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# Approximate nonclassical symmetries for the time-fractional KdV equations with the small parameter

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**Abstract** In this paper, the Lie symmetry analysis is presented for the time-fractional KdV equation with the Riemann-Liouville derivative. We introduce a generalized approximate nonclassical method that is applied to differential equations with fractional order. In the sense of this symmetry, the vector fields of fractional KdV equation are obtained. The similarity reduction corresponding to the symmetries of the equation is constructed.

**Keywords.** Approximate Lie symmetry analysis, Time-fractional KdV equation, Group-invariant solution.
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## 1. Introduction

Ordinary and partial differential equations with fractional derivatives have a wide range of applications as mathematical models of physical processes. These equations are one of the most accurate tools to refine the description of natural phenomena [4, 13, 16, 19]. Fractional differential equations have gained much attention since fractional order system response ultimately converges to the integer order response. The fractional derivatives are nonlocal operators and provide a powerful instrument for the description of memory and hereditary properties of different substances. This is the main advantage of models involving fractional operators in comparison to integer order models. The KdV equation was first derived as an evolution equation that governs a one-dimensional, small-amplitude, long surface gravity waves propagating in a shallow channel of water [14]. This equation has arisen in a number of other physical contexts as collision-free hydro-magnetic waves, stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics, aerodynamics and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behavior and mass transport, etc. [6]. The mathematical theory behind the KdV equation is rich and interesting and, in the broad sense, is a topic of active mathematical and physical research. In modeling of these physical phenomena, when memory effects are taken into account we obtain the KdV differential equation with nonlocal operators and fractional derivatives. Over the past few years, the fractional KdV equation has been

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studied extensively both theoretically and numerically. There are few studies, however, on this equation by employing analytical methods [5, 15, 20, 21]. Lie Symmetry analysis plays a fundamental role in the construction of the analytical and exact solutions of the differential equations [3, 7, 8, 10, 11, 18]. In Lie symmetry method, the purpose is to get vector fields and afterward to obtain exact solutions of corresponding vector fields. Analytical solutions for some fractional differential equations have been obtained by the classical and nonclassical Lie symmetry analysis[1, 9, 17]. To achieve new vector fields of a differential equation and consequently new solutions, one of the possible fields for perturbed differential equation of Lie symmetry method is the "approximate Lie symmetry method" which firstly is considered by Baikov et al. [2] . Some researchers have applied approximate classical Lie symmetry method to find the analytical solutions of some partial differential equations[7, 12]. Approximate nonclassical symmetry method has not been extended and applied to differential equations. In this study we first intend to express the nonclassical Lie symmetry analysis to the fractional differential equations. By applying this method, we achieve infinitesimal generators which are new in comparison with obtained symmetries by Lie symmetry method. The obtained results justify the applicability of the method. We hope this paper is a beginning for future research in this direction.

The rest of this paper is organized as follows: In Section 2, we recall some notations and introduce approximate nonclassical symmetry theory of a fractional differential equation with small parameter. An algorithm of calculating such symmetries were proposed in this section. In Section 3 we obtain the new vector fields by approximate nonclassical Lie symmetry method, reduction of order and exact solutions for time-fractional KdV equation.

## 2. The approximate nonclassical Lie symmetry analysis

The main purpose of this section is to introduce the approximate transformations group admitted by fractional differential equations with a small parameter $\varepsilon$. we will consider the approximation in the first order of precision in $\varepsilon$.

**Definition 2.1.** Let $\alpha > 0$ and $m = [\alpha] + 1$. The operator $D_t^\alpha$, defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-\alpha-1} f(s) ds,$$

is called the left-sided Riemann-Liouville fractional differential derivative of order $\alpha$ [4].

Let us consider the one-parameter group of approximate transformations $G$

$$\begin{aligned} \bar{x} &= x + a\xi(x,t,u,\varepsilon) + O(a^2), \\ \bar{t} &= t + a\tau(x,t,u,\varepsilon) + O(a^2), \\ \bar{u} &= u + a\varphi(x,t,u,\varepsilon) + O(a^2), \end{aligned} \tag{2.1}$$

such that

$$\xi(x,t,u,\varepsilon) \approx \xi_0(x,t,u) + \varepsilon\xi_1(x,t,u) + \ldots + \varepsilon^p \xi_p(x,t,u),$$

where $a, \varepsilon$ are the group parameter and the small parameter, respectively and the approximate equality $f \approx g$ means $f(x,\varepsilon) = g(x,\varepsilon) + o(\varepsilon^p)$. The functions $\tau$ and $\varphi$ are defined similarly. The infinitesimal generator of an approximate transformation group $G$ (2.1) is

$$V = \xi(x,t,u,\varepsilon)\frac{\partial}{\partial x} + \tau(x,t,u,\varepsilon)\frac{\partial}{\partial t} + \varphi(x,t,u,\varepsilon)\frac{\partial}{\partial u}.$$

In theoretical discussions, approximate equalities are considered with an error $o(\varepsilon^p)$ of an arbitrary order $p \geq 1$. However, in the most of applications the theory is simplified by letting $p = 1$. Consider one-parameter approximate transformation groups in the first order of precision. Let

$$V = V_0 + \varepsilon V_1, \tag{2.2}$$

be a given infinitesimal generator, where

$$\begin{aligned} V_0 &= \xi_0(x,t,u)\frac{\partial}{\partial x} + \tau_0(x,t,u)\frac{\partial}{\partial t} + \varphi_0(x,t,u)\frac{\partial}{\partial u}, \\ V_1 &= \xi_1(x,t,u)\frac{\partial}{\partial x} + \tau_1(x,t,u)\frac{\partial}{\partial t} + \varphi_1(x,t,u)\frac{\partial}{\partial u}. \end{aligned}$$

The basic idea of the approximate nonclassical Lie symmetry method is to require that the perturbed fractional differential equation

$$\begin{aligned} F(x,t,u,u_x,u_{xx},\cdots,D_t^\alpha u,\varepsilon) \equiv{}& F_0(x,t,u,u_x,u_{xx},\cdots,D_t^\alpha u) \\ &+\varepsilon F_1(x,t,u,u_x,u_{xx},\cdots,D_t^\alpha u) \approx 0, \end{aligned}$$

and the invariance surface condition

$$\Lambda : (\xi_0 + \varepsilon\xi_1)u_x + (\tau_0 + \varepsilon\tau_1)u_t = \varphi_0 + \varepsilon\varphi_1, \tag{2.3}$$

which is associated with the vector field (2.2) are both invariant under the approximate transformations group $G$ for every infinitesimal generator $V$ of $G$. Then the invariance is given by

$$Pr^{(\alpha,t)}V(\Lambda)\big|_{F\approx 0,\Lambda\approx 0} \approx 0, \qquad Pr^{(\alpha,t)}V(F)\big|_{F\approx 0,\Lambda\approx 0} \approx 0, \tag{2.4}$$

where

$$Pr^{(\alpha,t)}V = V + \varphi^x\frac{\partial}{\partial u_x} + \varphi^{xx}\frac{\partial}{\partial u_{xx}} + \cdots + \varphi^{(\alpha,t)}\frac{\partial}{\partial D_t^\alpha u},$$

with

$$\begin{aligned} &\varphi^{(\alpha,t)} = \mathcal{D}_t^\alpha\varphi + \xi D_t^\alpha u_x - \mathcal{D}_t^\alpha(\xi u_x) - \mathcal{D}_t^{\alpha+1}(\tau u) + \mathcal{D}_t^\alpha(\mathcal{D}_t(\tau)u) + \tau D_t^{\alpha+1}u, \\ &\varphi^x = \mathcal{D}_x\varphi - \mathcal{D}_x(\xi)u_x - \mathcal{D}_x(\tau)u_t, \quad \varphi^{xx} = \mathcal{D}_x(\varphi^x) - \mathcal{D}_x(\xi)u_{xx} - \mathcal{D}_x(\tau)u_{xt}, \\ &\varphi^{xxx} = \mathcal{D}_x(\varphi^{xx}) - \mathcal{D}_x(\xi)u_{xxx} - \mathcal{D}_x(\tau)u_{xxt}, \qquad \cdots \end{aligned} \tag{2.5}$$

**Remark 2.2.** In the case that the function $f$ is multivariable, $f(x,t,u)$, $D_t^\alpha f, D_t^n f$ stands for the partial fractional and integer derivative of $f$ w.r.t. $t$ where the other variables, $x$ and $u$, are constant and $\mathcal{D}_t^\alpha f, \mathcal{D}_t^n f$ denote the total fractional derivative and the total derivative of $f$ w.r.t. $t$, respectively.

It is easy to prove that the first condition (2.4) is identically satisfied and we only consider the second condition, thus

$$Pr^{(\alpha,t)}V(F)\big|_{F\approx 0,\Lambda\approx 0}\approx 0. \tag{2.6}$$

The approximate determining equation (2.6) is equivalent to condition

$$Pr^{(\alpha,t)}V_0(F_0)+\varepsilon\left[Pr^{(\alpha,t)}V_1(F_0)+Pr^{(\alpha,t)}V_0(F_1)\right]|_{F\approx 0,\Lambda\approx 0}\approx 0.$$

## 3. The approximate nonclassical symmetry for the time-fractional KdV equation

We consider the perturbed generalized KdV equation with fractional order

$$\Delta: D_t^{\alpha}u=u^p u_x+\varepsilon u_{xxx},\qquad 0<\alpha\leq 1,\quad p>0, \tag{3.1}$$

where $|\varepsilon|<1$ is a small parameter. A one-parameter Lie group of the approximate transformations (2.1) is admitted by Eq. (3.1) if and only if

$$Pr^{(\alpha,t)}V(\Delta)\big|_{\Delta\approx 0,\Lambda\approx 0}\approx 0.$$

Applying $Pr^{(\alpha,t)}V$ to Eq. (3.1), we find the Lie's invariance condition

$$p\varphi_0 u^{p-1}u_x+u^p\varphi_0^x-\varphi_0^{(\alpha,t)}+\varepsilon\varphi_0^{xxx}+\varepsilon p\varphi_1 u^{p-1}u_x+\varepsilon u^p\varphi_1^x-\varepsilon\varphi_1^{(\alpha,t)}\big|_{\Delta\approx 0,\Lambda\approx 0}\approx 0. \tag{3.2}$$

Taking into account the invariant surface condition (2.3), We can recognize two cases: $\xi_0+\varepsilon\xi_1\neq 0$ and $\xi_0+\varepsilon\xi_1=0$. In the case $\xi_0+\varepsilon\xi_1\neq 0$, without loss of generality, we can put $\xi_0=1$ and $\xi_1=0$, thus we have

$$u_x=\varphi_0+\varepsilon\varphi_1-(\tau_0+\varepsilon\tau_1)u_t. \tag{3.3}$$

Here it should be also mentioned that in Definition 2.1, the lower limit of the integral is fixed, hence it should be invariant with respect to group of transformation (2.1); i.e.

$$\tau_0(x,t,u)|_{t=0}=0,\qquad \tau_1(x,t,u)|_{t=0}=0.$$

Differentiating Eq. (3.3) w.r.t. $t$ and $x$, we get

$$\begin{aligned}
u_{xt}&=\mathcal{D}_t(\varphi_0+\varepsilon\varphi_1)-\mathcal{D}_t(\tau_0+\varepsilon\tau_1)u_t-(\tau_0+\varepsilon\tau_1)u_{tt},\\
u_{xx}&=\mathcal{D}_x(\varphi_0+\varepsilon\varphi_1)-(\tau_0+\varepsilon\tau_1)\mathcal{D}_t(\varphi_0+\varepsilon\varphi_1)-[\mathcal{D}_x(\tau_0+\varepsilon\tau_1)\\
&\quad-(\tau_0+\varepsilon\tau_1)\mathcal{D}_t(\tau_0+\varepsilon\tau_1)]u_t+(\tau_0+\varepsilon\tau_1)^2u_{tt},\\
u_{xxt}&=\mathcal{D}_t\mathcal{D}_x(\varphi_0+\varepsilon\varphi_1)-\mathcal{D}_t(\tau_0+\varepsilon\tau_1)\mathcal{D}_t(\varphi_0+\varepsilon\varphi_1)-(\tau_0+\varepsilon\tau_1)\mathcal{D}_t^2(\varphi_0+\varepsilon\varphi_1)\\
&\quad-\left\{\mathcal{D}_t\mathcal{D}_x(\tau_0+\varepsilon\tau_1)-[\mathcal{D}_t(\tau_0+\varepsilon\tau_1)]^2-(\tau_0+\varepsilon\tau_1)\mathcal{D}_t^2(\tau_0+\varepsilon\tau_1)\right\}u_t\\
&\quad-[\mathcal{D}_x(\tau_0+\varepsilon\tau_1)-3(\tau_0+\varepsilon\tau_1)\mathcal{D}_t(\tau_0+\varepsilon\tau_1)]u_{tt}+(\tau_0+\varepsilon\tau_1)^2u_{ttt},\\
u_{xxx}&=\mathcal{D}_x^2(\varphi_0+\varepsilon\varphi_1)-2\mathcal{D}_x(\tau_0+\varepsilon\tau_1)\mathcal{D}_t(\varphi_0+\varepsilon\varphi_1)-(\tau_0+\varepsilon\tau_1)\mathcal{D}_x\mathcal{D}_t(\varphi_0+\varepsilon\varphi_1)\\
&\quad+(\tau_0+\varepsilon\tau_1)\mathcal{D}_t(\tau_0+\varepsilon\tau_1)\mathcal{D}_t(\varphi_0+\varepsilon\varphi_1)+(\tau_0+\varepsilon\tau_1)^2\mathcal{D}_t^2(\varphi_0+\varepsilon\varphi_1)\\
&\quad-\{\mathcal{D}_x^2(\tau_0+\varepsilon\tau_1)-2\mathcal{D}_x(\tau_0+\varepsilon\tau_1)\mathcal{D}_t(\tau_0+\varepsilon\tau_1)-(\tau_0+\varepsilon\tau_1)\mathcal{D}_x\mathcal{D}_t(\tau_0+\varepsilon\tau_1)\\
&\quad+(\tau_0+\varepsilon\tau_1)[\mathcal{D}_t(\tau_0+\varepsilon\tau_1)]^2+(\tau_0+\varepsilon\tau_1)^2\mathcal{D}_t^2(\tau_0+\varepsilon\tau_1)\}u_t\\
&\quad+[3(\tau_0+\varepsilon\tau_1)\mathcal{D}_x(\tau_0+\varepsilon\tau_1)-3(\tau_0+\varepsilon\tau_1)^2\mathcal{D}_t(\tau_0+\varepsilon\tau_1)]u_{tt}-(\tau_0+\varepsilon\tau_1)^3u_{ttt}.
\end{aligned}$$

After substituting $\varphi_0^x, \varphi_1^x, \varphi_0^{xxx}, \varphi_0^{(\alpha,t)}, \varphi_1^{(\alpha,t)}$ into Eq. (3.2) (notice that these are the same as (2.5) except that $\xi, \tau, \varphi$ will be replaced by $\xi_0, \tau_0, \varphi_0$ or $\xi_1, \tau_1, \varphi_1$), replacing $D_t^\alpha u$ by $u^p u_x + \varepsilon u_{xxx}$, then re-substituting the expressions $u_x, u_{xt}, u_{xx}, u_{xtt}, u_{xxt}, u_{xxx}$ wherever it occurs, we obtain the approximate nonclassical determining equations for the symmetry group. To analyze these, we set the coefficients of $1, u_t, u_t^2, u_{tt}, \dots$ to zero. The coefficient of $u_{tt}^2$ is

$$3\tau_0^3 \tau_{0u} = 0,$$

and so we have the two situations: $\tau_0 = 0$ and $\tau_{0u} = 0$. If $\tau_0 = 0$, we equate the coefficients of the remaining partial derivatives of $u$ to zero and we find the determining equations as follows

$$p\varphi_0^2 + \varphi_{0x} u = 0, \quad D_t^\alpha \varphi_0 - u D_t^\alpha \varphi_{0u} = 0, \quad D_t^n \varphi_{0u} = 0, \quad n \in \mathbb{N},$$

$$\begin{aligned}
&2p\varphi_0\varphi_1 u^{p-1} + \varphi_{0xxx} + 3\varphi_0\varphi_{0xxu} + 3\varphi_0^2\varphi_{0xuu} + \varphi_0^3\varphi_{0uuu} + 3\varphi_{0x}\varphi_{0xu} \\
&\qquad +3\varphi_0\varphi_{0u}\varphi_{0xu} + 3\varphi_0\varphi_{0x}\varphi_{0uu} + 3\varphi_0^2\varphi_{0u}\varphi_{0uu} + \varphi_{1x}u^p + \alpha\tau_{1t}\varphi_0 u^p = 0 \\
&p\tau_1\varphi_0 + \tau_{1x}u + (1-\alpha)\tau_{1u}\varphi_0 u = 0, \quad D_t^\alpha \varphi_1 - u D_t^\alpha \varphi_{1u} = 0, \\
&\binom{\alpha}{n} D_t^n \varphi_{1u} - \binom{\alpha}{n+1} \mathcal{D}_t^{n+1} \tau_1 = 0, \quad n \in \mathbb{N}.
\end{aligned} \tag{3.4}$$

Solving the system of equations (3.4) for $\varphi_0, \tau_1$ and $\varphi_1$, we achieve for $p = 1$ and $p = \frac{1}{2}$

$$\varphi_0 = \frac{u}{px + \beta}, \qquad \tau_1 = \frac{c_1 t}{px + \beta}, \qquad \varphi_1 = \frac{(-\alpha c_1 x + c_2)u + c_3 t^{\alpha-1}}{(px+\beta)^2},$$

where $c_1, c_2, c_3, \beta$ are arbitrary constants. Therefore

$$V = V_0 + \varepsilon V_1 = \partial_x + \frac{\varepsilon c_1 t}{px + \beta}\partial_t + \left(\frac{u}{px+\beta} + \varepsilon \frac{(-\alpha c_1 x + c_2)u + c_3 t^{\alpha-1}}{(px+\beta)^2}\right)\partial_u,$$

and in this case, the approximate symmetry algebra of Eq. (3.1) is spanned by the four vector fields

$$\begin{aligned}
V_1 &= \partial_x + \frac{u}{px+\beta}\partial_u, \quad V_2 = \partial_x + \frac{\varepsilon t}{px+\beta}\partial_t + \frac{(px+\beta-\varepsilon\alpha x)u}{(px+\beta)^2}\partial_u, \\
V_3 &= \partial_x + \frac{(px+\beta+\varepsilon)u}{(px+\beta)^2}\partial_u, \quad V_4 = \partial_x + \frac{(px+\beta)u + \varepsilon t^{\alpha-1}}{(px+\beta)^2}\partial_u.
\end{aligned}$$

If $\tau_0 \neq 0$ and $\tau_{0u} = 0$, we find immediately the set of determining equations for the approximate nonclassical symmetry group of Eq. (3.1). To find $\tau_0$ and $\varphi_0$, after simplifying, these equations can be written as follows

$$\begin{aligned}
&\varphi_{0uu} = 0, \quad D_t^\alpha \varphi_0 - u D_t^\alpha \varphi_{0u} = 0, \quad \binom{\alpha}{n} D_t^n \varphi_{0u} - \binom{\alpha}{n+1} \mathcal{D}_t^{n+1}\tau_0 = 0, \; n \in \mathbb{N}, \\
&\alpha\tau_0\tau_{0t} + 3\tau_{0x} = 0, \quad \tau_{0x}\varphi_{0u} - \tau_0\varphi_{0xu} = 0, \quad p\tau_0\varphi_0 - 2\tau_{0x}u = 0,
\end{aligned}$$

which has solutions

$$\tau_0 = \frac{3t}{\alpha x + \gamma}, \qquad \varphi_0 = \frac{-2\alpha u}{p(\alpha x + \gamma)},$$

where $\gamma$ is arbitrary constant. Substituting $\tau_0$ and $\varphi_0$ into the remaining determining equations, we have

$$\begin{aligned}
&\tau_{1u} = 0, \quad 2p\varphi_0\varphi_1 + \alpha\tau_{0t}\varphi_1 u + \varphi_{1x}u + \alpha\tau_{1t}\varphi_0 u = 0\\
&p\tau_1\varphi_0 + \alpha\tau_{0t}\tau_1 u + p\tau_0\varphi_1 + \tau_{1x}u + \alpha\tau_0\tau_{1t} = 0, \quad D_t^\alpha\varphi_1 - uD_t^\alpha\varphi_{1u} = 0,\\
&\binom{\alpha}{n} D_t^n\varphi_{1u} - \binom{\alpha}{n+1}\mathcal{D}_t^{n+1}\tau_1 = 0, \quad n \in \mathbb{N}.
\end{aligned}$$

After solving this system of equations for $\tau_1$ and $\varphi_1$, we find

$$\tau_1 = \frac{c_4(\alpha x + \gamma)t + c_5 t}{(\alpha x + \gamma)^2}, \qquad \varphi_1 = -\frac{3c_4\alpha(\alpha x + \gamma)u + 2c_5\alpha u}{3p(\alpha x + \gamma)^2},$$

where $c_4, c_5$ are arbitrary constants. Hence

$$V = V_0 + \varepsilon V_1 = \partial_x + \frac{(3+\varepsilon c_4)(\alpha x+\gamma)t + \varepsilon c_5 t}{(\alpha x + \gamma)^2}\partial_t - \frac{\alpha(6+3\varepsilon c_4)(\alpha x+\gamma)u + 2\varepsilon c_5\alpha u}{3p(\alpha x + \gamma)^2}\partial_u.$$

In this case, the approximate symmetry algebra of Eq. (3.1) is spanned by the three vector fields

$$\begin{aligned}
&V_5 = \partial_x + \frac{3t}{\alpha x + \gamma}\partial_t - \frac{2\alpha u}{p(\alpha x + \gamma)}\partial_u, \quad V_6 = \partial_x + \frac{(3+\varepsilon)t}{\alpha x + \gamma}\partial_t - \frac{\alpha(2+\varepsilon)u}{p(\alpha x + \gamma)}\partial_u,\\
&V_7 = \partial_x + \frac{3(\alpha x + \gamma)t + \varepsilon t}{(\alpha x + \gamma)^2}\partial_t - \frac{6\alpha(\alpha x + \gamma)u + 2\varepsilon\alpha u}{3p(\alpha x + \gamma)^2}\partial_u.
\end{aligned}$$

In the case $\xi_0 + \varepsilon\xi_1 = 0$ and $\tau_0 + \varepsilon\tau_1 \neq 0$, similar to the previous case, we have $(\tau_0 + \varepsilon\tau_1)u_t = \varphi_0 + \varepsilon\varphi_1$. Differentiating this equation and get $u_{xt}, u_{xxt}$ and also re-substituting these expressions wherever it occurs, we obtain the nonclassical determining equations which are difficult to solve.

**Remark 3.1.** Notice that $V_1$ and $V_5$ are the nonclassical infinitesimal generators of Eq. (3.1) [17].

These vector fields can be used to construct the exact solution of Eq. (3.1).

**Example 3.2.** We consider the KdV equation

$$D_t^\alpha u = uu_x + \varepsilon u_{xxx}, \tag{3.5}$$

with approximate infinitesimal generator $V = V_3$ with invariant solution

$$u(x,t) = (x+\beta)e^{-\frac{\varepsilon}{x+\beta}}f(t).$$

Now we use the Taylor expansion of $e^{-\frac{\varepsilon}{x+\beta}}$ about $\varepsilon = 0$ and we have

$$\begin{aligned}
u(x,t) &= (x+\beta)\left[1 - \frac{\varepsilon}{x+\beta} + \cdots\right] f(t) = (x+\beta-\varepsilon)\,f(t) + o(\varepsilon)\\
&\approx (x+\beta-\varepsilon)\,f(t).
\end{aligned}$$

Substituting this approximate solution into Eq. (3.5), we obtain

$$D_t^\alpha f(t) = f^2(t) + o(\varepsilon), \qquad \text{or} \qquad D_t^\alpha f(t) \approx f^2(t).$$

Then the approximate invariant solution of Eq. (3.5) is

$$u(x,t) = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}(x+\beta-\varepsilon)t^{-\alpha}.$$

## 4. Conclusion

In this paper, we applied the approximate nonclassical Lie symmetry analysis to a class of time-fractional differential equations with the small parameter. By employing these method and some technical calculations, new infinitesimal generators are obtained for time-fractional KdV equations. The basic idea described in this paper is an efficient and powerful method for solving wide classes of perturbed fractional differential equations.

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