Haar wavelet iteration method for solving time fractional Fisher’s equation

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Abstract
In this work, we investigate fractional version of the Fisher equation and solve it by using an efficient iteration technique based on the Haar wavelet operational matrices. In fact, we convert the nonlinear equation into a Sylvester equation by the Haar wavelet collocation iteration method (HWCIM) to obtain the solution. We provide four numerical examples to illustrate the simplicity and efficiency of the technique.

Keywords. Fractional differential equation, Haar wavelet, Operational matrices, Numerical solution, iterative technique, Sylvester equation.

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1. Introduction

Fractional calculation has different applications in physics and engineering such as electrodynamics, acoustics, viscoelasticity, electrochemistry and material science (see for example [1, 13, 18, 23]). In 1937 Fisher, Kolmogorov, Petrovsky and Piscounov [5, 10] investigated independently the Fisher-KPP equation, after than, it is extensively known as the Fisher’s equation. Fisher’s equation belong to the class of reaction-diffusion equation, it is one of the simplest nonlinear r.d.e. Fisher proposed this equation in his article, "The wave of advance of advantageous genes" in 1937, which is about of population dynamics to describe the spatial spread of an advantageous allele and explored its travelling wave solutions.

Distinct numerical methods have been applied for approximate solutions of Fisher’s equation including the Adomian decomposition method (ADM) [9], the homotopy perturbation method (HPM) [24], the variational iteration method (VIM) [15], the Haar...
wavelet method [3, 4, 7, 8, 11, 17, 20] and the tanh method [21]. As you know the time
fractional Fisher’s equation, which is a mathematical model for wide range of impor-
tant physical phenomena, is a partial differential equation obtained from the classical
Fisher’s equation by replacing the time derivative with a fractional derivative of order
\(0 < \alpha \leq 1\). In fact, the Fisher’s equation as a model for the propagation of a mutant
gene, is encountered in chemical kinetics and population dynamics which includes
problems such as nonlinear evolution of a population in a one-dimensional habitat,
neutron population in a nuclear reaction. Moreover, the same equation occurs in
logistic population growth models, flame propagation, neurophysiology, autocatalytic
chemical reactions and branching Brownian motion processes [14, 16]. Due to the
occurrence of Fisher’s equation in many biological and chemical processes, it is one
of the most important classes of nonlinear equations [6].

There are different kinds of wavelets. The simplest orthogonal wavelet with com-
pact support is the Haar wavelet. Since the derivatives do not exist in the points of
 discontinuity, it is not possible to apply the Haar wavelets for solving differential equa-
tion directly. We use Haar wavelet based on the collocation method. In this paper,
we use combination of Haar wavelet and iteration technique for numerical solutions
of the modified Fisher’s equation with time fractional derivative of the form

\[
  D_\alpha^t u(x, t) = D_{xx} u(x, t) + \lambda u(x, t)(1 - u^n(x, t)) + q(x, t),
\]

\( (x, t) \in [0, 1] \times [0, 1], u(x, 0) = f(x), u(0, t) = y_1(t), u(1, t) = y_2(t), \)

where \(0 < \alpha \leq 1\), \(\lambda\) is real parameter, \(D_\alpha^t\) denotes the Caputo fractional derivative in
time and \(y_1(t), y_2(t), f(x)\) and \(q(x, t)\) are known functions. We discretize the nonlinear
fractional partial differential equation by iteration technique (Picard iteration) and
then convert the obtained discritized equation into a Sylvester equation by the Haar
wavelet collocation method to obtain the solution.

2. Operational matrices

2.1. Integer and fractional integration of operational matrix. The \(i\)-th uni-
form Haar wavelet \(h_i(x)\) is defined by

\[
  h_i(x) = \begin{cases} 
    1 & a(i) \leq x \leq b(i) \\
    -1 & b(i) \leq x < c(i) \\
    0 & \text{otherwise,}
  \end{cases}
\]

where \(a(i) = \frac{k-1}{m}, b(i) = \frac{k-0.5}{m}, c(i) = \frac{k}{m}, i = 2^j + k + 1, j = 0, 1, 2, 3, \ldots, J,\)
\(x \in [0, 1]\), \(j\) is a dilation parameters, \(m = 2^j\) and \(k = 0, 1, 2, \ldots, 2^j - 1\) are translation
parameter. The Maximum level of resolution is \(J\). In particular \(h_1(x) = \chi_{[0,1]}(x)\) is
the Haar scaling function, where \(\chi_{[0,1]}(x)\) is characteristic function on interval \([0,1]\).
Let us define the collocation points \(x_l = \frac{l-0.5}{m}\) where \(l = 1, 2, 3, \ldots, m\). We establish
an operational matrix for integration via Haar wavelet. The operational matrix of
integration of integer order is obtained by integrating Eq. (2.1) as following
\[
P_{\alpha,1}(x) = I_{a(1)}^{\alpha}h_1(x) = \frac{1}{\Gamma(\alpha)} \int_{a(1)}^{x} (x - s)^{\alpha-1}ds, \quad \alpha > 0
\]
\[
P_{\alpha,i}(x) = I_{a(i)}^{\alpha}h_i(x) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_{a(i)}^{b(i)} (x - s)^{\alpha-1}ds & a(i) \leq x < b(i), \\
\frac{1}{\Gamma(\alpha)} \int_{a(i)}^{b(i)} (x - s)^{\alpha-1}ds - \int_{b(i)}^{x} (x - s)^{\alpha-1}ds & b(i) \leq x < c(i), \\
\int_{a(i)}^{b(i)} (x - s)^{\alpha-1}ds - \int_{b(i)}^{c(i)} (x - s)^{\alpha-1}ds & x \geq c(i) \end{cases}
\]
(2.2)

By simplifying, we get
\[
P_{\alpha,1}(x) = \frac{(x - a(1))^{\alpha}}{\Gamma(\alpha + 1)}
\]
(2.3)
and
\[
P_{\alpha,i}(x) = I_{a(i)}^{\alpha}h_i(x) = \frac{1}{\Gamma(\alpha)} \begin{cases} 
(x - a(i))^{\alpha}, & a(i) \leq x < b(i); \\
(x - a(i))^{\alpha} - 2(x - b(i))^{\alpha}, & b(i) \leq x < c(i); \\
(x - a(i))^{\alpha} - 2(x - b(i))^{\alpha} + (x - c(i))^{\alpha}, & x \geq c(i) \end{cases}
\]
(2.4)

Each function \(y \in L_2[0, 1]\) can be expressed in terms of the Haar wavelet by:
\[
y(x) = \sum_{i=1}^{\infty} c_i h_i(x),
\]
where \(c_i\)s are the Haar wavelet coefficients, given by \(c_i = \int_{0}^{1} y(x)h_i(x)dx\). We can approximate the function \(y(x)\) by the truncated series:
\[
y(x) \approx \sum_{i=1}^{m-1} c_i h_i(x).
\]
(2.5)

By taking the collocation points \(x(l) = \frac{l - 0.5}{m}\) for \(l = 1, 2, \ldots, m\), we define Haar wavelet matrix \(H_{m \times m}\) by
\[
H_{m \times m} = \begin{pmatrix}
    h_1(x(1)) & h_1(x(2)) & \cdots & h_1(x(m)) \\
    h_2(x(1)) & h_2(x(2)) & \cdots & h_2(x(m)) \\
    \vdots & \vdots & \ddots & \vdots \\
    h_m(x(1)) & h_m(x(2)) & \cdots & h_m(x(m))
\end{pmatrix}
\]

We can represent Eq. (2.5) in vector form as \(y = cH\), where \(c = [c_1, c_2, \ldots, c_m]\).

The Haar coefficients \(c_i\) can be evaluated by \(c = yH^{-1}\) where \(H^{-1}\) is inverse of \(H\).

Similarly, we can obtain the fractional order integration operational matrix \(P\) of Haar function by substituting the collocation points in Eq. (2.3) and Eq. (2.4) as following:
\[
P_{\alpha}^{m \times m} = \begin{pmatrix}
P_{\alpha,1}(x(1)) & P_{\alpha,1}(x(2)) & \cdots & P_{\alpha,1}(x(m)) \\
P_{\alpha,2}(x(1)) & P_{\alpha,2}(x(2)) & \cdots & P_{\alpha,2}(x(m)) \\
\vdots & \vdots & \ddots & \vdots \\
P_{\alpha,m}(x(1)) & P_{\alpha,m}(x(2)) & \cdots & P_{\alpha,m}(x(m))
\end{pmatrix}
\]
For example, if \( m = 8 \) and \( \alpha = 0.9 \), the Haar wavelet operational matrix of fractional integration is given by:

\[
P^{0.9}_{8 \times 8} = \begin{pmatrix}
0.0857 & 0.2305 & 0.3650 & 0.4941 & 0.6195 & 0.7421 & 0.8625 & 0.9811 \\
0.0857 & 0.2305 & 0.3650 & 0.4941 & 0.4480 & 0.2812 & 0.1325 & -0.0071 \\
0.0857 & 0.2305 & 0.1935 & 0.0331 & -0.0248 & -0.156 & -0.115 & -0.0091 \\
0.0857 & 0.0590 & -0.0102 & -0.0054 & -0.0037 & -0.0028 & -0.0022 & -0.0018 \\
0 & 0 & 0 & 0 & 0.0857 & 0.2305 & 0.1935 & 0.0331 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0857 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0857 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0857
\end{pmatrix}
\]

2.2. Haar wavelet operational matrix of fractional integration for boundary value problems. We derive another operational matrix of fractional integration to solve the fractional boundary value problems. Let \( \eta > 0 \), \( g : [0, \eta] \to \mathbb{R} \) be a continuous function and \([0, \eta]\) be the Haar function support. Put

\[
g(x)I_0^\alpha h_1(\eta) = g(x) \int_0^\eta (\eta - s)^{\alpha-1} ds,
\]

\[
v^{\alpha,1} = g(x)C_{\alpha,1}
\]

and

\[
g(x)I_0^\alpha h_i(\eta) = g(x) \left\{ \int_{a(i)}^{b(i)} (\eta - s)^{\alpha-1} ds - \int_{c(i)}^{d(i)} (\eta - s)^{\alpha-1} ds \right\},
\]

\[
v^{\alpha,i} = g(x)C_{\alpha,i},
\]

where \( C_{\alpha,1} = \frac{\eta^\alpha}{\Gamma(1+\alpha)} \) and \( C_{\alpha,i} = \frac{1}{\Gamma(1+\alpha)} \left[ (\eta - a(i))^\alpha - 2(\eta - b(i))^\alpha + (\eta - c(i))^\alpha \right] \). By using the introduced collocation points, we obtain

\[
V_{m \times m}^{\alpha,\eta}g(x) = \begin{pmatrix}
g(x(1))I_0^\alpha h_1(\eta) & g(x(2))I_0^\alpha h_1(\eta) & \cdots & g(x(m))I_0^\alpha h_1(\eta) \\
g(x(1))I_0^\alpha h_2(\eta) & g(x(2))I_0^\alpha h_2(\eta) & \cdots & g(x(m))I_0^\alpha h_2(\eta) \\
\vdots & \vdots & \ddots & \vdots \\
g(x(1))I_0^\alpha h_m(\eta) & g(x(2))I_0^\alpha h_m(\eta) & \cdots & g(x(m))I_0^\alpha h_m(\eta)
\end{pmatrix}
\]

3. Method of solution

In this section, we describe the solving method of the time fractional modified Fisher’s equation. The first step is to convert the fractional nonlinear partial differential equation to a linear partial differential equation by using the iteration technique. The second step is to solve the obtained discretized fractional partial differential equation by Haar wavelet operational matrix method.

Consider the problem:

\[
D_t^\alpha u(x,t) = D_{xx}u(x,t) + \lambda u(x,t)(1-u^n(x,t)) + q(x,t), \quad (3.1)
\]

\[0 < \alpha \leq 1, \quad \lambda \in \mathbb{R}, \quad n \geq 1,
\]

with the initial and boundary conditions:

\[
u(x,0) = g(x), \quad u(0,t) = y_1(t), \quad u(1,t) = y_2(t) \quad (t \geq 0, \quad 0 \leq x \leq 1).
\]
By applying the iteration method of Picard to Eq. (3.1) (see [2]), we get the following problem:

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} - \frac{\partial^2 u_{r+1}}{\partial x^2} = \lambda u_r(x, t)(1 - u_r(x, t)) + q(x, t), \quad (0 < \alpha \leq 1, \ \ r \geq 0)$$  \hspace{1cm} (3.2)

with the initial and boundary conditions:

$$u_{r+1}(x, 0) = g(x), \quad u_{r+1}(0, t) = y_1(t), \quad u_{r+1}(1, t) = y_2(t), \quad (t \geq 0, \ 0 \leq x \leq 1).$$

For applying the fractional integral operator

By using the boundary conditions and putting

Thus, the Eq. (3.4) can be rewritten as:

Now for simplicity, we denote the right side of Eq. (3.2) by $S(x, t)$, that is,

where $m_{i,j} = \langle h_i(x), \langle S(x, t), h_j(t)\rangle \rangle$. By substituting Eq. (3.6) and Eq. (3.3) in Eq. (3.2), we get:

By applying fractional integral operator $I^\alpha_x$ to Eq. (3.7) and using the initial condition, we obtain:

Now, from Eqs. (3.8) and (3.5) we get:

where $K(x, t) = -g(x) + x(y_2(t) - y_1(t)) + y_1(t)$. By discarding Eq. (3.9) and using collocation points, we obtain the matrix form:

$$\left((P^2_x)^T - V^{2,1,g(x)}\right) C^{r+1} H - H^T C^{r+1} P^\alpha_t - H^T M P^\alpha_t + K = 0,$$

$$\frac{\partial^2 u_{r+1}}{\partial x^2} = \sum_{i=1}^{m} \sum_{j=1}^{m} C^{r+1}_{i,j} h_i(x) h_j(t) = H^T(x) C^{r+1} H(t).$$  \hspace{1cm} (3.3)

By applying the fractional integral operator $I^\alpha_x$ on the Eq. (3.3), we obtain:

$$u_{r+1}(x, t) = (P^2_x)^T C^{r+1} H(t) + p(t)x + q(t).$$  \hspace{1cm} (3.4)

By using the boundary conditions and putting $x = 0$ and $x = 1$, we get $q(t) = y_1(t)$ for $x = 0$ and $p(t) = y_2(t) - y_1(t) - (P^2_x(1))^T C^{r+1} H(t)$ for $x = 1$. Thus, the Eq. (3.4) can be rewritten as:

$$u_{r+1}(x, t) = (P^2_x)^T C^{r+1} H(t) - x\{ (P^2_x(1))^T C^{r+1} H(t) \} + x(y_2(t) - y_1(t)) + y_1(t).$$  \hspace{1cm} (3.5)

$$S(x, t) = \lambda u_r(x, t)(1 - u_r(x, t)) + q(x, t)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} m_{i,j} h_i(x) h_j(t) = H^T(x) M H(t),$$

$$K(x, t) + (P^2_x)^T C^{r+1} H(t) - x\left( (P^2_x(1))^T C^{r+1} H(t) \right)$$

$$- H^T(x) C^{r+1} P^\alpha_t - H^T(x) M P^\alpha_t = 0,$$
Proof. Since Theorem 4.1. Consider the functions $u_m(x, t)$ obtained by the Haar wavelet with approximation $u(x, t)$. Then, $\|u(x, t) - u_m(x, t)\|_E \leq \frac{K_m}{\sqrt{2^m}}$, where

$$\|u(x, t)\|_E = \left( \int_0^1 \int_0^1 u^2(x, t) dt dx \right)^{1/2}$$

Proof. Since $u_m(x, t) = \sum_{n=0}^{m-1} \sum_{l=0}^{l-1} u_n h_n(x) h_l(t)$, we have

$$u(x, t) - u_m(x, t) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} u_n h_n(x) h_l(t) = \sum_{n=2^p+1}^{\infty} \sum_{l=2^p+1}^{\infty} u_n h_n(x) h_l(t).$$

The orthogonality of the functions $h_k(x)$ on $[0, 1)$ implies that $h_l(\cdot) = 2^k h(2^k \cdot -k)$. Hence,

$$\|u(x, t) - u_m(x, t)\|_E^2 = \int_0^1 \int_0^1 (u(x, t) - u_m(x, t))^2 dx dt$$

$$= 2^l \sum_{n=2^p+1}^{\infty} \sum_{l=2^p+1}^{\infty} \sum_{n'=2^p+1}^{\infty} \sum_{l'=2^p+1}^{\infty} u_{n'l'} u_{n'l'} \left( \int_0^1 h_n(x) h_{n'}(x) dx \right) \left( \int_0^1 h_l(t) h_{l'}(t) dt \right)$$

$$= 2^l \sum_{n=2^p+1}^{\infty} \sum_{l=2^p+1}^{\infty} u_{n'l'}^2$$
where \( u_{nl} = \left\langle h_n(x), (u(x,t), h_t(t)) \right\rangle \). By using Eq. (2.1), we get:

\[
\left\langle u(x,t), h_t(t) \right\rangle = \int_0^1 u(x,t)h_t(t)dt = 2^j \left( \int_{k-0.5 \over 2j}^{k \over 2j} u(x,t)dt - \int_{k-0.5 \over 2j}^{k-1 \over 2j} u(x,t)dt \right).
\]

By using mean value theorem of integrals, there exist \( {k-1 \over 2j} \leq t_1 < {k-0.5 \over 2j} \) and \( {k-0.5 \over 2j} < t_2 < {k \over 2j} \) such that

\[
\left\langle u(x,t), h_t(t) \right\rangle = 2^j \left( \left( k - 0.5 \over 2j - \left( k - 1 \over 2j \right) u(x,t_1) - \left( k - 0.5 \over 2j \right) u(x,t_2) \right)
\]

and

\[
u_{nl} = \left\langle h_n(x), {1 \over 2j+1} (u(x,t_1) - u(x,t_2)) \right\rangle
\]

By using Eq. (4.2), we get:

\[
u_{nl} = {1 \over 2j+1} \int_0^1 h_n(x)(u(x,t_1) - u(x,t_2))dx = 2^j \left( \int_{k-0.5 \over 2j}^{k \over 2j} u(x,t_1)dx - \int_{k-0.5 \over 2j}^{k\over 2j} u(x,t_1)dx \right)
\]

\[
- \int_{k-1 \over 2j}^{k \over 2j} u(x,t_2)dx + \int_{k-1 \over 2j}^{k \over 2j} u(x,t_1)dx.
\]

By using mean value theorem of integrals, choose \( {k-1 \over 2j} \leq x_1, x_2 < {k-0.5 \over 2j} \) and \( {k-0.5 \over 2j} \leq x_3, x_4 < {k \over 2j} \) such that

\[
u_{nl} = \left\{ \left( k - 0.5 \over 2j - \left( k - 1 \over 2j \right) u(x_1,t_1) - \left( k - 0.5 \over 2j \right) u(x_2,t_1) - \left( k - 0.5 \over 2j \right) u(x_3,t_2) + \left( k \over 2j - \left( k - 0.5 \over 2j \right) u(x_4,t_2) \right) \right\}
\]

and so \( u_{nl}^2 \leq \left\{ \left( u(x_1,t_1) - u(x_2,t_1) \right) - \left( u(x_3,t_2) - u(x_4,t_2) \right) \right\}^2 \). By using mean value theorem of derivatives, there exist \( x_1 \leq \xi_1 < x_2 \) and \( x_3 \leq \xi_2 < x_4 \) such that

\[
u_{nl}^2 \leq \left\{ \left( x_2 - x_1 \right)^2 \left( \left| \frac{\partial u(\xi_1,t_1)}{\partial x} \right| \right)^2 + \left( x_4 - x_3 \right)^2 \left( \left| \frac{\partial u(\xi_1,t_1)}{\partial x} \right| \right)^2 + \left( x_2 - x_1 \right) \left( x_4 - x_3 \right) \left( \left| \frac{\partial u(\xi_2,t_2)}{\partial x} \right| \right)^2 \right\}.
\]
By using a similar procedure, we can show that \( \frac{\partial u(x, t)}{\partial x} \) is bounded on \((0, 1) \times (0, 1)\), there exists \( K > 0 \) such that \( \left| \frac{\partial u(x, t)}{\partial x} \right| \leq K \) for all \( x, t \in (0, 1) \times (0, 1) \). This implies that \( u_{n+1} \leq (\frac{1}{2^{2r+\alpha}}) \frac{4K^2}{2^{2r+\alpha}} = \frac{4K^2}{2^{2r+\alpha}} \) and so

\[
\|u(x, t) - u_m(x, t)\|_E = \sum_{j=p+1}^{\infty} \left( \sum_{n=2^j}^{2^{j+1}-2} \sum_{m=2^j}^{2^{j+1}-1} u_{n+1}^2 \right) \leq \sum_{j=p+1}^{\infty} \left( \sum_{n=2^j}^{2^{j+1}-2} \sum_{m=2^j}^{2^{j+1}-1} \frac{4K^2}{2^{2j+\alpha}} \right)
\]

\[
= 4K^2 \sum_{j=p+1}^{\infty} \left( \sum_{n=2^j}^{2^{j+1}-2} \sum_{m=2^j}^{2^{j+1}-1} \frac{1}{2^{4j+4}} \right)
\]

\[
= K^2 \frac{1}{4^{p+1}} = K^2 \frac{1}{3^{p+1}}.
\]

Thus, \( \|u(x, t) - u_m(x, t)\|_E \leq \frac{K}{\sqrt{3}m} \) and so \( \|u(x, t) - u_m(x, t)\|_E \to 0 \) when \( m \to \infty \). By using a similar procedure, we can show that \( \|u_{r+1}(x, t) - u_{m+1}(x, t)\|_E \leq \frac{K}{\sqrt{3}m} \) which implies that error between the exact and approximate solution at the \( (r+1) \)th iteration is inversely proportional to the maximal level of resolution. This shows that \( u_{m+1}(x, t) \) converges to \( u_{r+1}(x, t) \) as \( m \to \infty \). Since \( u_{r+1}(x, t) \) is obtained at \( (r+1) \)th iteration of Picard technique, we conclude that \( u_{r+1}(x, t) \) converges to \( u(x, t) \) as \( r \to \infty \). Thus, \( \lim_{m, r \to \infty} u_{m+1}(x, t) = u(x, t) \).

5. Experiments and results

In this section, we use the Haar wavelet collocation iteration method (HWCIM) to solve the fractional Fisher’s equations. We provide four examples to illustrate the method.

**Example 1.** Consider the problem (1.1) for \( \lambda = 1, n = 6, g(x, t) = 0 \).

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(x, t) \left( 1 - u^6(x, t) \right)
\]

with the initial and boundary conditions:

\[
u(x, 0) = g(x) = \left( \frac{1}{1 + e^{\frac{x}{2}}} \right)^\frac{1}{2}, \quad u(0, t) = y_1(t) = \left( \frac{1}{1 + e^{\frac{t}{2}}} \right)^\frac{1}{2},
\]

\[
u(1, t) = y_2(t) = \left( \frac{1}{1 + e^{\frac{t}{2}}} \right)^\frac{1}{2}.
\]

It is known that the problem has the exact solution \( u_{exact} = \left( \frac{1}{1 + e^{\frac{t}{2}}} \right)^\frac{1}{2} \) for \( \alpha = 1 \).

We put \( u_0(x, t) = \left( \frac{1}{1 + e^{\frac{t}{2}}} \right)^\frac{1}{2} \) and apply the Haar wavelet iteration technique. By applying the iteration technique of Picard, we obtain:

\[
\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} - \frac{\partial^2 u_{r+1}}{\partial x^2} = u_r \left( 1 - (u_r)^6 \right)
\]

(5.2)
with the initial and boundary conditions: \( u_{r+1}(x,0) = g(x) \), \( u_{r+1}(0,t) = y_1(t) \) and \( u_{r+1}(1,t) = y_2(t) \). We suppose that
\[
\frac{\partial^2 u_{r+1}}{\partial x^2} = H^T(x)C^{r+1}H(t) \tag{5.3}
\]
Now, by applying the fractional integral operator \( I_t^\alpha \) on the Eq. (5.3), we get:
\[
u_{r+1}(x,t) = (P_x^2)^T C^{r+1}H(t) + p(x)t + q(t). \tag{5.4}
\]
By using the boundary conditions and putting \( x = 0 \) and \( x = 1 \), we get \( q(t) = y_1(t) \) for \( x = 0 \) and \( p(t) = y_2(t) - y_1(t) - (P_x^2)^T C^{r+1}H(t) \) for \( x = 1 \). Thus, the Eq. (5.4) can be rewritten as:
\[
\frac{\partial^2 u_{r+1}}{\partial x^2} = H^T(x)C^{r+1}H(t) - x(P_x^2)^T C^{r+1}H(t) + x(y_2(t) - y_1(t)) + y_1(t). \tag{5.5}
\]
We put the right of Eq. (5.2) by \( S(x,t) \), and we estimate by Haar wavelet, that is,
\[
u_r(x,t)(1 - (\nu_r(x,t))^6) = S(x,t) \tag{5.6}
\]
\[
= \sum_{i=1}^{m} \sum_{j=1}^{m} m_{i,j} h_i(x) h_j(t) = H^T M H(t),
\]
where \( m_{i,j} = \langle h_i(x), \langle S(x,t), h_j(t) \rangle \rangle \). We substitute Eqs. (5.6) and (5.3) in Eq. (5.2),
\[
\frac{\partial^2 u_{r+1}}{\partial t^\alpha} = H^T(x)C_{r+1}H(t) + H^T(x)MH(t). \tag{5.7}
\]
We apply fractional integral operator \( I_t^\alpha \) to Eq. (5.7) and use the initial condition,
\[
u_{r+1}(x,t) = H^T(x)C_{r+1}P_t^\alpha + H^T(x)MP_t^\alpha + g(x). \tag{5.8}
\]
Now, from Eqs. (5.8) and (5.4) we get:
\[
K(x,t) + (P_x^2)^T C^{r+1}H(t) - x((P_x^2(1))^T C^{r+1}H(t)) \tag{5.9}
\]
\[-H^T(x)C^{r+1}P_t^\alpha - H^T(x)MP_t^\alpha = 0,
\]
where \( K(x,t) = -q(x) + x(y_2(t) - y_1(t)) + y_1(t) \). By discarding Eq. (5.9) and using collocation points, we obtain the matrix form as follow:
\[
\left( (P_x^2)^T - V_{21, g(x)} \right) C^{r+1}H - H^T C^{r+1}P_t^\alpha - H^T MP_t^\alpha + K = 0, \tag{5.10}
\]
where \( H \) is the \( m \times m \) Haar matrix, \( V_{21, g(x)} = g(x)I_t^\alpha \) \( H^T(x) = x(P_1^2(1))^T \) is the \( m \times m \) fractional integration matrix for boundary value problem, \( P_t^\alpha = I_t^\alpha H^T \) and \( P_t^\alpha = I_t^\alpha H \) are the \( m \times m \) matrices of fractional integration of the Haar function. Also, \( K = K(x(i), t(i)) (i = 1, 2, \ldots, m) \) is the matrix determined by using the collocation points. Note that, the Eq. (5.10) can be rewritten by:
\[
(H^T)^{-1} \left( (P_x^2)^T - V_{21, g(x)} \right) C^{r+1} - C^{r+1} P_t^\alpha (H^{-1}) = (H^T)^{-1} \left( H^T M P_t^\alpha - K \right) (H^{-1}), \tag{5.11}
\]
where is the Sylvester equation. By considering \( r = 0 \) and initial approximation \( u_0(x,t) \) and solving the Sylvester equation by Matlab software, we obtain \( C^1 \). Then,
by substituting $C^1$ to Eq. (5.8) or Eq. (5.4) we get $u_1(x, t)$ at the collocation points. Similarly for $r = 2$, we obtain $u_2(x, t)$ and so on.

The numerical results for $m = 64$ are given in Figure 1 and Table 1. Note that, Table 1 shows that the solutions obtained by present method for different value of $\alpha$, convergence to the exact solution whenever $\alpha$ tends to 1. Also, the obtained numerical results have been compared with HPM [12], [22] and MVIM [16].

**Example 2.** Consider the problem (1.1) for $\lambda = 1$, $n = 1$ and $q(x, t) = 0$ ,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(x, t)(1 - u(x, t))$$

with the initial and boundary conditions

$$u(x, 0) = \beta, u(0,t) = \frac{\beta e^t}{1-\beta+\beta e^t}, u(1,t) = \frac{\beta e^t}{1-\beta+\beta e^t},$$

Where $\beta$ be a constant. It is known that the problem has the exact solution $u_{\text{exact}} = \frac{\beta e^t}{1-\beta+\beta e^t}$ for $\alpha = 1$.

Put $u_0(x, t) = \beta$ and apply the Haar wavelet iteration technique. The numerical results for $m = 64$ and $\beta = 2/3$ at 4 iterations are given in Figure 3, Figure 5 and Table 2. Note that, Table 2 shows that the solutions obtained by present method for different value of $\alpha$, convergence to the exact solution whenever $\alpha$ tends to 1. Also, the obtained numerical results have been compared with HPM [24] and MVIM [16] and Figure 5 shows that the approximate results are closer to the exact solution by increasing the resolution ($m$).

**Example 3.** Consider the problem (1.1) for $\lambda = 6$, $n = 1$ and $q(x, t) = 0$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + 6u(x, t)(1 - u(x, t))$$

with the initial and boundary conditions:

$$u(x, 0) = \frac{1}{(1+e^x)^2}, u(0,t) = \frac{1}{(1+e^{-5t})^2}, u(1,t) = \frac{1}{(1+e^{-5t})^2}.$$

<table>
<thead>
<tr>
<th>method</th>
<th>HW CIM</th>
<th>HW CIM</th>
<th>HW CIM</th>
<th>HW CIM</th>
<th>MVIM [16]</th>
<th>HPM [12]</th>
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</tbody>
</table>
It is known that the problem has the exact solution $u_{\text{exact}} = \frac{1}{(1+e^{-\alpha t})^2}$ for $\alpha = 1$. 
Table 2. Absolute error for Example 2, \( m = 64 \) and \( \alpha \rightarrow 1 \).

<table>
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<th>HW CIM</th>
<th>HW CIM</th>
<th>HW CIM</th>
<th>HPM [24]</th>
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<td>u_4 - u_{\text{ex}}</td>
<td>)</td>
<td>(</td>
<td>u_4 - u_{\text{ex}}</td>
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<td>(</td>
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<td>(8.88 \times 10^{-4})</td>
<td>(1.02 \times 10^{-3})</td>
<td>(5.60 \times 10^{-4})</td>
<td>(5.01 \times 10^{-7})</td>
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<td>(7.24 \times 10^{-7})</td>
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<td>(4.80 \times 10^{-5})</td>
<td>(2.57 \times 10^{-5})</td>
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<td>(9.10 \times 10^{-7})</td>
<td>(3.68 \times 10^{-2})</td>
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</tbody>
</table>

We put \(u_0(x, t) = \frac{1}{(1 + x^2)^{\frac{1}{2}}}\) and apply the Haar wavelet iteration technique. The numerical results included absolute error and approximate solutions for \( m = 64 \) at 4 iteration are given in Figure 7 which shows that, numerical solution is in very good coincide with exact solution by increasing iterations. The present method solution (HW CIM) has been compared with HPM [24] and MVIM [16] at Table (3), which shows Haar wavelet collocation iteration method is more accurate than the HPM and MVIM methods.

Example 4. Consider the problem (1.1) (non-homogenous time fractional fisher equation) for \( \lambda = 1, n = 3 \) and

\[
q(x, t) = t \left( -2 - x(t + x)(1 - t^3x^3(t + x)^3) + \frac{x^{2t-\alpha}}{\Gamma(2 - \alpha)} + \frac{2xt^{1-\alpha}}{\Gamma(3 - \alpha)} \right),
\]

with the initial and boundary conditions \( u(x, 0) = 0, u(0, t) = 0 \) and \( u(1, t) = t^2 + t \), where it is known that the problem has the exact solution \( u_{\text{exact}}(x, t) = xt^2 + tx^2 \).

Put \(u_0(x, t)\) and apply the Haar wavelet collocation iteration technique. The numerical results included absolute error and approximate solutions for \( m = 64 \) at 4 iterations are shown in Figure 9 and Table 4.
### Table 3. Absolute error for Example 3, \( m = 64 \) and \( \alpha \to 1 \)

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<th>HWCIM</th>
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<th>HWCIM</th>
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<td>(</td>
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<td>1.12 × 10⁻¹</td>
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### Table 4. Absolute error for Example 4, \( m = 64 \) and \( \alpha \to 1 \)

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<td>2.33 × 10⁻²</td>
<td>8.77 × 10⁻³</td>
<td>2.89 × 10⁻⁶</td>
</tr>
<tr>
<td>((\frac{63}{127}, \frac{63}{127}))</td>
<td>3.57 × 10⁻²</td>
<td>2.25 × 10⁻²</td>
<td>7.96 × 10⁻³</td>
<td>4.12 × 10⁻⁶</td>
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<td>((\frac{63}{127}, \frac{63}{127}))</td>
<td>3.00 × 10⁻²</td>
<td>1.81 × 10⁻²</td>
<td>6.01 × 10⁻³</td>
<td>3.33 × 10⁻⁶</td>
</tr>
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<td>((\frac{63}{127}, \frac{63}{127}))</td>
<td>1.16 × 10⁻²</td>
<td>6.40 × 10⁻³</td>
<td>1.22 × 10⁻³</td>
<td>1.19 × 10⁻³</td>
</tr>
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<td>5.45 × 10⁻³</td>
<td>3.56 × 10⁻⁴</td>
<td>1.28 × 10⁻⁴</td>
<td>3.93 × 10⁻⁸</td>
</tr>
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<td>((\frac{63}{127}, \frac{63}{127}))</td>
<td>5.00 × 10⁻³</td>
<td>3.03 × 10⁻⁴</td>
<td>1.00 × 10⁻⁴</td>
<td>1.18 × 10⁻⁷</td>
</tr>
<tr>
<td>((\frac{63}{127}, \frac{63}{127}))</td>
<td>3.92 × 10⁻³</td>
<td>2.21 × 10⁻⁴</td>
<td>6.67 × 10⁻⁴</td>
<td>2.38 × 10⁻⁷</td>
</tr>
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<td>((\frac{63}{127}, \frac{63}{127}))</td>
<td>1.07 × 10⁻³</td>
<td>1.99 × 10⁻⁴</td>
<td>3.92 × 10⁻⁴</td>
<td>5.52 × 10⁻⁴</td>
</tr>
</tbody>
</table>
6. Conclusion

In this work, we successfully apply the combination of Haar wavelet operational matrices method and iteration Picard technique to obtain the solution of fractional
fisher’s equation. By the use of iteration technique, we transform the nonlinear fractional partial differential equation to the linear equation and Sylvester equation. The obtained results have been compared with exact solutions, the HPM method and the MVIM method, which shows that numerical solutions are in very good coincide with the exact solution by increasing iterations or level of resolution or both. These results have been cited in the tables in order to justify the accuracy and efficiency of the proposed method.
Figure 7. Comparison of the exact solution and the Haar wavelet collocation iteration solution for Example 3

(A) exact solution for $\alpha = 1$

(B) approximate solution at the 4th iteration with $m = 64, \alpha = 1$

(C) absolute error for the 1st iteration with $m = 64, \alpha = 1$

(D) absolute error for the 2nd iteration with $m = 64, \alpha = 1$

(E) absolute error for the 3rd iteration with $m = 64, \alpha = 1$

(F) absolute error for the 4th iteration with $m = 64, \alpha = 1$
Figure 9. Comparison of the exact solution and the Haar wavelet collocation iteration solution for Example 4.

(A) exact solution for \( m = 64, \alpha = 1 \)

(B) approximate solution at the 4th iteration with \( m = 64, \alpha = 1 \)

(C) absolute error for the 1st iteration with \( m = 64, \alpha = 1 \)

(D) absolute error for the 2nd iteration with \( m = 64, \alpha = 1 \)

(E) absolute error for the 3rd iteration with \( m = 64, \alpha = 1 \)

(F) absolute error for the 4th iteration with \( m = 64, \alpha = 1 \)
REFERENCES


