Legendre-collocation spectral solver for variable-order fractional functional differential equations

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Abstract A numerical method for the variable-order fractional functional differential equations (VO-FFDEs) has been developed. This method is based on approximation with shifted Legendre polynomials. The properties of the latter were stated, first. These properties, together with the shifted Gauss-Legendre nodes were then utilized to reduce the VO-FFDEs into a solution of matrix equation. Sequentially, the error estimation of the proposed method was investigated. The validity and efficiency of our method were examined and verified via numerical examples.

Keywords. Variable-order fractional functional differential equations; shifted Legendre polynomials; Gauss-Legendre nodes; matrix equation.

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1. Introduction

A brief history of the development of fractional differential operators can be found in [23, 28]. Podlubny [30] deals lengthily with the theory of fractional order (non-integer) derivatives and integrals. Nowaday, research on fractional calculus is a hot topic (see for example [5, 17]).

Fractional calculus is currently being employed in several fields, including economics, engineering, and science [5]. However, problems involving fractional differential equations (FDEs) are already extensive and still growing, including interdisciplinary applications. FDEs provide much accurate models for systems under consideration. Applications of FDEs in the anomalous transport [22], bioengineering [20], colored noise [21], dynamics of interfaces between nanoparticles and substrates [4], economics [1], fluid-dynamic traffic model [14], frequency-dependent damping behavior of viscoelastic materials [39], and nonlinear oscillation of earthquakes [13], solid mechanics [31] are fast-growing. The analytic results on the existence and uniqueness of FDEs solutions have been investigated [17, 30].
Adomian’s decomposition [26], Bernstein polynomials [3], collocation [11], finite difference method [7], Galerkin [40], He’s variational iteration [15, 24, 27], homotopy analysis [12], homotopy perturbation [25], Laplace transform [10], reproducing kernel method [18], reproducing kernel splines method [38], robust meshless method based on the moving least squares approximation and finite difference scheme [37], shifted Chebyshev operational matrix [2, 9, 33] and orthogonal spectral [32, 34] are well-studied examples of numerical methods to handle FDEs.

This study aimed to introduce a numerical method to enhance the accuracy of the numerical solution of the VO-FFDEs Dirichlet boundary value problem [38]. Consider

$$D^\mu(z)v(z) + \alpha(z)v'(z) + \beta(z)v(z) + \gamma(z)v(\tau(z)) = g(z), \quad z \in [0, \ell],$$

subject to

$$v(0) = \kappa_0, \quad v(\ell) = \kappa_1,$$

where $\alpha(z)$, $\beta(z)$, $\gamma(z) \in C^2[0, \ell]$, $\mu(z)$, $\tau(z)$, $g(z) \in C[0, \ell]$, $1 \leq \mu(z) < 2$, $0 \leq \tau \leq \ell$, $\kappa_0$ and $\kappa_1$ are constant, $D^\mu(z)$ denotes the variable fractional order derivative in Caputo’s sense defined as follows

$$D^\mu(z)v(z) = \frac{1}{\Gamma(2 - \mu(z))} \int_0^z (z-t)^{1-\mu(z)}v''(t)dt,$$

where $\Gamma(.)$ is Gamma function.

The proposed algorithm converts the VO-FFDEs (1.1) to a system of algebraic equations by combining the basis functions of shifted Legendre polynomials and the Gauss-shifted Legendre nodes as the collocation points.

The organization of the paper encompass: In Section 2, an overview of shifted Legendre polynomials and their relevant properties required henceforward are presented, and in Section 3, the way of constructing the collocation technique for VO-FFDEs is described by using the shifted Legendre polynomials. In Section 4, we give a detailed study of the convergence and error analyses In Section 5, the proposed method was applied to solve two examples. Finally, a conclusion is given in Section 6.

2. Mathematical preliminaries

The well-known Legendre polynomials $L_i(y)$ are defined on the interval $[-1, 1]$. Firstly, some properties about the standard Legendre polynomials have been recalled in this section. The Legendre polynomials satisfy

$$L_0(y) = 1, \quad L_1(y) = y, \quad L_{k+2}(y) = \frac{2k + 3}{k + 2} y L_{k+1}(y) - \frac{k + 1}{k + 2} L_k(y).$$

Let the shifted Legendre polynomials $L_i(\frac{2z}{\ell} - 1)$ be denoted by $P_i(z)$. Then $P_i(z)$ can be obtained as follows

$$(i + 1)P_{i+1}(z) = (2i + 1)(2z - 1)P_i(z) - iP_{i-1}(z), \quad i = 1, 2, \ldots.$$

Legendre polynomials have the following analytic form

$$P_i(z) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)! \cdot (k)!^2 \ell^k} z^k.$$
and
\[ P_i(0) = (-1)^i, \quad P_i(\ell) = 1. \] (2.2)

We used \( z_j \), and \( \varpi_j \), \( 0 \leq j \leq N \), as the nodes and Christoffel numbers of the standard Legendre-Gauss interpolation in the interval \([-1, 1]\).

The corresponding nodes and corresponding Christoffel numbers of the shifted Legendre-Gauss interpolation in the interval \([0, \ell]\) can be given by
\[ z_{\ell,j} = \frac{\ell}{2}(z_j + 1), \quad \varpi_{\ell,j} = \left(\frac{\ell}{2}\right)\varpi_j, \quad 0 \leq j \leq N. \]

3. Shifting Legendre collocation method (SLCM)

Let
\[ a_0 = -\kappa_0, \quad a_1 = \frac{\kappa_0 - \kappa_1}{\ell}. \]
then by using the transformation
\[ V(z) = v(z) + a_0 + a_1z, \] (3.1)
The boundary conditions (1.1) will be
\[ V(0) = V(\ell) = 0. \] (3.2)

Hence it suffices to solve the modified variable-order fractional functional boundary value problem
\[ D^{\mu(z)}V(z) + \alpha(z)V'(z) + \beta(z)V(z) + \gamma(z)V(\tau(z)) = f(z), \quad z \in [0, \ell], \] (3.3)
subject to the homogeneous boundary conditions (3.2), where \( V(z) \) is given by (3.1), and
\[ f(z) = g(z) + \alpha(z)a_1 + \beta(z)(a_0 + a_1z) + \gamma(z)(a_0 + a_1\tau(z)). \]

Thus the approximate solution will be extended by using combination of basis functions of shifted Legendre polynomials, in the form
\[ V_N(z) \approx \sum_{i=0}^{N} c_i \phi_i(z) = C^T \varphi(z), \] (3.4)
where the shifted Legendre coefficient vector \( C \) is given by
\[ C^T = [c_0, c_1, \ldots, c_N] \] (3.5)
\( N \) is any arbitrary positive integer, and
\[ \phi_i(z) = P_i(z) + \zeta_i P_{i+1}(z) + \eta_i P_{i+2}(z). \] (3.6)
From the boundary conditions; \( V(0) = V(\ell) = 0 \) and the two relations (2.2), we have the accompanying framework
\[ 1 - \zeta_i + \eta_i = 0, \] (3.7)
\[ 1 + \zeta_i + \eta_i = 0. \] (3.8)
Thus $\zeta_i$ and $\eta_i$ can be remarkably resolved to give [35],

$$\zeta_i = 0, \quad \eta_i = -1.$$  
(3.9)

Also $\varphi(z)$ is given by

$$\varphi(z) = [\phi_0, \phi_1, \ldots, \phi_N]^T.$$  
(3.10)

Using (3.4) we can consider that

$$V_N(\tau(z)) \approx \sum_{i=0}^{N} c_i \phi_i(\tau(z)).$$  
(3.11)

Substituting Eqs. (1.2), (3.4) and (3.11) into Eq. (3.3) we will have:

$$1 = \frac{1}{\Gamma(2-\mu(z))} \int_0^z (z-t)^{1-\mu(z)} \sum_{i=0}^{N} c_i \phi_i''(t) dt + \alpha(z) \sum_{i=0}^{N} c_i \phi_i'(z) + \beta(z) \sum_{i=0}^{N} c_i \phi_i(z)$$

$$+ \gamma(z) \sum_{i=0}^{N} c_i \phi_i(\tau(z)) \approx f(z).$$  
(3.12)

Let

$$h_i(z) = \frac{1}{\Gamma(2-\mu(z))} \int_0^z (z-t)^{1-\mu(z)} \phi_i''(t) dt + \alpha(z) \phi_i'(z) + \beta(z) \phi_i(z) + \gamma(z) \phi_i(\tau(z))$$

Then, Eq. (3.12) can be rewritten as:

$$\sum_{i=0}^{N} c_i h_i(z) = f(z).$$  
(3.13)

Collocating Eq. (3.13) in $N+1$ roots of the shifted Legendre polynomial $P_{N+1}(z)$, the shifted Gauss-Legendre nodes, we will obtain:

$$\sum_{i=0}^{N} c_i h_i(z_j) = f(z_j), \text{ for } j = 0, 1, \ldots, N,$$  
(3.14)

which can be written in the following matrix form:

$$H^T C = F,$$

where

$$F = [f(z_0), f(z_1), \ldots, f(z_N)]^T,$$

and

$$H = (h_{ij}), \quad i, j = 0, 1, \ldots, N,$$  
(3.15)

in which the entries of the matrix $F$ are determined as follows:

$$h_{ij} = h_i(z_j), \quad i, j = 0, 1, \ldots, N.$$
Finally, the unknown vector $C$ can be computed by:

$$C = (H^T)^{-1}F.$$  

Therefore, the approximate solution of Eq. (3.3) is given by $V_N(z) = C^T\varphi(z)$.

In the following algorithm, we present the necessary steps of the proposed scheme.

**Remark 1.** The choice of nodes to be the roots of the shifted Legendre polynomials has the attraction, noted previously, that polynomial interpolation based on this set is relatively well behaved; in sharp contrast to this is the known very bad behavior of interpolation based on the equally spaced points (see [6]). The equally spaced case here yields very bad errors; similarly, it is not covered by existing approach to the collocation method [29,36], except in very special cases.

**Algorithm 1** Coding algorithm for the proposed scheme

**Input** $N \in \mathbb{N}$, $\ell \in \mathbb{R}^+$; the functions $\alpha(z)$, $\beta(z)$, $\gamma(z)$, $\mu(z)$, $\tau(z)$ and $g(z)$.

**Step 1.** Define the shifted Legendre polynomials by (2.1).

**Step 2.** Compute the basis function of shifted Legendre polynomials by (3.6).

**Step 3.** Define the basis function vector $\phi(z)$ by (3.10).

**Step 4.** Substituting Eqs. (1.2), (3.4) and (3.11) into Eq. (3.3).

**Step 5.** Collocating Eq. (3.13) in $N + 1$ roots of the polynomial $P_{N+1}(z)$.

**Step 6.** Compute the matrix $H$ using (3.15).

**Step 7.** Define the $(N + 1)$ unknown vector $C^T$.

**Step 8.** Use NSolve command to solve the system $H^T C = F$.

**Output** The approximate solution: $V_N(z) \simeq C^T\varphi(z)$.

4. **Convergence and Error Analysis**

In this part of the paper, we state and prove two theorems ascertain the convergence of the proposed approximate solution, to be more precise, in the first theorem we find an upper estimate for the expansion coefficients, in the second theorem, we find an estimate for the $L_2$–norm of the error.

**Lemma 1.** The basis functions $\{\phi_i(z)\}$ are orthogonal w.r.t. the positive weight function $w(z) = \frac{1}{z(\ell-z)}$, namely,

$$\int_0^\ell \frac{\phi_i(z)\phi_j(z)}{z(\ell-z)} \, dz = \frac{4(2i + 3)}{\ell(i + 1)(i + 2)} \delta_{ij}. \quad (4.1)$$

**Proof.** By noting that $\phi_i(z)$ are related with shifted Jacobi polynomials as follows

$$\phi_i(z) = \frac{2(2i + 3)}{\ell^2(i + 1)} z(\ell - z) J_i^{(1,1)}(z).$$

\[\Box\]
Theorem 4.1. [8] The repeated integration of shifted Legendre polynomials is given by

\[
\int \cdots \int P_i(z) \, dz \, \cdots \, dz = \frac{\ell^4}{4^r} \sum_{j=0}^r \binom{r}{j} (-1)^j \frac{(i + r - 2j + \frac{1}{2})}{\Gamma(i + r - j + \frac{3}{2})} P_{i+r-2j}(z). \tag{4.2}
\]

Theorem 4.2. Let \( V(z) \) is the exact solution of (3.3) which satisfy the homogenous boundary conditions (3.2), \( V(z) = z(\ell - z)u(z) \), \( |u^{(3)}(z)| \leq M \), and \( V(z) \) is approximated by \( V_N(z) = \sum_{i=0}^N c_i \phi_i(z) \). Then we will have

\[
| c_i | \leq \frac{\ell^4 M}{16 \, i^2}, \quad \forall i \geq 3.
\]

Proof. From Lemma 1 we have,

\[
c_i = \frac{\ell(i+1)(i+2)}{4(2i+3)} \int_0^\ell \phi_i(z) V_N(z) \frac{dz}{z(\ell - z)},
\]

and therefore by the hypothesis of the theorem we have

\[
c_i = \frac{\ell(i+1)(i+2)}{4(2i+3)} \int_0^\ell u(z)(P_i(z) - P_{i+2}(z)) \, dz.
\]

By applying integration by parts 3-times and using Theorem 4.1 for \( r = 3 \) we have

\[
| c_i | \leq \frac{M \ell^4(i+1)(i+2)}{2(2i-3)(2i+1)(2i+5)(2i+9)} \leq \frac{\ell^4 M}{16 \, i^2}.
\]

\[ \square \]

Theorem 4.3. Let \( V(z) = \sum_{i=0}^\infty c_i \phi_i(z) \) satisfies the hypothesis of theorem 4.2 and \( V_N(z) = \sum_{i=0}^N c_i \phi_i(z) \). Then we will have

\[
\| V - V_N \|_2 < \frac{\ell^2 M}{4N^2}.
\]
Proof. We have $V = \sum_{i=0}^{\infty} c_i \phi_i$ and $V_N = \sum_{i=0}^{N} c_i \phi_i$, and therefore $V - V_N = \sum_{i=N+1}^{\infty} c_i \phi_i$. Now, let
\[
\| V - V_N \|_2 = \sqrt{\langle V - V_N, V - V_N \rangle} = \sqrt{\left\| \sum_{i=N+1}^{\infty} c_i \phi_i(z) \right\|_2}
\]
by the orthogonality of $\{\phi_i\}$
\[
= \sqrt{\sum_{i=N+1}^{\infty} c_i^2 \| \phi_i(z) \|_2^2}
\]
\[
= \sqrt{\sum_{i=N+1}^{\infty} \frac{4(2i + 3)}{\ell(i + 1)(i + 2)} c_i^2}.
\]
Then by the result of theorem 4.2 we will have
\[
\| V - V_N \|_2^2 < \sum_{i=N+1}^{\infty} \frac{M^2 \ell^8 (2i + 3)}{4^4 \ell(i + 1)(i + 2)i^4}
\]
\[
< \sum_{i=N+1}^{\infty} \frac{M^2 \ell^7}{4i(i + 1)(i + 2)(i + 3)(i + 4)}
\]
\[
= \frac{M^2 \ell^7}{16(N + 1)(N + 2)(N + 3)(N + 4)}
\]
\[
< \frac{M^2 \ell^7}{16N^4},
\]
which completes the proof of the theorem.

5. Numerical results

In this section two numerical examples are presented to confirm the accuracy of the proposed method. Here, all the computations are carried out by using Mathematica, version 8.0, and all counts are completed in a PC of CPU Intel(R) Core(TM) i3-2350M 2 Duo CPU 2.30 GHz, 6.00 GB of RAM.

Example 1. ([19]). Consider the accompanying variable-order fractional functional boundary value problem of the form
\[
\begin{cases}
D^{\mu(z)}v(z) + \cos(z)v'(z) + 4v(z) + 5v(z^2) = g(z), & z \in [0, \ell], \\
v(0) = 0, \ v(\ell) = \ell^2,
\end{cases}
\]
Table 1. Comparison of the absolute errors at various choices of $z$, for Example 1.

<table>
<thead>
<tr>
<th>$z$</th>
<th>RKM [19] $n = 40$</th>
<th>SRKM [16] $n = 20$</th>
<th>RKSM [38] $n = 40$</th>
<th>our method $N = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$1.27 \times 10^{-8}$</td>
<td>$1.17 \times 10^{-8}$</td>
<td>$1.53 \times 10^{-14}$</td>
<td>$1.38 \times 10^{-11}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$2.14 \times 10^{-8}$</td>
<td>$1.77 \times 10^{-8}$</td>
<td>$1.13 \times 10^{-14}$</td>
<td>$5.55 \times 10^{-17}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$2.12 \times 10^{-8}$</td>
<td>$2.17 \times 10^{-8}$</td>
<td>$7.66 \times 10^{-14}$</td>
<td>$5.55 \times 10^{-17}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$3.05 \times 10^{-8}$</td>
<td>$2.39 \times 10^{-8}$</td>
<td>$4.08 \times 10^{-14}$</td>
<td>$5.55 \times 10^{-17}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$3.21 \times 10^{-8}$</td>
<td>$2.45 \times 10^{-8}$</td>
<td>$5.27 \times 10^{-16}$</td>
<td>0</td>
</tr>
<tr>
<td>0.6</td>
<td>$3.25 \times 10^{-8}$</td>
<td>$2.34 \times 10^{-8}$</td>
<td>$2.66 \times 10^{-15}$</td>
<td>$2.77 \times 10^{-17}$</td>
</tr>
<tr>
<td>0.7</td>
<td>$3.20 \times 10^{-8}$</td>
<td>$2.07 \times 10^{-8}$</td>
<td>$6.21 \times 10^{-15}$</td>
<td>$2.77 \times 10^{-17}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$3.87 \times 10^{-8}$</td>
<td>$1.59 \times 10^{-8}$</td>
<td>$9.65 \times 10^{-15}$</td>
<td>$8.32 \times 10^{-17}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$5.30 \times 10^{-8}$</td>
<td>$1.11 \times 10^{-8}$</td>
<td>$1.28 \times 10^{-14}$</td>
<td>$4.16 \times 10^{-17}$</td>
</tr>
</tbody>
</table>

Figure 1. Graph of exact solution and approximate solution at $\ell = 100$ and $N = 4$ for Example 1.

where $\mu(z) = \frac{5 + \sin(z)}{4}$, $g(z) = \frac{2z^2 - \mu(z)}{\Gamma(3 - \mu(z))} + \frac{z^4}{2^4} + 4z^2 + 2\cos(z)$. The exact solution is $v(z) = z^2$. Table 1 shows the maximum absolute errors by SLCM at $N = 2$. Our results also are compared with the reproducing kernel method (RKM) in [19], the simplified reproducing kernel method (SRKM) in [16] and the reproducing kernel splines method (RKSM) in [38]. It is confirmed that the proposed method is more accurate than the RKM [19], SRKM [16] and RKSM [38]. The Graph of analytical solution and approximate solution at $\ell = 100$ and $N = 4$ is displayed in Fig 1 to make it easier to compare with analytical solution. Absolute errors obtained by SLCM, with $\ell = 100$ and $N = 4$ are plotted in Fig 2.
Example 2. ([19]). Consider the accompanying variable-order fractional functional boundary value problem of the form

\[
\begin{cases}
D^{\mu(z)}v(z) + e^z v'(z) + 2v(z) + 8v(e^{z-1}) = g(z), \quad z \in [0, 1], \\
v(0) = 4, \quad v(1) = 9,
\end{cases}
\]

where \( g(z) = \frac{2z^{2-\mu(z)}}{\Gamma(3-\mu(z))} + 2(z^2 + 4z + 4) + 8(4e^{z-1} + e^{2z-2} + 4) + e^z(2z + 4). \) The exact solution is \( v(z) = z^2 + 4z + 4. \) The proposed shifted Legendre collocation method with \( \mu(z) = 6 + \cos(\frac{z}{4}), \quad N = 2 \) was compared to the shifted Chebyshev operational matrix (SCOM) [2]. Table 2 shows that the absolute errors obtained by the SLCM using few numbers of the shifted Legendre polynomials is significantly better than that obtained by SCOM [2]. Figure 3 shows the graphs of the absolute errors function between the exact and approximate solutions with \( \mu(z) = \frac{20 - e^z}{10}, \quad N = 2. \)

6. Conclusions

A shifted Legendre collocation method for solving variable-order fractional functional boundary value problem has been developed. This method uses shifted Gauss-Legendre nodes to reduce the considered VO-FFDEs boundary value problem to the solution of a matrix equation. The main advantage of the developed method relates to its high accurate solutions with few numbers of retained modes. Numerical illustrations were given to demonstrate the validity and applicability of the method. The results show that the presented method is simple and truthful.
Table 2. Comparison of the absolute errors at various choices of \( z \), for Example 2.

<table>
<thead>
<tr>
<th>( z )</th>
<th>SCOM [2] ( N = 10 )</th>
<th>our method ( N = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>( 3.059 \times 10^{-16} )</td>
<td>( 2.024 \times 10^{-18} )</td>
</tr>
<tr>
<td>0.2</td>
<td>( 6.009 \times 10^{-17} )</td>
<td>( 2.921 \times 10^{-17} )</td>
</tr>
<tr>
<td>0.3</td>
<td>( 7.999 \times 10^{-18} )</td>
<td>( 1.227 \times 10^{-19} )</td>
</tr>
<tr>
<td>0.4</td>
<td>( 1.620 \times 10^{-16} )</td>
<td>( 2.615 \times 10^{-17} )</td>
</tr>
<tr>
<td>0.5</td>
<td>( 1.124 \times 10^{-16} )</td>
<td>( 2.274 \times 10^{-18} )</td>
</tr>
<tr>
<td>0.6</td>
<td>( 8.261 \times 10^{-18} )</td>
<td>( 1.895 \times 10^{-18} )</td>
</tr>
<tr>
<td>0.7</td>
<td>( 1.106 \times 10^{-16} )</td>
<td>( 6.415 \times 10^{-19} )</td>
</tr>
<tr>
<td>0.8</td>
<td>( 2.170 \times 10^{-16} )</td>
<td>( 5.637 \times 10^{-17} )</td>
</tr>
<tr>
<td>0.9</td>
<td>( 1.007 \times 10^{-16} )</td>
<td>( 2.933 \times 10^{-17} )</td>
</tr>
</tbody>
</table>

Figure 3. Graph of absolute errors for Example 2 with \( \mu(z) = \frac{20 - e^z}{10} \), \( N = 2 \).

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