define Lyapunov exponents for discontinuous dynamical systems of Filippov type. Consider the following Filippov system
\[
    \dot{x} = \begin{cases} 
    f^-(x); & x \in S^-, \quad f^- \in C^1 \\
    f^+(x); & x \in S^+, \quad f^+ \in C^1 
    \end{cases} \tag{3.1}
\]
such that
\[
    S^+ = \{ x \in U \subseteq \mathbb{R}^n; \ h(x) > 0 \}, \tag{3.2}
\]
\[
    S^- = \{ x \in U \subseteq \mathbb{R}^n; \ h(x) < 0 \}, \tag{3.3}
\]
where \(h : U' \rightarrow \mathbb{R}\) satisfies in the hypothesis of Section 2.1. Moreover, the discontinuity boundary \(\Sigma\) which separates two regions \(S^+\) and \(S^-\) is
\[
    \Sigma = \{ x \in U \subseteq \mathbb{R}^n; \ h(x) = 0 \}. \tag{3.4}
\]
for more details see Section 2.1.

Here we suppose that orbits of (3.1) transversally cross the discontinuity boundary Σ, i.e., we assume that the orbits of (3.1) satisfy in relation (2.8) of Definition 2.2.

Now for computing the complete spectrum of Lyapunov exponents of system (3.1), we have to linearize equation (3.1). Indeed our aim is doing a linearization similar to equation (2.10) for the Filippov system (3.1) by using the properties of discontinuous systems. For this purpose, we do the following process:

Let \( x(t) \) be a solution of system (3.1) with \( x(t_0) = x_0 \in S^- \). Moreover, it is assumed that \( x(t) \) reaches the boundary \( \Sigma \) at time \( t_1 > t_0 \). For this solution we can write

\[
\begin{cases}
  \dot{x}(t) = f^-(x(t)); & t_0 \leq t < t_1 \\
  h(x(t_1)) = 0; & t = t_1 \\
  \dot{x}(t) = f^+(x(t)); & t > t_1
\end{cases}
\]  

(3.5)

Note that the existence of transversality condition for all solutions of system (3.1) plays an important role in our process. Here transversality condition for \( x(t) \) results

\[ \exists t^+_1 > t_1 \text{ s.t. } h(x(t^+_1)) > 0. \]  

(3.6)

We put

\[ t^{inf}_1 := \inf \{ t^+_1 | h(x(t^+_1)) > 0 \}. \]  

(3.7)

To check chaotic behaviors of the system (3.1), we have to investigate the behavior of trajectories which are close to the solutions of the system (3.1). So in order to discuss the behavior of nearby trajectories, we compare \( x(t) \) with a trajectory \( \hat{x}(t) \) such that \( \hat{x}(t_0) = x_0 + \delta x_0 \in S^- \). As well as two trajectories \( x(t) \) and \( \hat{x}(t) \), satisfies in the following relation

\[ \hat{x}(t) - x(t) = \delta x(t), \quad \forall t \]  

(3.8)

For \( t^-_1 < t_1 \) and \( \hat{t}^+_1 > \hat{t}_1 \) we define

\[ \delta x_- := \delta x(t^-_1) = \hat{x}(t^-_1) - x(t^-_1), \]  

\[ \delta x_+ := \delta x(\hat{t}^+_1) = \hat{x}(\hat{t}^+_1) - x(\hat{t}^+_1). \]  

(3.10)
Figure 1. Nearby trajectories $x(t)$ and $\dot{x}(t)$ of system (3.1)

Applying a Taylor series expansion up to the first order terms to $h(\dot{x}(t_1))$ yields

$$0 = h(\dot{x}(t_1)) = h(\dot{x}(t_1 + kt)) \approx h(\dot{x}(t_1) + kt \dot{x}(t_1))$$

$$= h\left(\dot{x}(t_1) + kt f^- (\dot{x}(t_1))\right)$$

(3.11)

$x(t)$ is continuous for $t_0 \leq t < t_1$ and also by the first equation of system (3.5), we can obtain $x(t_1^-)$, for some $t_1^- < t_1$ sufficiently close to $t_1$. This implies that the last line of (3.11) becomes

$$\approx h\left(\dot{x}(t_1^-) + kt f^- (\dot{x}(t_1^-))\right) = h\left(x(t_1^-) + \delta x_- + kt f^- (x(t_1^-) + \delta x_-)\right)$$

$$\approx h\left(x(t_1^-) + \delta x_- + kt \left[f^- (x(t_1^-)) + \delta x_- \frac{\partial f(x)}{\partial x} \bigg|_{x=x(t_1^-)}\right]\right)$$

$$\approx h(x(t_1^-)) + \frac{\partial h(x)}{\partial x} \bigg|_{x=x(t_1^-)} \left[\delta x_- + kt f^- (x(t_1^-))\right]$$

$$\approx H(x(t_1^-)) \left[\delta x_- + kt f^- (x(t_1^-))\right],$$

where

$$H(x(t_1^-)) = \frac{\partial h(x)}{\partial x} \bigg|_{x=x(t_1^-)}.$$
Therefore,
\[ H(x(t_1^{-})), \left[ \delta x_- + kt. f^-(x(t_1^-)) \right] = 0. \] (3.13)

Hence we have
\[ kt = -\frac{H(x(t_1^-)). \delta x_-}{H(x(t_1^-)). f^-(x(t_1^-))}. \] (3.14)

On the other hand, for \( \hat{t}_1^+ \) sufficiently close to \( \hat{t}_1 \)
\[ \hat{x}(\hat{t}_1^+) = \hat{x}(t_1^+ + kt) \approx \hat{x}(t_1^+) + kt. \hat{x}(t_1^+) = \hat{x}(t_1^+) + kt. f^-(\hat{x}(t_1^+)) \]
\[ \approx \hat{x}(t_1^+) + kt. f^-(x(t_1^+) + \delta x(t_1^+)) \approx \hat{x}(t_1^+) + kt. f^-(x(t_1^+)), \] (3.15)

where \( t_1 < t_1^+ < \hat{t}_1^+ \). Also,
\[ x(\hat{t}_1^+) = x(t_1^+ + kt) \approx x(t_1^+) + kt. \hat{x}(t_1^+) = x(t_1^+) + kt. f^+(x(t_1^+)), \] (3.16)

where \( t_1 < t_1^+ \). Thus,
\[ \delta x_+ = \hat{x}(\hat{t}_1^+) - x(\hat{t}_1^+) = \delta x(t_1^+) + kt \left[ f^-(x(t_1^+)) - f^+(x(t_1^+)) \right]. \] (3.17)

By substituting \( kt \) from (3.14) we have
\[ \delta x_+ = \delta x(t_1^+) - \frac{H(x(t_1^-)). \delta x_-}{H(x(t_1^-)). f^-(x(t_1^-))}. \left[ f^-(x(t_1^+)) - f^+(x(t_1^+)) \right]. \] (3.18)

The above computations which led to calculate \( \delta x_+ \) and \( kt \), will help us to perform our desired linearization for Filippov system (3.1). In fact by the aid of these calculations we can define a linearization for the system (3.1) in the form of the following set
\[ \begin{cases} 
\delta \dot{x} = F^-(x(t)) \delta x; & t_0 \leq t < \hat{t}_1 \\
\delta \dot{x} = F^+(x(t)) \delta x; & t > \hat{t}_1 
\end{cases} \] (3.19)

in which \( F^- \) and \( F^+ \) and also initial conditions for both equations must be determined. Notice that the first equation of (3.19) is the linearization of the system (3.1) before the discontinuity, and the second one is the linearization of (3.1) after the discontinuity. We also need a suitable transition condition of the linearized equations at the instant of discontinuity. Putting
\[ F^-(x(t)) = \frac{\partial f^- (x)}{\partial x^F} \bigg|_{x=x(t)}; \] (3.20)
and considering \( \delta x(t_0) = \delta x_0 \) as an initial condition for the first equation of (3.19), will complete the first part of linearization. To accomplish the second part of linearization, if we set
\[ F^+(x(t)) = \frac{\partial f^+(x)}{\partial x^F} \bigg|_{x=x(t)}; \] (3.21)

just defining a reasonable initial condition for the second equation of (3.19) will be needed. The second equation of (3.19) is defined for \( t > t_1 \), and the time starts from \( t_1^{inf} \). Therefore by choosing \( \delta x(t_1^{inf}) = \delta x_+ \), we can find a suitable initial condition for the second part of linearization. Moreover, the relation (3.18) is the transition condition of the linearized equations at the instant of discontinuity.
Due to the mentioned explanations, the following set together with the transition condition (3.18) can describe a linearization for the Filippov system (3.1)

\[
\begin{align*}
\delta \dot{x} &= F^- (x(t)) \delta x; & \delta x(t_0) &= \delta x_0, & t_0 \leq t < t_1, \\
\delta \dot{x} &= F^+ (x(t)) \delta x; & \delta x(t_{inf}) &= \delta x_+ + \delta x_+, & t > t_1,
\end{align*}
\]

(3.22)

where \( F^- \), \( F^+ \) and \( \delta x_+ \) are stated by the relations (3.20), (3.21) and (3.18).

Using this linearization, we can calculate the spectrum of Lyapunov exponents in the case of Filippov systems. It should be mentioned that here we supposed \( kt > 0 \). Notice that the same results can be obtained for \( kt < 0 \).

4. Conclusion

Due to the wide applications of Filippov systems, here we defined Lyapunov exponents for these systems in order to investigate chaos for them. In fact to compute the complete spectrum of Lyapunov exponents of the discontinuous system (3.1), we performed a linearization for that by the aid of the properties of discontinuous systems.

REFERENCES


