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Algorithm for solving the Cauchy problem for stationary systems of fractional order linear ordinary differential equations

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Abstract A new simplified analytical formula is given for solving the Cauchy problem for a homogeneous system of fractional order linear differential equations with constant coefficients (SFOLDECC). The exponential function matrix in this formula is replaced by a Taylor series. Next, an analytical expression of the integral is obtained, with the help of which, for the transition matrix, a relation is obtained that allows one to obtain a solution of the Cauchy problem with high accuracy. The results also apply to the case of inhomogeneous systems with constant perturbations and are illustrated by numerical examples.

Keywords. Cauchy problem, linear fractional derivative system, Mittag-Leffler function, constant matrix coefficients.

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1. Introduction

In work [12], the solution of the (SFOLDECC) based on the Mittag-Leffler function is considered. However, in [5,6] such solution is given for the first time on the basis of an exponential function, which from a computational point of view is more appropriate, since the exponential function can be calculated quite accurately [2–4,7,8,10]. At the same time, in addition to the exponential function, this solution also includes an integral expression, which presents certain difficulties for the calculation [1,11,13,15]. Therefore, if instead of the exponential function we substitute its Taylor expansion,

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then we can obtain these integrals analytically, but the errors of the solution depend on the selected number of terms from the reduced series.

In this paper, we obtain analytical solution formulas, which, in contrast to [6], contain only one integral expression, which also makes it easier to obtain the solution of SFOLDECC. Next, substituting the expansion of the exponential function in the integrand, we obtain the numerical-analytical formula for the solution in the form of a Taylor series. The results are illustrated with a numerical example and a comparative analysis with the results of [12] is given.

2. Simplified Formula of Solution for the Cauchy Problem

As is known for solving of Cauchy problem SFOLDECC

$$D^{\alpha}x(t) = Ax(t), \quad x(t_0) = x_0, \quad t > t_0 > 0, \tag{2.1}$$

there is the following analytical formula [6]

$$\begin{aligned} x(t) = &\left\{ \sum_{s=0}^{2q} A^{\frac{s+2q+1}{2p+1}} \left[E\frac{t_0^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!} \right.\right. \\ &\left. + A^{\frac{2q+1}{2p+1}} \int_0^{t_0} \frac{(t_0-\tau)^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!} e^{\tau A^{\frac{2q+1}{2p+1}}} d\tau \right] \\ &\left. + \sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} \right\}^{-1} \times \left\{ \sum_{s=0}^{2q} A^{\frac{s+2q+1}{2p+1}} \left[E\frac{t^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!} \right.\right. \\ &\left. + A^{\frac{2q+1}{2p+1}} \int_0^{t} \frac{(t-\tau)^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!} e^{\tau A^{\frac{2q+1}{2p+1}}} d\tau \right] \\ &\left. + \sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} \right\} x(t_0), \end{aligned} \tag{2.2}$$

where A is the constant matrix of dimension $n \times n$, $x(t)$ is state vector of dimension n, $x(t_0)$ is the initial vector, $\alpha \in (0, 1)$ and $\alpha = \frac{2p+1}{2q+1}$, here p and q are natural numbers, E is an unit matrix of dimension $n \times n$.

As shown in [5], any real number can be approximated to a rational one with any accuracy , and any rational number to a number equal to the ratio of two odd numbers.

To simplify formulas (2.2), we use the relation [17]

$$D_{x_0}^{\alpha}u(\xi) = \frac{u(x_0)\,\xi^{-\alpha}}{(-\alpha)!} + \int_{x_0}^{\xi} \frac{(\xi - t)^{-\alpha}}{(-\alpha)!} u'(t)\,dt, \tag{2.3}$$

which defines derivatives of $u(x)$ with order α. Using the expression (2.3) in (2.2), after simple transformations [5,6] we reduce the relation (2.2) to the form

$$
\begin{aligned}
x(t) &= \left[e^{t_0 A^{\frac{2q+1}{2p+1}}} \sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} \right]^{-1} \sum_{s=0}^{2q} \left[A^{\frac{s}{2p+1}} e^{t_0 A^{\frac{2q+1}{2p+1}}} \frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} \right. \\
&\quad \left. + A^{\frac{s+2q+1}{2p+1}} \int_{t_0}^{t} \frac{(t-\tau)^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} e^{A^{\frac{2q+1}{2p+1}}\tau} d\tau \right] x_0 \\
&= \left[\sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} \right]^{-1} \sum_{s=0}^{2q} \left[A^{\frac{s}{2p+1}} \frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} \right. \\
&\quad \left. + A^{\frac{s+2q+1}{2p+1}} \int_{t_0}^{t} \frac{(t-\tau)^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} e^{A^{\frac{2q+1}{2p+1}}(\tau - t_0)} d\tau \right] x_0.
\end{aligned}
\tag{2.4}
$$

Note that in formula (2.4) the integrand function contains a weak singularity, since $\frac{s-2q}{2q+1} < 1$, and this allows the disappearance of such singularity after integration.

Thus, the following statement holds:

Theorem 2.1. *Consider Cauchy problem (2.1), such that A is constant matrix. Then the solution of (2.1) is represented in the form of (2.4). Note that formula (2.4) in the classical case, i.e. when $\alpha = 1$ coincides with the known [10]. Indeed in this case $ap = 0$, $q = 0$ and*

$$
\begin{aligned}
x(t) &= e^{-At_0} \left[e^{At_0} + A \int_{t_0}^{t} e^{A\tau} d\tau \right] x_0 \\
&= \left[E + e^{-At_0} \left(e^{At} - e^{At_0} \right) \right] x_0 = e^{A(t-t_0)} x_0.
\end{aligned}
\tag{2.5}
$$

3. Method of Calculations

The integrand in (2.4) is such that its integration, both analytically and numerically, facing serious difficulties. To avoid this, we expand the exponential function $e^{A^{\frac{2q+1}{2p+1}}(\tau - t_0)}$ in (2.4) by the Taylor series in a neighborhood of $\tau = t$, as

$$
e^{A^{\frac{2q+1}{2p+1}}(\tau - t_0)} = \sum_{k=0}^{\infty} A^{k\frac{2q+1}{2p+1}} e^{(t-t_0)A^{\frac{2q+1}{2p+1}}} \frac{(\tau - t)^k}{k!}. \tag{3.1}
$$

Substituting the series (3.1) into (2.4) we obtain

$$
\begin{aligned}
x\left(t\right) = & \left[\sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right]^{-1} \sum_{s=0}^{2q} \left[A^{\frac{s}{2p+1}} \frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right. \\
& + \sum_{k=0}^{\infty} \left(-1\right)^k \frac{A^{\frac{s+(k+1)(2q+1)}{2p+1}}}{k!} e^{(t-t_0)A^{\frac{2q+1}{2p+1}}} \\
& \left. \times \int_{t_0}^{t} \frac{\left(t-\tau\right)^{k+\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} d\tau x_0. \right.
\end{aligned} \tag{3.2}
$$

By integrating the power function in (3.2), finally, to solve the Cauchy problem (2.1), we get

$$
\begin{aligned}
x\left(t\right) = & \left[\sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right]^{-1} \sum_{s=0}^{2q} \left[A^{\frac{s}{2p+1}} \frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right. \\
& - \sum_{k=0}^{\infty} \frac{\left(-1\right)^{k+1}}{k!} A^{\frac{s+(k+1)(2q+1)}{2p+1}} e^{(t-t_0)A^{\frac{2q+1}{2p+1}}} \\
& \left. \times \frac{\left(t-t_0\right)^{k+\frac{s+1}{2q+1}}}{\frac{s-2q}{2q+1}!\left(k+\frac{s+1}{2q+1}\right)}\right] x_0.
\end{aligned} \tag{3.3}
$$

Let

$$
\begin{aligned}
x_r\left(t\right) = & \left[\sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right]^{-1} \sum_{s=0}^{2q} \left[A^{\frac{s}{2p+1}} \frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right. \\
& - \frac{e^{(t-t_0)A^{\frac{2q+1}{2p+1}}}}{\frac{s-2q}{2q+1}!} \sum_{k=0}^{r} \frac{\left(-1\right)^{k+1}}{k!} A^{\frac{s+(k+1)(2q+1)}{2p+1}} \\
& \left. \times \frac{\left(t-t_0\right)^{k+\frac{s+1}{2q+1}}}{\left(k+\frac{s+1}{2q+1}\right)}\right] x_0, \quad r \in N.
\end{aligned} \tag{3.4}
$$

In practice, the computation (3.4) stops when this condition

$$
\left\|x_l\left(t\right) - x_{l-1}\left(t\right)\right\| > \left\|x_{l+1}\left(t\right) - x_l\left(t\right)\right\|, \tag{3.5}
$$

does not hold.

corresponding function. So we have the following

Algorithm 1.

1. The matrix A and the initial condition x_0 from (2.1) are given.
2. In $x_r, r = 1,\ 2,\ 3,$ are calculated using (3.4), i.e. $l =\ 3$.

3. The condition $\|x_l(t) - x_{l-1}(t)\| < \|x_{l+1}(t) - x_l(t)\|$ is checking. If it is satisfied, the calculation process is stopped and the $x_l(t)$ is accepted as the solution. Otherwise, go to step 4.

4. Put $l = l + 1$ and calculate $x_l(t)$ by (3.4) and go to step 3.

We illustrate the results in the following example from [2].

Example 1. The scalar case is considered, i.e. in equation (2.1), let $A = -1$, and $x_0 = 1$. The calculation is carried out on the interval $[1, 3]$. The interval step is $h = \frac{3-1}{5}$, i.e. the segment is divided into 5 points. For various values of α , the $x_r(t)$ is calculated by the formula (3.4). According to the formula (3.117) given in [16], the $x_M(t)$ is calculated. In Table 1, a number of a comparison between iteration steps of our method and the method is presented in M have been given (3.5).

Table 1. Comparison of the number of iteration steps

α	$\|x_r(t) - x_M$ 1001[1] $(t)\|$	Iteration steps r	Iteration steps M
1/3	0.314	23	7
1/5	0.237	23	11
5/7	0.4248	37	4
1/7	0.1707	37	15
3/7	0.262	27	5

At $\alpha = 1$ the $x_r(t)$ is calculated by the formula (3.4), and we get

$$\|x_{37}(t) - x_{36}(t)\| < \|x_{38}(t) - x_{37}(t)\|,$$

so we take $x_{37}(t)$ as a solution. For this case the analytical solution $x_A(t)$ is calculated by the formula (2.4).

4. Solution of the Cauchy Problem (2.1) with Constant Perturbations

Let consider the following Cauchy problem

$$D^{\alpha}x(t) = Ax(t) + B(t), \quad x(t_0) = x_0, \tag{4.1}$$

where $B(t)$ is the vector of dimension n.

As is known [12], the solution of the Cauchy problem (4.1) has the form [1]

$$x(t) = \sum_{k=0}^{\infty} A^k \frac{(t-t_0)^{(k+1)\alpha-1}}{[(k+1)\alpha-1]!} x_0 + \int_{t_0}^{t} \sum_{k=0}^{\infty} A^k \frac{(t-\xi)^{(k+1)\alpha-1}}{[(k+1)\alpha-1]!} B(\xi)\, d\xi. \tag{4.2}$$

In the case when the vector $B(t) = B$ is constant, it is easy to show that in (4.2) the integral can be calculated and we have the following analytical formula

$$x(t) = \sum_{k=0}^{\infty} A^k \frac{(t-t_0)^{(k+1)\alpha-1}}{[(k+1)\alpha-1]!} x_0 + \sum_{k=0}^{\infty} A^k \frac{(t-t_0)^{(k+1)\alpha}}{[(k+1)\alpha]!} B. \tag{4.3}$$

[1] The relation (4.1) at $t = t_0$ has the singularities that turns into infinity [12]

Function (4.3) is a solution of the Cauchy problem (4.1) using the Mittag-Lefler function. Now, using expressions (2.4), we transform function (4.3) through the exponential function in the following form

$$
\begin{aligned}
x(t) = & \left[\sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right]^{-1} \left\{\sum_{s=0}^{2q}\left[A^{\frac{s}{2p+1}} \frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right.\right. \\
& \left. + A^{\frac{s+2q+1}{2p+1}} \int_{t_0}^{t} \frac{(t-\tau)^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} e^{(\tau-t_0)\cdot A^{\frac{2q+1}{2p+1}}} d\tau\right] x_0 + \\
& + \left[\int_{t_0}^{t} \sum_{s=0}^{2q} \left(A^{\frac{s}{2p+1}} \frac{\eta^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right.\right. \\
& \left.\left.\left. + A^{\frac{s+2q+1}{2p+1}} \int_{t_0}^{\eta} \frac{(\eta-\tau)^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} e^{(\tau-t_0)A^{\frac{2q+1}{2p+1}}} d\tau\right) d\eta\right] B\right\}.
\end{aligned}
\tag{4.4}
$$

Thus, the following theorem is proved.

Theorem 4.1. *Let in the Cauchy problem (4.1) $B(t) = B$ is the constant matrix. Then the solution of the corresponding Cauchy problem is represented in the form of (4.4).*

Note that for $p = 0,\ q = 0$ (i.e. $\alpha = 1$) from (4.4) we have

$$
\begin{aligned}
x(t) &= \left\{\left[E + A\int_{t_0}^{t} e^{(\tau-t_0)A} d\tau\right] x_0 + \int_{t_0}^{t}\left[E + A\int_{t_0}^{\eta} e^{(\tau-t_0)A} d\tau\right] d\eta \cdot B\right\} \\
&= e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(\eta-t_0)} d\eta \cdot B = e^{A(t-t_0)} x_0 + A^{-1}\left(e^{A(t-t_0)} - E\right)\cdot B,
\end{aligned}
\tag{4.5}
$$

which coincides with the classical solution (at $\alpha = 1$) from [9, 10, 14].

Note that, in (4.4), similarly to §3, by expanding $e^{(\tau-t_0)A^{\frac{2q+1}{2p+1}}}$ at a nearby point $\tau = \eta$ whit a Taylor series (3.1), we have (the first term before the initial data x_0, has already been transformed in §3, therefore only the last integral term is considered here)

$$\begin{aligned} J_1 &= \int\limits_{t_0}^{t} A^{\frac{s+2q+1}{2p+1}} d\eta \int\limits_{t_0}^{\eta} \frac{(\eta-\tau)^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} e^{(\tau-t_0)A^{\frac{2q+1}{2p+1}}} d\tau \\ &= \int\limits_{t_0}^{t} A^{\frac{s+2q+1}{2p+1}} \left(\int\limits_{t_0}^{\eta} \frac{(\eta-\tau)^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} \sum_{k=0}^{\infty} A^{k\frac{2q+1}{2p+1}} e^{(\eta-t_0)A^{\frac{2q+1}{2p+1}}} \frac{(\tau-\eta)^k}{k!} d\tau \right) d\eta \\ &= -\sum_{k=0}^{\infty} (-1)^k A^{\frac{(k+1)(2q+1)+s}{2p+1}} \int_{t_0}^{t} e^{(\eta-t_0)A^{\frac{2q+1}{2p+1}}} d\eta \int\limits_{t_0}^{\eta} \frac{(\eta-\tau)^{k+\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!k!} d(\eta-\tau) \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{A^{\frac{(k+1)(2q+1)+s}{2p+1}}}{\frac{s-2q}{2q+1}!k!} \int\limits_{t_0}^{t} e^{(\eta-t_0)A^{\frac{2q+1}{2p+1}}} \frac{(\eta-t_0)^{k+\frac{s+1}{2q+1}}}{k+\frac{s+1}{2q+1}} d\eta. \end{aligned} \tag{4.6}$$

Now by expanding the function $e^{(\eta-t_0)A^{\frac{2q+1}{2p+1}}}$ to Macloren's series as

$$e^{(\eta-t_0)A^{\frac{2q+1}{2p+1}}} = \sum_{l=0}^{\infty} \frac{\left[(\eta-t_0)A^{\frac{2q+1}{2p+1}}\right]^l}{l!}, \tag{4.7}$$

and substituting (4.7) into (4.6), we have

$$J_1 = \sum_{k=0}^{\infty} (-1)^k \frac{A^{(k+1)(2q+1)+s}}{\frac{s-2q}{2q+1}!k!\left(k+\frac{s+1}{2q+1}\right)} \sum_{l=0}^{\infty} \frac{A^{l\frac{2q+1}{2p+1}}}{l!} \frac{(t-t_0)^{l+k+\frac{s+1}{2q+1}+1}}{l+k+\frac{s+1}{2q+1}+1}. \tag{4.8}$$

Taking into account (4.8) in the last integral (4.4) we have

$$\begin{aligned} &\int\limits_{t_0}^{t} \sum_{s=0}^{2q} \left[A^{\frac{s}{2p+1}} \frac{\eta^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} + A^{\frac{s+2q+1}{2p+1}} \int_{t_0}^{\eta} \frac{(\eta-\tau)^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} e^{(\tau-t_0)A^{\frac{2q+1}{2p+1}}} d\tau \right] d\eta \cdot B \\ &= \sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \left\{ \left[\frac{t^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!} - \frac{t_0^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!} \right] + J_1 \right\} B. \end{aligned} \tag{4.9}$$

Taking into account (3.3) and (4.9) in (4.4), we obtain

$$x(t)=\left(\sum_{s=0}^{2q}A^{\frac{s}{2p+1}}\frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right)^{-1}\left\{\sum_{s=0}^{2q}\left(A^{\frac{s}{2p+1}}\frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right.\right.$$
$$\left.-\frac{e^{(t-t_0)A^{\frac{2q+1}{2p+1}}}}{\frac{s-2q}{2q+1}!}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k!}A^{\frac{s+(k+1)(2q+1)}{2p+1}}\cdot\frac{(t-t_0)^{k+\frac{s+1}{2q+1}}}{k+\frac{s+1}{2q+1}}\right)x_0$$
$$+\sum_{s=0}^{2q}A^{\frac{s}{2p+1}}\left[\left(\sum_{k=0}^{\infty}(-1)^k\frac{A^{(k+1)(2q+1)+s}}{\frac{s-2q}{2q+1}!\cdot k!\cdot\left(k+\frac{s+1}{2q+1}\right)}\right.\right.$$
$$\left.\left.\left.\times\sum_{l=0}^{\infty}\frac{A^{l\frac{2q+1}{2p+1}}}{l!}\frac{(t-t_0)^{l+k+\frac{s+1}{2q+1}+1}}{\left(l+k+1+\frac{s+1}{2q+1}\right)}\right)+\left(\frac{t^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!}-\frac{t_0^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!}\right)\right]B\right\}. \tag{4.10}$$

as the solution of the Cauchy problem (4.1).In (4.10), we replace the infinite sums with the $r+1$ terms and denote the obtained expressions by $x_r(t)$. In this case, as in §3, if

$$\|x_r(t)-x_{r-1}(t)\|>\|x_{r+1}(t)-x_r(t)\|, \tag{4.11}$$

then we continue the calculation process, otherwise the process are stopped and we put $x=x_r$ as a solution. So our algorithm will be as the following:

Algorithm 2.

1. From (4.1) the matrices A , perturbation vector B and the initial data x_0 are formed.
2. $A^{\frac{s+(k+1)(2q+1)}{2p+1}}$ are calculated (at different values of k and s all degrees of the multiplier A are obtained from (4.10)), $t^{k+\frac{s+1}{2q+1}}$ (at different values of k, s and q all possible degrees t_0, $(t-t_0)$, t), $\frac{s-2q}{2q+1}$, k, $l+k+1+\frac{s+1}{2q+1}$, $e^{(t-t_0)}A^{\frac{2q+1}{2p+1}}$ are obtained.
3. $x_r(t)$ are calculated from (4.10) (instead of the infinite sum from (4.10), $r+1$ terms are taken).
4. The condition (4.11) is checked, if it is satisfied, the calculation process continues, otherwise the calculation process stops and $x_r(t)$ is accepted as the solution.

Example 2. In this paper, it is shown that the reduced solution (64) for the initial problem (53) from [12] is not true. Indeed, in [12] it is shown that the solution of the initial problem

$$\begin{aligned}&y''+3D^{13/2}y+y=8\\&y(0)=0,\ y'(0)=0,\quad t\in(0,1)\end{aligned} \tag{4.12}$$

has the form (in [12] this is numbered (64)).

$$y(t) = 10^{-2} Re[(-4.8+1.4i)E_{1/2}(\lambda_1 t^{1/2}) + (0.9-7.4i)E_{1/2}(\lambda_2 t^{1/2}) + \\ +0.348E_{1/2}(\lambda_3 t^{1/2}) - 0.59E_{1/2}(\lambda_4 t^{1/2})], \tag{4.13}$$

where

$$\lambda_1 = 0.363 - 0.556i, \quad \lambda_2 = 0.363 + 0.556i, \\ \lambda_3 = -2.962, \quad \lambda_4 = -0.765$$

and

$$E_{1/2}\left(\lambda_k t^{1/2}\right) = \sum_{m=0}^{\infty} \frac{\left(\lambda_k t^{1/2}\right)^m}{\Gamma\left(\frac{m}{2}+1\right)}, \quad k = \overline{1,4}. \tag{4.14}$$

Let us show that from (4.13) both, $y(0)$ and $y'(0)$ don't satisfy the initial conditions (4.12). Indeed, taking into account (4.14) in solution (4.13), after some simple transformations for $y(t)$ we have the following representation

$$y(t) = 10^{-2} Re\left\{\sum_{m=0}^{\infty} \frac{t^{m/2}}{\Gamma\left(\frac{m}{2}+1\right)}\left[(-4.8+1.4i)(0.363-0.556i)^m + \right.\right.$$

$$\left.\left. + (0.9-7.4i)(0.363+0.556i)^m + 0.348(-2.962)^m - 0.59(-0.765)^m\right]\right\} =$$

$$= -0.0412 + 10^{-2} Re\left\{\sum_{m=1}^{\infty} \frac{t^{m/2}}{\Gamma\left(\frac{m}{2}+1\right)}\left[(-4.8+1.4i)(0.363-0.556i)^m + \right.\right.$$

$$\left.\left. + (0.9-7.4i)(0.363+0.556i)^m + 0.348(-2.962)^m - 0.59(-0.765)^m\right]\right\}. \tag{4.15}$$

At $t=0$ from (4.15) for $y(0)$ we obtain

$$y(0) = -0.0412, \tag{4.16}$$

i.e., the first initial condition of the Cauchy problem (4.12) is not satisfied [18].

5. Conclusion

In this paper, a solution of the Cauchy problem for homogeneous and inhomogeneous systems of fractional-order linear ordinary differential equations with constant matrix coefficients and perturbations is given. It is shown that, in contrast to the known [10], this approach is based on the exponential function obtained from the Mittag-Lefler function. Unlike existing results [12, 16, 18], this method is better implemented on computer and gives more accurate results.

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