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An efficient high-order compact finite difference method for the Helmholtz equation

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Abstract This paper is devoted to applying the sixth-order compact finite difference approach to the Helmholtz equation. Instead of using matrix inversion, a discrete sinusoidal transform is used as a quick solver to solve the discretized system resulted from the compact finite difference method. Through this way, the computational costs of the method with large numbers of nodes are greatly reduced. The efficiency and accuracy of the scheme are investigated by solving some illustrative examples, having the exact solutions.

Keywords. Helmholtz equation, Compact finite difference method, Fast discrete sine transform.

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1. Introduction

The Helmholtz equation, is named after Hermann von Helmholtz, as a partial differential equation, has the following form

$$\Delta u + k^2 u = g, \tag{1.1}$$

where Δ is the Laplacian operator, k is the wave number, unknown u usually represents a pressure field in the frequency domain, and g denotes the source function [35]. The Helmholtz equation governs some important physical phenomena, such as the potential in time-harmonic acoustic and electromagnetic fields, acoustic wave scattering, noise reduction in silencers, water wave propagation, membrane vibration, and radar scattering [23]. This equation appears in a natural way in the solution of the wave equation, where $k = i\omega/c_0$ is the wave number in a dispersive medium (ω is the wave frequency and c_0 the speed of light or sound) or in the solution of the linearised Poisson-Boltzmann equation, where $k = q\sqrt{8\pi\beta c/\varepsilon}$ (q is the charge of an ion, c the ionic concentration, ε the dielectric constant of the solvent, and β denotes the inverse thermal energy). Due to its importance, great effort has been made to

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develop fast and accurate methods to solve it[30]. It has been solved numerically by finite difference methods [4, 11, 14, 23, 28, 30, 35, 36, 37], finite element methods [2, 3, 5, 13, 29, 34, 39] and spectral method [21].
There are several ways to apply a finite difference approximation to Eq. (1.1). In the usual finite difference method, a second-order spatial derivative is approximated by central difference quotient which is of the second order accuracy [1]. In order to increase the order of accuracy of an approximation by the traditional finite-difference methods, the computational stencil must be widened. Whilst, adding more nodes and using smaller mesh sizes, require more computation time and storage space. This is considered a major disadvantage of a finite difference approach. Thereby, to obtain satisfactory numerical results with reasonable computational cost, high-order compact finite difference (CFD) methods are developed. Compared to the traditional explicit finite difference schemes of the same order, compact schemes are significantly more accurate, along with the benefit of using smaller stencil sizes [9]. CFD scheme was proposed for the first time by Collatz in 1966 [7]. Within the recent fifty years, various versions of the compact schemes have been analyzed and implemented successfully by many researchers, e.g. see [6, 8-10, 12, 15-20, 22-28, 30-33, 35, 36, 38]. A comprehensive investigation of the CFD methods has been performed in [17].
Solving partial differential equations by CFD schemes results in a tridiagonal or pentadiagonal systems of linear equations. Such systems are usually solved by inversion of the matrix of coefficients directly. In the present paper, an algorithm which does not need matrix inversion is proposed for employing high-order CFD schemes to solve the Helmholtz equation numerically. This algorithm utilizes the fast discrete sinusoidal transform as a fast solver and makes the application of the compact scheme more computationally cost-effectiv. It has been proposed in [32] for solving the Poisson equation. It is also used for Gross–Pitaevskii equation in [33]. In both [32] and [33] the fourth-order CFD is implemented. Here, the Dirichlet boundary value problem of the Helmholtz equation in two dimensions is discretized with the sixth-order CFD method.
The outline of the paper is as follows. Section 2 is devoted to describing the CFD scheme. In section 3, some formulas of the sixth-order CFD method based on the fast discrete sine transform are developed. In section 4, the Helmholtz equation in two dimensions is solved by the proposed method. In section 5, numerical results are demonstrated. Finally, the conclusions are presented in section 6.

2. The CFD scheme

Consider a uniformly spaced mesh consisting of the grid points $x_0, ..., x_N$ constructed by dividing the interval $[a, b]$ into N subintervals $[x_i, x_{i+1}]$ so that $x_{i+1} - x_i = h, i = 0, ..., N-1$ where $h = (b-a)/N$ is the mesh size. In the standard compact finite difference formula, a linear combination of the values of the function $u(x_j)$ at three (a few) successive points is approximated by a linear combination of the values of derivatives of the function at three (a few) successive points, with high accuracy when the derivatives are known and the function values of $u(x_i), i = 0, ..., N$ are unknown, or conversely the function values are known and derivatives are unknown [10]. The general formula of this scheme for approximating the first-order derivative, at the internal nodes, is as follows.

$$\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} = b\frac{u_{i+2} - u_{i-2}}{4h} + a\frac{u_{i+1} - u_{i-1}}{2h}, \tag{2.1}$$

where α, a, and b are the constants to be determined. Notice that u_i and u'_i denote $u(x_i)$ and $u'(x_i)$, respectively. An α-family of fourth-order tri-diagonal schemes can be applied if the following relationships are satisfied.

$$a = \frac{2}{3}(\alpha + 2), \qquad b = \frac{1}{3}(4\alpha - 1). \tag{2.2}$$

In this case, the truncation error of Eq. (2.1) is $(4/5!)(3\alpha - 1)h^4 u^{(5)}$ [17]. Setting $b = 0$ and plugging it into Eqs (2.2) lead to $\alpha = 1/4$, and $a = 3/2$. As a result, a CFD scheme with fourth-order accuracy for u'_i is given as the following

$$\frac{1}{4}u'_{i-1} + u'_i + \frac{1}{4}u'_{i+1} = \frac{3}{2}\frac{u_{i+1} - u_{i-1}}{2h}. \tag{2.3}$$

In order to obtain a sixth-order tridiagonal scheme for u'_i, consider $\alpha = 1/3$ which leads to $a = 14/9$, and $b = 1/9$. Thus, a CFD scheme with sixth-order accuracy for u'_i is achieved as follows

$$\frac{1}{3}u'_{i-1} + u'_i + \frac{1}{3}u'_{i+1} = \frac{1}{9}\frac{u_{i+2} - u_{i-2}}{4h} + \frac{14}{9}\frac{u_{i+1} - u_{i-1}}{2h}. \tag{2.4}$$

An approximation for the second-order derivative at internal nodes is obtained by the following general CDF formula

$$\alpha u''_{i-1} + u''_i + \alpha u''_{i+1} = b\frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2} + a\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \tag{2.5}$$

where α and b are again the constants that should be determined. An α-family of fourth-order tridiagonal schemes can be provided when a and b are chosen as

$$a = \frac{4}{3}(1 - \alpha), \qquad b = \frac{1}{3}(10\alpha - 1). \tag{2.6}$$

In this case, the truncation error of Eq. (2.5) is $(-4/6!)(11\alpha - 2)h^4 u^{(6)}$ [17]. If $\alpha = 1/10$, $a = 6/5$, and $b = 0$ are chosen, the CFD scheme with fourth-order accuracy can be derived for u''_i, as the following

$$\frac{1}{10}u''_{i-1} + u''_i + \frac{1}{10}u''_{i+1} = \frac{6}{5}\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \tag{2.7}$$

Choosing $\alpha = 2/11$ leads to obtain a sixth-order scheme for u''_i. Putting this value of α into Eq. (2.6) gives $a = 12/11$, and $b = 3/11$. As a result, the following CFD scheme with sixth-order accuracy is obtained for u''_i

$$\begin{aligned} \frac{2}{11}u''_{i-1} + u''_i + \frac{2}{11}u''_{i+1} = {} & \frac{3}{11}\frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2} \\ & + \frac{12}{11}\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \end{aligned} \tag{2.8}$$

3. Formulation of the CFD scheme by the discrete sine transform

In this section, the second-order partial derivatives with respect to x and y are approximated by the sixth-order CFD method in combination with the discrete sine transform. For this purpose, let $h_x = (b-a)/M$, and $h_y = (c-d)/N$ be the mesh sizes in x and y directions, respectively. In this case, the points $x_i = a + ih_x, i = 0, 1, ..., M$ and $y_j = c + jh_y, i = 0, 1, ..., N$ are assumed as the grid points. The values u_{ij} and g_{ij} approximate $u(x_i, y_j)$ and $g(x_i, y_j)$, respectively. Moreover, the values u_{ij}^{xx} and u_{ij}^{yy} approximate the second-order partial derivatives $\frac{\partial^2 u}{\partial x^2}(x_i, y_j)$ and $\frac{\partial^2 u}{\partial y^2}(x_i, y_j)$, respectively. Taking into consideration of the compact finite difference scheme of Eq. (2.8), the following approximations are obtained

$$\begin{aligned}&2u^{xx}_{i-1,j} + 11u^{xx}_{ij} + 2u^{xx}_{i+1,j} = \\ &\frac{3}{4h^2}(u_{i-2,j} + 16u_{i-1,j} - 34u_{i,j} + 16u_{i+1,j} + u_{i+2,j}),\end{aligned} \tag{3.1}$$

$$\begin{aligned}&2u^{yy}_{i,j-1} + 11u^{yy}_{i,j} + 2u^{yy}_{i,j+1} = \\ &\frac{3}{4h^2}(u_{i,j-2} + 16u_{i,j-1} - 34u_{i,j} + 16u_{i,j+1} + u_{i,j+2}).\end{aligned} \tag{3.2}$$

The two-dimensional discrete sine transform for u_{ij} and its inverse for $\widehat{u}_{kl}$ are defined as follows [32]

$$u_{ij} = \sum_{k=1}^{M-1}\sum_{l=1}^{N-1} \widehat{u}_{kl} \sin(\frac{ik\pi}{M}) \sin(\frac{jl\pi}{N}), \tag{3.3}$$

$$\widehat{u}_{kl} = \frac{2}{M}\frac{2}{N}\sum_{i=1}^{M-1}\sum_{j=1}^{N-1} \widehat{u}_{ij} \sin(\frac{ik\pi}{M}) \sin(\frac{jl\pi}{N}). \tag{3.4}$$

Applying (3.3) to Eq. (3.1) leads to

$$\begin{aligned}&2\sum_{k=1}^{M-1}\sum_{l=1}^{N-1} \widehat{u}^{xx}_{kl} \sin(\frac{(i-1)k\pi}{M}) \sin(\frac{jl\pi}{N}) + 11\sum_{k=1}^{M-1}\sum_{l=1}^{N-1} \widehat{u}^{xx}_{kl} \sin(\frac{ik\pi}{M}) \sin(\frac{jl\pi}{N}) + \\ &2\sum_{k=1}^{M-1}\sum_{l=1}^{N-1} \widehat{u}^{xx}_{kl} \sin(\frac{(i+1)k\pi}{M}) \sin(\frac{jl\pi}{N}) = \\ &\frac{3}{4h_x^2}(\sum_{k=1}^{M-1}\sum_{l=1}^{N-1} \widehat{u}_{kl} \sin(\frac{(i-2)k\pi}{M}) \sin(\frac{jl\pi}{N}) + \\ &16\sum_{k=1}^{M-1}\sum_{l=1}^{N-1} \widehat{u}_{kl} \sin(\frac{(i-1)k\pi}{M}) \sin(\frac{jl\pi}{N}) - 34\sum_{k=1}^{M-1}\sum_{l=1}^{N-1} \widehat{u}_{kl} \sin(\frac{ik\pi}{M}) \sin(\frac{jl\pi}{N}) + \\ &16\sum_{k=1}^{M-1}\sum_{l=1}^{N-1} \widehat{u}_{kl} \sin(\frac{(i+1)k\pi}{M}) \sin(\frac{jl\pi}{N}) + \sum_{k=1}^{M-1}\sum_{l=1}^{N-1} \widehat{u}_{kl} \sin(\frac{(i+2)k\pi}{M}) \sin(\frac{jl\pi}{N})),\end{aligned}$$

This equation can be rewritten as

$$
\begin{aligned}
&\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}(2\sin(\frac{(i-1)k\pi}{M})+11\sin(\frac{ik\pi}{M})+\\
&2\sin(\frac{(i+1)k\pi}{M}))\sin(\frac{jl\pi}{N}))\widehat{u}_{kl}^{xx}=\\
&\frac{3}{4h_x^2}\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}(\sin(\frac{(i-2)k\pi}{M})+16\sin(\frac{(i-1)k\pi}{M})+\\
&-34\sin(\frac{ik\pi}{M})+16\sin(\frac{(i+1)k\pi}{M})+\sin(\frac{(i+2)k\pi}{M}))\sin(\frac{jl\pi}{N}))\widehat{u}_{kl}.
\end{aligned}
$$

After some simplifications, the following equation is obtained

$$
(4\cos(\frac{k\pi}{M})+11))\widehat{u}_{kl}^{xx}=\frac{3}{2h_x^2}(\cos(\frac{2k\pi}{M})+16\cos(\frac{k\pi}{M})-17))\widehat{u}_{kl},
$$

from which, $\widehat{u}_{kl}^{xx}$ is given as

$$
\widehat{u}_{kl}^{xx}=\frac{3}{2h_x^2}(4\cos(\frac{k\pi}{M})+11)^{-1}(\cos(\frac{2k\pi}{M})+16\cos(\frac{k\pi}{M})-17)\widehat{u}_{kl}. \tag{3.5}
$$

Similarly, applying the discrete sine transform (3.3) to Eq. (3.2) leads to the following equation

$$
\begin{aligned}
&2\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}\widehat{u}_{kl}^{yy}\sin(\frac{ik\pi}{M})\sin(\frac{(j-1)l\pi}{N})+11\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}\widehat{u}_{kl}^{yy}\sin(\frac{ik\pi}{M})\sin(\frac{jl\pi}{N})+\\
&2\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}\widehat{u}_{kl}^{yy}\sin(\frac{ik\pi}{M})\sin(\frac{(j+1)l\pi}{N})=\\
&\frac{3}{4h_y^2}(\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}\widehat{u}_{kl}\sin(\frac{ik\pi}{M})\sin(\frac{(j-2)l\pi}{N})+\\
&16\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}\widehat{u}_{kl}\sin(\frac{ik\pi}{M})\sin(\frac{(j-1)l\pi}{N})-34\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}\widehat{u}_{kl}\sin(\frac{ik\pi}{M})\sin(\frac{jl\pi}{N})+\\
&16\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}\widehat{u}_{kl}\sin(\frac{ik\pi}{M})\sin(\frac{(j+1)l\pi}{N})+\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}\widehat{u}_{kl}\sin(\frac{ik\pi}{M})\sin(\frac{(i+2)l\pi}{N})).
\end{aligned}
$$

After some manipulations and simplifying, $\widehat{u}_{kl}^{yy}$ can be obtained as follows

$$
\widehat{u}_{kl}^{yy}=\frac{3}{2h_y^2}(4\cos(\frac{l\pi}{N})+11)^{-1}(\cos(\frac{2l\pi}{N})+16\cos(\frac{l\pi}{N})-17)\widehat{u}_{kl}. \tag{3.6}
$$

4. Approximation of the solution of the Helmholtz equation

The two-dimensional Helmholtz equation with zero Dirichlet boundary condition is considered as follows.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+k^2u=g, \quad in\ \Omega:=(a,b)\times(c,d), \tag{4.1}$$

with boundary condition

$$u=0, \quad on\ \Gamma:=\partial\Omega,$$

where $u=u(x,y)$ is an unknown, and $g=g(x,y)$ is a given function.
Now, the Helmholtz equation is discretized at the grid point (x_i,y_j) as follows

$$u_{ij}^{xx}+u_{ij}^{yy}+k^2u_{ij}=g_{ij}. \tag{4.2}$$

Applying the inverse sine transform to this equation yields to the following equations

$$\widehat{u}_{kl}^{xx}+\widehat{u}_{kl}^{yy}+k^2\widehat{u}_{kl}=\widehat{g}_{kl}, \qquad 1\le k\le M-1,\ 1\le l\le N-1, \tag{4.3}$$

where

$$\widehat{g}_{kl}=\frac{2}{M}\frac{2}{N}\sum_{k=1}^{M-1}\sum_{l=1}^{N-1}g_{ij}\sin(\frac{ik\pi}{M})\sin(\frac{jl\pi}{N}).$$

If Eqs.(3.5) and (3.6) are substituted into (4.3), $\widehat{u}_{kl}$ can be obtained as

$$\widehat{u}_{kl}=\frac{2\widehat{g}_{kl}}{3\left(\frac{\cos(\frac{2k\pi}{M})+16\cos(\frac{k\pi}{M})-17}{h_x^2(4\cos(\frac{k\pi}{M})+11)}+\frac{\cos(\frac{2l\pi}{N})+16\cos(\frac{l\pi}{N})-17}{h_y^2(4\cos(\frac{l\pi}{N})+11)}+\frac{2}{3}k^2\right)}. \tag{4.4}$$

Finally, by means of the discrete sine transform (3.3), u_{ij} can be calculated from $\widehat{u}_{kl}$. It is worth to notice that if one solves the discretized system (4.2) by direct matrix inversion, then the computational cost will be $O(M^2N^2)$ while formula (4.4) reduces the computational cost to $O(MNlog(MN))$.

5. Numerical experiments

In this section, a numerical example that includes the Helmholtz problem, with a constant wave number is presented to illustrate the efficiency of the proposed scheme. All computations are performed by Matlab 2017. The Helmholtz equation is considered as follows.

$$\Delta u+k^2u=-k^2\sin(kx)\sin(ky), \quad in\ \Omega:=(0,\pi)\times(0,\pi), \tag{5.1}$$

with the boundary condition

$$u=0, \qquad on\ \Gamma:=\partial\Omega,$$

where k is assumed as a positive integer. The exact solution to this problem has the following form [35].

$$u(x,y)=\sin(kx)\sin(ky), \qquad (x,y)\in[0,\pi]\times[0,\pi].$$

The L_∞-norm is used to measure the error of approximated solution. Numerical results are shown in Figures 1-4. In both Figures 1 and 2, we take $M=N=64$. Figure 1 shows the approximated solution of the problem (5.1) with $k=30$ at the

Table 1. Error analysis for $k = 30$.

$M = N$	32	64	128	256	512
Proposed method	0.8099	0.0070	1.0015e-04	1.5275e-06	2.4115e-08
Optim. Compact 9 p	4.3584	0.0911	0.0034	1.8977e-04	1.550e-05

Table 2. Error analysis for $k = 50$.

$M = N$	64	128	256	512	1024
Proposed method	0.1944	0.0023	3.3215e-05	5.1029e-07	8.1124e-09
Optim. Compact 9 p	17.2907	0.0349	0.0016	9.0539e-05	5.5554e-06

grid points (x_i, y_j). In Figure 2, the error of the approximated solution is shown. Maximum error is less than 8×10^{-3}. In both Figures 3 and, 4 we take $M = N = 128$. Figure 3 shows the approximated solution of the problem (5.1) with $k = 50$ at the grid points (x_i, y_j). In Figure 4, the error of the approximated solution is shown. Maximum error is less than 2.5×10^{-3}. Furthermore, In Tables 1 and 2, the results are compared with those reported in [35] where an optimal compact 9 points method is proposed to solve the Helmholtz equation (5.1). At both tables, number of grid points are doubly increased. Tables 1 and 2 show that the accuracy of the proposed method in this study is higher than the optimal compact 9 points method.

Figure 1. Approximated solution of problem (5.1) in grid points $(x_i, y_j)(i = 1, ..., 64 - 1, j = 1, ..., 64 - 1)$, with wave number $k = 30$.

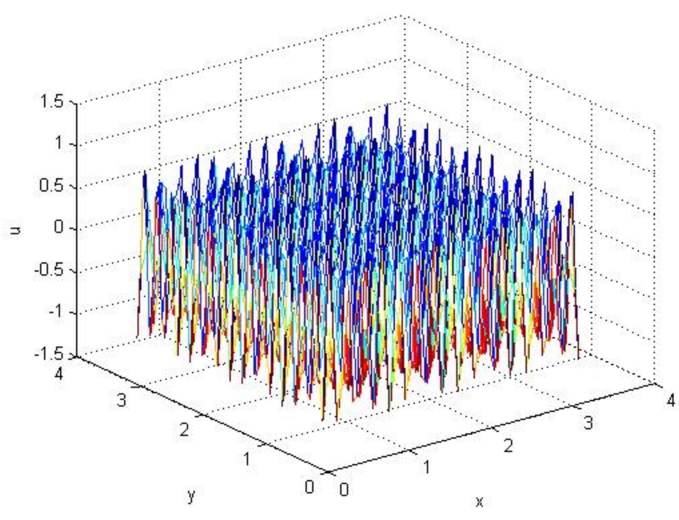


FIGURE 2. Error in L_∞-norm between approximated solution and exact solution of problem (5.1) at grid points, with wave number $k = 30$.

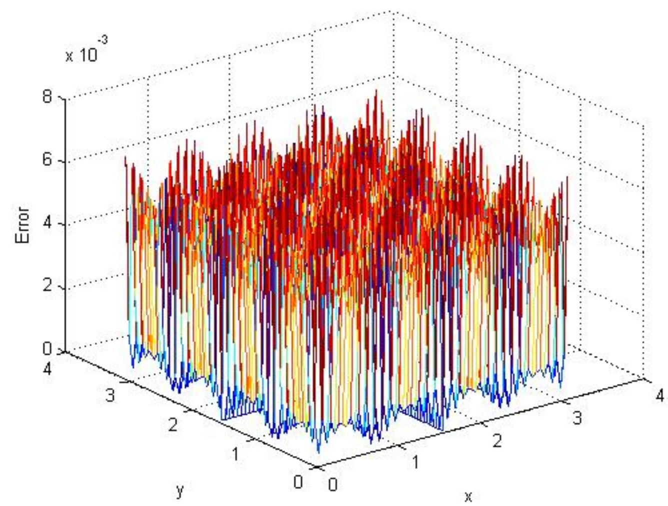


FIGURE 3. Approximated solution of problem (5.1) in grid points $(x_i, y_j)(i = 1, ..., 128-1, j = 1, ..., 128-1)$, with wave number $k = 50$.

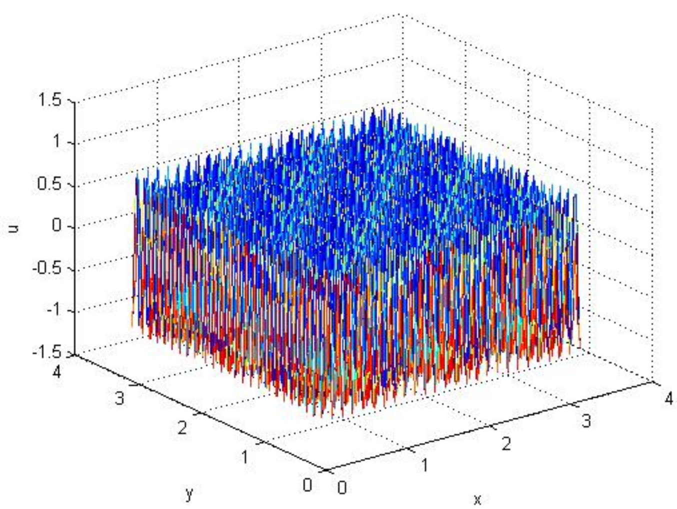
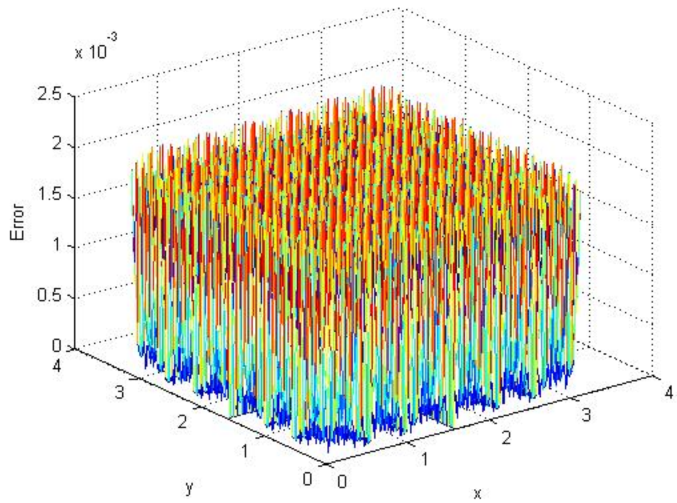


FIGURE 4. Error in L_∞-norm between approximated solution and exact solution of problem (5.1) at grid points, with wave number $k = 50$.



6. Conclusion

Compact finite difference methods are well-known tools to discretize and solve partial differential equations. Applying such methods for partial differential equations leads to a tridiagonal or pentadiagonal system of linear equations which is usually solved by matrix inversion. However, the cost of this method increases when it is applied to a higher-dimensional problem. In this study, an algorithm, based on the sixth-order CFD method, is designed for solving the Helmholtz equation with Dirichlet boundary conditions. This algorithm does not use matrix inversion. It uses the fast discrete sine transform. This makes a compact scheme more cost-effective. The numerical results for two-dimensional equations confirm the acuracy of the proposed sixth-order compact finite difference scheme.

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