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Legendre-collocation spectral solver for variable-order fractional functional differential equations

Ramy Mahmoud Hafez
Department of Mathematics, Faculty of Education,
Matrouh University, Egypt.
E-mail: r_mhafez@yahoo.com

Youssri Hassan Youssri*
Department of Mathematics, Faculty of Science,
Cairo University, Giza-Egypt.
E-mail: youssri@sci.cu.edu.eg; youssri@aucegypt.edu

Abstract A numerical method for the variable-order fractional functional differential equations (VO-FFDEs) has been developed. This method is based on approximation with shifted Legendre polynomials. The properties of the latter were stated, first. These properties, together with the shifted Gauss-Legendre nodes were then utilized to reduce the VO-FFDEs into a solution of matrix equation. Sequentially, the error estimation of the proposed method was investigated. The validity and efficiency of our method were examined and verified via numerical examples.

Keywords. Variable-order fractional functional differential equations; shifted Legendre polynomials; Gauss-Legendre nodes; matrix equation.

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1. Introduction

A brief history of the development of fractional differential operators can be found in [23, 28]. Podlubny [30] deals lengthily with the theory of fractional order (non-integer) derivatives and integrals. Nowaday, research on fractional calculus is a hot topic (see for example [5, 17]).

Fractional calculus is currently being employed in several fields, including economics, engineering, and science [5]. However, problems involving fractional differential equations (FDEs) are already extensive and still growing, including interdisciplinary applications. FDEs provide much accurate models for systems under consideration. Applications of FDEs in the anomalous transport [22], bioengineering [20], colored noise [21], dynamics of interfaces between nanoparticles and substrates [4], economics [1], fluid-dynamic traffic model [14], frequency-dependent damping behavior of viscoelastic materials [39], and non-linear oscillation of earthquakes [13], solid mechanics [31] are fast-growing. The analytic results on the existence and uniqueness of FDEs solutions have been investigated [17, 30].

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∗ Corresponding author.

Adomian's decomposition [26], Bernstein polynomials [3], collocation [11], finite difference method [7], Galerkin [40], He's variational iteration [15,24,27], homotopy analysis [12], homotopy perturbation [25], Laplace transform [10], reproducing kernel method [18], reproducing kernel splines method [38], robust meshless method based on the moving least squares approximation and finite difference scheme [37], shifted Chebyshev operational matrix [2,9,33] and orthogonal spectral [32,34] are well-studied examples of numerical methods to handle FDEs.

This study aimed to introduce a numerical method to enhance the accuracy of the numerical solution of the VO-FFDEs Dirichlet boundary value problem [38]. Consider

$$D^{\mu(z)}v(z) + \alpha(z)v^{'}(z) + \beta(z)v(z) + \gamma(z)v(\tau(z)) = g(z),\ z \in [0, \ell], \tag{1.1}$$

subject to

$$v(0) = \kappa_0,\ v(\ell) = \kappa_1,$$

where $\alpha(z),\ \beta(z),\ \gamma(z) \in C^2[0, \ell]$, $\mu(z),\ \tau(z),\ g(z) \in C[0, \ell]$, $1 \le \mu(z) < 2$, $0 \le \tau \le \ell$, κ_0 and κ_1 are constant, $D^{\mu(z)}$ denotes the variable fractional order derivative in Caputo's sense defned as follows

$$D^{\mu(z)}v(z) = \frac{1}{\Gamma(2-\mu(z))}\int_0^z (z-t)^{1-\mu(z)}v^{''}(t)dt, \tag{1.2}$$

where $\Gamma(.)$ is Gamma function.

The proposed algorithm converts the VO-FFDEs (1.1) to a system of algebraic equations by combining the basis functions of shifted Legendre polynomials and the Gauss-shifted Legendre nodes as the collocation points.

The organization of the paper encompass; In Section 2, an overview of shifted Legendre polynomials and their relevant properties required henceforward are presented, and in Section 3, the way of constructing the collocation technique for VO-FFDEs is described by using the shifted Legendre polynomials. In Section 4, we give a detailed study of the convergence and error analyses In Section 5, the proposed method was applied to solve two examples. Finally, a conclusion is given in Section 6.

2. Mathematical preliminaries

The well-known Legendre polynomials $L_i(y)$ are defined on the interval $[-1, 1]$. Firstly, some properties about the standard Legendre polynomials have been recalled in this section. The Legendre polynomials satisfy

$$L_0(y) = 1, \quad L_1(y) = y, \quad L_{k+2}(y) = \frac{2k+3}{k+2}yL_{k+1}(y) - \frac{k+1}{k+2}L_k(y).$$

Let the shifted Legendre polynomials $L_i(\frac{2z}{\ell} - 1)$ be denoted by $P_i(z)$. Then $P_i(z)$ can be obtained as follows

$$(i+1)P_{i+1}(z) = (2i+1)(2z-1)P_i(z) - iP_{i-1}(z), \qquad i = 1, 2, \cdots.$$

Legendre polynomials have the following analytic form

$$P_i(z) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!\ (k!)^2\ \ell^k}\ z^k. \tag{2.1}$$

and

$$P_i(0) = (-1)^i, \; P_i(\ell) = 1. \tag{2.2}$$

We used z_j, and ϖ_j, $0 \le j \le N$, as the nodes and Christoffel numbers of the standard Legendre-Gauss interpolation in the interval $[-1, 1]$.

The corresponding nodes and corresponding Christoffel numbers of the shifted Legendre-Gauss interpolation in the interval $[0, \ell]$ can be given by

$$z_{\ell,j} = \frac{\ell}{2}(z_j + 1), \; \varpi_{\ell,j} = (\frac{\ell}{2})\varpi_j, \; 0 \le j \le N.$$

3. Shifted Legendre collocation method (SLCM)

Let

$$a_0 = -\kappa_0, \; a_1 = \frac{\kappa_0 - \kappa_1}{\ell}.$$

then by using the transformation

$$V(z) = v(z) + a_0 + a_1 z, \tag{3.1}$$

The boundary conditions (1.1) will be

$$V(0) = V(\ell) = 0. \tag{3.2}$$

Hence it suffices to solve the modified variable-order fractional functional boundary value problem

$$D^{\mu(z)}V(z) + \alpha(z)V^{'}(z) + \beta(z)V(z) + \gamma(z)V(\tau(z)) = f(z), \; z \in [0, \ell], \tag{3.3}$$

subject to the homogeneous boundary conditions (3.2), where $V(z)$ is given by (3.1), and

$$f(z) = g(z) + \alpha(z)a_1 + \beta(z)(a_0 + a_1 z) + \gamma(z)(a_0 + a_1\tau(z)).$$

Thus the approximate solution will be extended by using combination of basis functions of shifted Legendre polynomials, in the form

$$V_N(z) \approx \sum_{i=0}^{N} c_i\phi_i(z) = \mathbf{C}^T\varphi(z), \tag{3.4}$$

where the shifted Legendre coefficient vector C is given by

$$\mathbf{C}^T = [c_0, c_1, \ldots, c_N] \tag{3.5}$$

N is any arbitrary positive integer, and

$$\phi_i(z) = P_i(z) + \zeta_i P_{i+1}(z) + \eta_i P_{i+2}(z). \tag{3.6}$$

From the boundary conditions; $V(0) = V(\ell) = 0$ and the two relations (2.2), we have the accompanying framework

$$1 - \zeta_i + \eta_i = 0, \tag{3.7}$$

$$1 + \zeta_i + \eta_i = 0. \tag{3.8}$$

Thus ζ_i and η_i can be remarkably resolved to give [35],

$$\zeta_i = 0, \ \eta_i = -1. \tag{3.9}$$

Also $\varphi(z)$ is given by

$$\varphi(z) = [\phi_0, \phi_1, \dots, \phi_N]^T. \tag{3.10}$$

Using (3.4) we can consider that

$$V_N(\tau(z)) \approx \sum_{i=0}^{N} c_i \phi_i(\tau(z)). \tag{3.11}$$

Substituting Eqs. (1.2), (3.4) and (3.11) into Eq. (3.3) we will have:

$$\frac{1}{\Gamma(2-\mu(z))} \int_0^z (z-t)^{1-\mu(z)} \sum_{i=0}^{N} c_i \phi_i^{''}(t) dt + \alpha(z) \sum_{i=0}^{N} c_i \phi_i^{'}(z) + \beta(z) \sum_{i=0}^{N} c_i \phi_i(z) + \gamma(z) \sum_{i=0}^{N} c_i \phi_i(\tau(z)) \approx f(z). \tag{3.12}$$

Let

$$h_i(z) = \frac{1}{\Gamma(2-\mu(z))} \int_0^z (z-t)^{1-\mu(z)} \phi_i^{''}(t) dt + \alpha(z)\phi_i^{'}(z) + \beta(z)\phi_i(z) + \gamma(z)\phi_i(\tau(z))$$

Then, Eq. (3.12) can be rewritten as:

$$\sum_{i=0}^{N} c_i h_i(z) = f(z). \tag{3.13}$$

Collocating Eq. (3.13) in $N+1$ roots of the shifted Legendre polynomial $P_{N+1}(z)$, the shifted Gauss-Legendre nodes, we will obtain:

$$\sum_{i=0}^{N} c_i h_i(z_j) = f(z_j), \ \text{for}\, j = 0, 1, \dots, N, \tag{3.14}$$

which can be written in the following matrix form:

$$\mathbf{H}^T \mathbf{C} = \mathbf{F},$$

where

$$\mathbf{F} = [f(z_0), f(z_1), \dots, f(z_N)]^T,$$

and

$$\mathbf{H} = (h_{ij}), \ i, j = 0, 1, \dots, N, \tag{3.15}$$

in which the entries of the matrix F are determined as follows:

$$h_{ij} = h_i(z_j), \ i, j = 0, 1, \dots, N.$$

Finally, the unknown vector C can be computed by:

$$\mathbf{C} = (\mathbf{H}^T)^{-1}\mathbf{F}.$$

Therefore, the approximate solution of Eq. (3.3) is given by $V_N(z) = \mathbf{C}^T\varphi(z)$.
In the following algorithm, we present the necessary steps of the proposed scheme.

Remark 1. The choice of nodes to be the roots of the shifted Legendre polynomials has the attraction, noted previously, that polynomial interpolation based on this set is relatively well behaved; in sharp contrast to this is the known very bad behavior of interpolation based on the equally spaced points (see [6]). The equally spaced case here yields very bad errors; similarly, it is not covered by existing approach to the collocation method [29,36], except in very special cases.

Algorithm 1 Coding algorithm for the proposed scheme

Input $N \in \mathbb{N}$, $\ell \in \mathbb{R}^+$; the functions $\alpha(z)$, $\beta(z)$, $\gamma(z)$, $\mu(z)$, $\tau(z)$ and $g(z)$.
Step 1. Define the shifted Legendre polynomials by (2.1).
Step 2. Compute the basis function of shifted Legendre polynomials by (3.6).
Step 3. Define the basis function vector $\varphi(z)$ by (3.10).
Step 4. Substituting Eqs. (1.2), (3.4) and (3.11) into Eq. (3.3).
Step 5. Collocating Eq. (3.13) in $N+1$ roots of the polynomial $P_{N+1}(z)$.
Step 6. Compute the matrix $\mathbf{H}$ using (3.15).
Step 7. Define the $(N+1)$ unknown vector $\mathbf{C}^T$.
Step 8. Use *NSolve* command to solve the system $\mathbf{H}^T\mathbf{C} = \mathbf{F}$.
Output The approximate solution: $V_N(z) \simeq \mathbf{C}^T\varphi(z)$.

4. Convergence and Error Analysis

In this part of the paper, we state and prove two theorems ascertain the convergence of the proposed approximate solution, to be more precise, in the first theorem we find an upper estimate for the expansion coefficients, in the second theorem, we find an estimate for the L_2-norm of the error.

Lemma 1. *The basis functions $\{\phi_i(z)\}$ are orthogonal w.r.t. the positive weight function $w(z) = \frac{1}{z(\ell-z)}$, namely,*

$$\int_0^\ell \frac{\phi_i(z)\phi_j(z)}{z(\ell-z)}dz = \frac{4(2i+3)}{\ell(i+1)(i+2)}\delta_{ij}. \tag{4.1}$$

Proof. By noting that $\phi_i(z)$ are related with shifted Jacobi polynomials as follows

$$\phi_i(z) = \frac{2(2i+3)}{\ell^2(i+1)}z(\ell-z)J_i^{(1,1)}(z).$$

□

Theorem 4.1. *[8]The repeated integration of shifted Legendre polynomials is given by*

$$\underbrace{\int\int\dots\int}_{r-times} P_i(z)\,\underbrace{dzdz\dots dz}_{r-times} = \frac{\ell^4}{4^r}\sum_{j=0}^{r}\binom{r}{j}(-1)^j\frac{(i+r-2j+\frac{1}{2})}{\Gamma(i+r-j+\frac{3}{2})}P_{i+r-2j}(z). \qquad (4.2)$$

Theorem 4.2. *Let $V(z)$ is the exact solution of (3.3) which satisfy the homogenous boundary conditions (3.2), $V(z) = z(\ell - z)u(z)$, $|u^{(3)}(z)| \leq M$, and $V(z)$ is approximated by $V_N(z) = \sum_{i=0}^{N} c_i\phi_i(z)$. Then we will have*

$$|\, c_i \,| \leq \frac{\ell^4 M}{16\ i^2}, \quad \forall i \geq 3.$$

Proof. From Lemma 1 we have,

$$c_i = \frac{\ell(i+1)(i+2)}{4(2i+3)}\int_0^{\ell}\frac{\phi_i(z)V_N(z)}{z(\ell-z)}dz,$$

and therefore by the hypothesis of the theorem we have

$$c_i = \frac{\ell(i+1)(i+2)}{4(2i+3)}\int_0^{\ell} u(z)(P_i(z) - P_{i+2}(z))dz.$$

By applying integration by parts 3-times and using Theorem 4.1 for $r = 3$ we have

$$|\, c_i \,| \leq \frac{M\ell^4(i+1)(i+2)}{2(2i-3)(2i+1)(2i+5)(2i+9)} < \frac{\ell^4 M}{16\ i^2}.$$

□

Theorem 4.3. *Let $V(z) = \sum_{i=0}^{\infty} c_i\phi_i(z)$ satisfies the hypothesis of theorem 4.2 and $V_N(z) = \sum_{i=0}^{N} c_i\phi_i(z)$. Then we will have*

$$\|\, V - V_N \,\|_2 < \frac{\ell^{\frac{7}{2}} M}{4N^2}.$$

Proof. We have $V = \sum_{i=0}^{\infty} c_i \, \phi_i$ and $V_N = \sum_{i=0}^{N} c_i \, \phi_i$,

and therefore $V - V_N = \sum_{i=N+1}^{\infty} c_i \, \phi_i$. Now, let

$$
\begin{aligned}
\| V - V_N \|_2 &= \sqrt{< V - V_N, V - V_N >} \\
&= \| \sum_{i=N+1}^{\infty} c_i \phi_i(z) \|_2 \\
&= \sqrt{< \sum_{i=N+1}^{\infty} c_i \, \phi_i, \sum_{j=N+1}^{\infty} c_j \, \phi_j >} \\
&\text{by the orthogonality of } \{\phi_i\} \\
&= \sqrt{\sum_{i=N+1}^{\infty} c_i^2 \| \phi_i(z) \|_2^2} \\
&= \sqrt{\sum_{i=N+1}^{\infty} \frac{4(2i+3)}{\ell(i+1)(i+2)} c_i^2}.
\end{aligned}
$$

Then by the result of theorem 4.2 we will have

$$
\begin{aligned}
\| V - V_N \|_2^2 &< \sum_{i=N+1}^{\infty} \frac{M^2 \ell^8 (2i+3)}{4^3 \ell (i+1)(i+2) i^4} \\
&< \sum_{i=N+1}^{\infty} \frac{M^2 \ell^7}{4i(i+1)(i+2)(i+3)(i+4)} \\
&= \frac{M^2 \ell^7}{16(N+1)(N+2)(N+3)(N+4)} \\
&< \frac{M^2 \ell^7}{16 N^4},
\end{aligned}
$$

which completes the proof of the theorem. □

5. Numerical results

In this section two numerical examples are presented to confirm the accuracy of the proposed method. Here, all the computations are carried out by using Mathematica, version 8.0,and all counts are completed in a PC of CPU Intel(R) Core(TM) i3-2350M 2 Duo CPU 2.30 GHz, 6.00 GB of RAM.

Example 1. *([19]). Consider the accompanying variable-order fractional functional boundary value problem of the form*

$$
\begin{cases}
D^{\mu(z)} v(z) + \cos(z) v^{'}(z) + 4v(z) + 5v(\frac{z^2}{\ell^2}) = g(z), \quad z \in [0, \ell], \\
v(0) = 0, \;\; v(\ell) = \ell^2,
\end{cases}
$$

TABLE 1. Comparison of the absolute errors at various choices of z, for Example 1.

z	RKM [19] $n = 40$	SRKM [16] $n = 20$	RKSM [38] $n = 40$	our method $N = 2$
0.1	1.27×10^{-8}	1.17×10^{-8}	1.53×10^{-14}	1.38×10^{-17}
0.2	2.14×10^{-8}	1.77×10^{-8}	1.13×10^{-14}	5.55×10^{-17}
0.3	2.12×10^{-8}	2.17×10^{-8}	7.66×10^{-15}	5.55×10^{-17}
0.4	3.05×10^{-8}	2.39×10^{-8}	4.08×10^{-15}	5.55×10^{-17}
0.5	3.21×10^{-8}	2.45×10^{-8}	5.27×10^{-16}	0
0.6	3.25×10^{-8}	2.34×10^{-8}	2.66×10^{-15}	2.77×10^{-17}
0.7	3.20×10^{-8}	2.07×10^{-8}	6.21×10^{-15}	2.77×10^{-17}
0.8	3.87×10^{-8}	1.59×10^{-8}	9.65×10^{-15}	8.32×10^{-17}
0.9	5.30×10^{-8}	1.11×10^{-8}	1.28×10^{-14}	4.16×10^{-17}

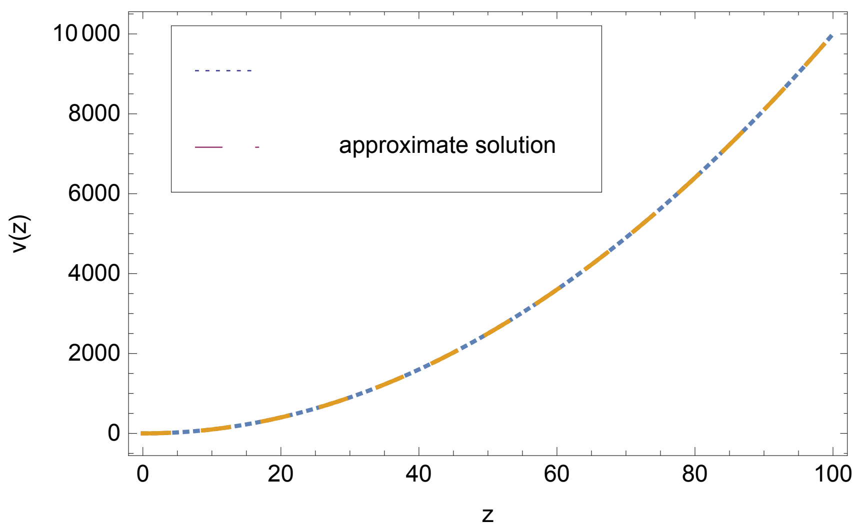

FIGURE 1. Graph of exact solution and approximate solution at $\ell = 100$ and $N = 4$ for Example 1.

where $\mu(z) = \dfrac{5 + \sin(z)}{4}$, $g(z) = \dfrac{2z^{2-\mu(z)}}{\Gamma(3 - \mu(z))} + \frac{5z^4}{\ell^4} + 4z^2 + 2z\cos(z)$. The exact solution is $v(z) = z^2$. Table 1 shows the maximum absolute errors by SLCM at $N = 2$. Our results also are compared with the reproducing kernel method (RKM) in [19], the simplifed reproducing kernel method (SRKM) in [16] and the reproducing kernel splines method (RKSM) in [38]. It is confirmed that the proposed method is more accurate than the RKM [19], SRKM [16] and RKSM [38]. The Graph of analytical solution and approximate solution at $\ell = 100$ and $N = 4$ is displayed in Fig 1 to make it easier to compare with analytical solution. Absolute errors obtained by SLCM, with $\ell = 100$ and $N = 4$ are plotted in Fig 2.

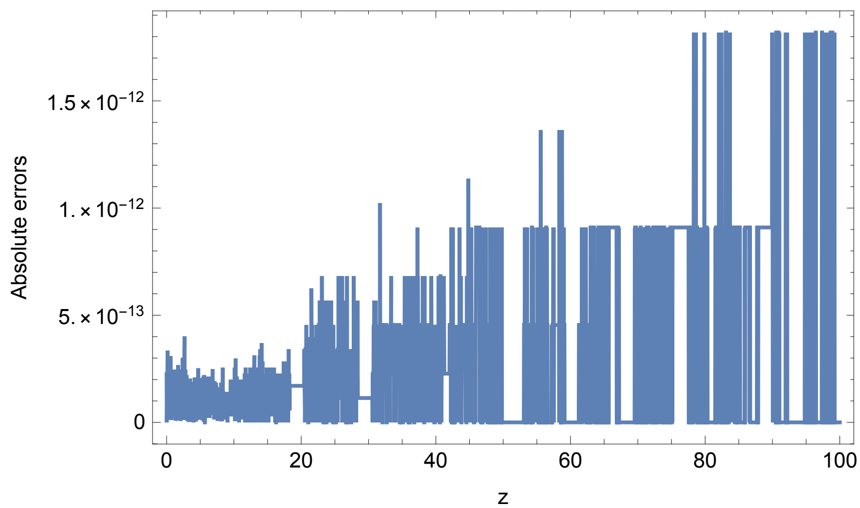

FIGURE 2. Graph of absolute errors at $\ell = 100$ and $N = 4$ for Example 1.

Example 2. *([19]). Consider the accompanying variable-order fractional functional boundary value problem of the form*

$$\begin{cases} D^{\mu(z)}v(z) + e^z v'(z) + 2v(z) + 8v(e^{z-1}) = g(z), & z \in [0,1], \\ v(0) = 4, \ v(1) = 9, \end{cases}$$

where $g(z) = \dfrac{2z^{2-\mu(z)}}{\Gamma(3-\mu(z))} + 2(z^2 + 4z + 4) + 8(4e^{z-1} + e^{2z-2} + 4) + e^z(2z+4)$. The exact solution is $v(z) = z^2 + 4z + 4$. The proposed shifted Legendre collocation method with $\mu(z) = \dfrac{6+\cos(z)}{4}$, $N = 2$ was compared to the shifted Chebyshev operational matrix (SCOM) [2]. Table 2 shows that the absolute errors obtained by the SLCM using few numbers of the shifted Legendre polynomials is significantly better than that obtained by SCOM [2]. Figure 3 shows the graphs of the absolute errors function between the exact and approximate solutions with $\mu(z) = \dfrac{20-e^z}{10}$, $N = 2$.

6. Conclusions

A shifted Legendre collocation method for solving variable-order fractional functional boundary value problem has been developed. This method uses shifted Gauss-Legendre nodes to reduce the considered VO-FFDEs boundary value problem to the solution of a matrix equation. The main advantage of the developed method relates to its high accurate solutions with few numbers of retained modes. Numerical illustrations were given to demonstrate the validity and applicability of the method. The results show that the presented method is simple and truthful.

TABLE 2. Comparison of the absolute errors at various choices of z, for Example 2.

z	SCOM [2] $N = 10$	our method $N = 2$
0.1	3.059×10^{-16}	2.024×10^{-18}
0.2	6.009×10^{-17}	2.921×10^{-17}
0.3	7.999×10^{-18}	1.227×10^{-19}
0.4	1.620×10^{-16}	2.615×10^{-17}
0.5	1.124×10^{-16}	2.274×10^{-18}
0.6	8.261×10^{-18}	1.895×10^{-18}
0.7	1.106×10^{-16}	6.415×10^{-19}
0.8	2.170×10^{-16}	5.637×10^{-17}
0.9	1.007×10^{-16}	2.933×10^{-17}

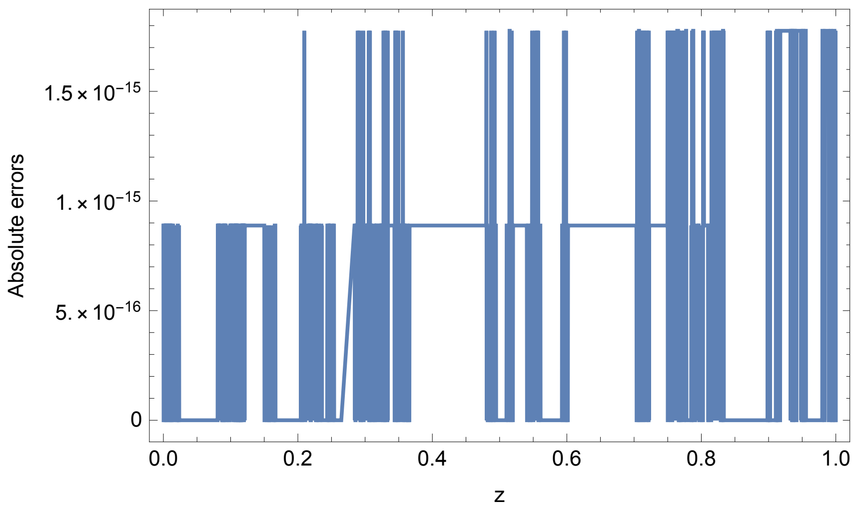

FIGURE 3. Graph of absolute errors for Example 2 with $\mu(z) = \dfrac{20 - e^z}{10}$, $N = 2$.

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References

[1] R.T. Baillie, *Long memory processes and fractional integration in econometrices*, J. Econometrices, *73* (1996), 5–59.

[2] A.H. Bhrawy and M.A. Zaky, *Numerical algorithm for the variable-order Caputo fractional functional differential equation*, Nonlinear Dyn, *85* (2016), 1815–1823.

[3] Y.M. Chen, L.Q. Liu, D. Liu, and et al, *Numerical study of a class of variable order nonlinear fractional differential equation in terms of Bernstein polynomials*, Ain Shams Eng. J, 2016.

[4] T.S. Chow, *Fractional dynamics of interfaces etween soft-nanoparticls and rough sustrates*, Phys. Lett. A, *342* (2005), 148–155.

[5] S. Das, *Functional Fractional Calculus for System Identification and Controls*, Springer, New York, 2008.

[6] P.J. Davis, *Interpolation and approximation*, Courier Corporation, 1975.

[7] W.H. Deng, S.D. Du, and Y.J. Wu, *High order fnite difference WENO schemes for fractional differential equations*, Appl. Math. Lett, *26* (2013), 362–366.

[8] E.H. Doha, *On the coefficients of integrated expansions and integrals of ultraspherical polynomials and their applications for solving differential equations*, J. Comput. Appl. Math, *139*(2) (2002), 275–298.

[9] E.H. Doha, W.M. Abd-Elhameed, N.A. Elkot, and Y.H. Youssri, *Integral spectral Tchebyshev approach for solving space Riemann-Liouville and Riesz fractional advection-dispersion problems*, Advances in Difference Equations, *284* (2017), DOI: 10.1186/s13662-017-1336-6.

[10] S. Gupta, D. Kumar, and J. Singh, *Numerical study for systems of fractional differential equations via Laplace transform*, J. Egyptian Math. Soc, *23* (2015), 256–262.

[11] R.M. Hafez and Y.H. Youssri, *Jacobi collocation scheme for variable-order fractional reaction-subdiffusion equation*, Computational and Applied Mathematics *37*(4)(2018), 5315–5333.

[12] I. Hashim, O. Abdulaziz, and S. Momani, *Homotopy analysis method for fractional IVPs*, Commun Nonlinear Sci. Numer. Simul, *14* (2009), 674–684.

[13] J.H. He, *Nonlinear oscillation with fractional derivatives and its applications*, International Conference on Vibrating Engineering'98, Dalia, Chinan, (1998), 288–291.

[14] J.H. He, *Some applications of nonlinear fractional differential equations and thier approximations Bull*, Sci. Technol, *15* (1999), 86–90.

[15] M. Inc, *The approximate andexat solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method*, J. Math. Anal Appl, *345* (2008), 476–484.

[16] Y.T. Jia, M.Q. Xu, and Y.Z. Lin, *A numerical solution for variable order fractional functional differential equation*, Appl. Math. Lett, *64* (2017), 125–130.

[17] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, San Diego, (2006).

[18] X. Li, H. Li, and B. Wu, *A new numerical method for variable order fractional functional differential equations*, Appl. Math. Lett, *68* (2017), 80–86.

[19] X. Li and B. Wu, *A numerical technique for variable fractional functional boundary value problems*, Appl. Math. Lett, *43* (2015), 108–113.

[20] R.L. Magin, *Frational calculus in bioengineering*, Crit. Rev. Biomed. Eng, *32* (2004), 1–104.

[21] B. Mandelbort, Some noises with 1/f spectrum, a bridge between direct current and white noise IEEE Trans. Inform. Theory, *13* (1967), 289–298.

[22] R. Metzler and J. Klafter, *The restaurant at the end of the random walk: Recent developments in the description of anomalos transport by fractional dyuamics*, J. Phys. A, *37* (2004), 161–208.

[23] K.S. Miller and B. Ross, *An Introduaction to the Frational Calculus and Fractional Differential Equations*, Wiley, New York 1993.

[24] S. Momani and Z. Odia, *Analytical approach to linear fractional partial differential equations arising in fluid mechanics*, Phys. Lett. A, *355* (2006), 271–279.

[25] S. Momani and Z. Odibat, *Homotopy perturpation method for nonlinear partial differential equations of fractional order*, Phys. Lett. A, *365* (2007), 345–350.

[26] S. Momani and N.T. Shawagfeh, *Decomposition method for solving fractional Riccati differential equations*, Appl. Math. Comput, *182* (2006), 1083–1092.

[27] Z. Odibat and S. Momani, *Applications of variational iteration method to nonlinear differential equations of fractional order*, IntJ. Nonlinear Sci. Numer. Simul, *7* (2006), 271–279.
[28] K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.
[29] J.L. Phillips, *The use of collocation as a projection method for solving linear operator equations*, SIAM Journal on Numerical Analysis *9*(1) (1972), 14–28.
[30] I. Podluny,*Fractional Differential Equations*, Academic Press, San Diego, 1999.
[31] Y.A. Rossikhin and M.V. Shitikova, *Applications of fractional calculus to dynamic prolems of linear and nonlinear hereditary mechanics of solids*, Appl. Mech. Rev, *50* (1997), 15–67.
[32] A. Saadatmandi and M. Dehghan, *A new operational matrix for solving fractional-order differential equations*, Computers and Mathematics with Applications, *59*(3) (2010), 1326-1336.
[33] A. Saadatmandi and M. Dehghan, *A Legendre collocation method for fractional integro-differential equations*, Journal of Vibration and Control, *17*(13) (2011), 2050-2058.
[34] A. Saadatmandi and M. Dehghan, *A tau approach for solution of the space fractional diffusion equation*, Computers and Mathematics with Applications, *62*(3) (2011), 1135-1142.
[35] J. Shen, *Efficient Spectral-Galerkin Method I. Direct Solvers for the Second and Fourth Order Equations Using Legendre Polynomials*, Siam J. Sci. Comput, *15* (1994), 1489-1505.
[36] I.H. Sloan and B. J. Burn, *Collocation with polynomials for integral equations of the second kind: a new approach to the theory*, The Journal of Integral Equations (1979), 77-94.
[37] A. Tayebi, Y. Shekari, and M.H. Heydari, *A meshless method for solving two-dimensional variable-order time fractional advection-diffusion equation*, J. Comput. Phys, *340* (2017), 655-669.
[38] J. Yang, H. Yao, and B. Wu, *An efcient numerical method for variable order fractional functional differential equation*, Appl. Math. Lett, *76* (2018), 221-226.
[39] Y.H. Youssri, *A new operational matrix of Caputo fractional derivatives of Fermat polynomials: an application for solving the Bagley-Torvik equation*, Advances in Difference Equations *2017*(1) (2017), 73.
[40] Y.H. Youssri and W.M. Abd-Elhameed, *Numerical Spectral Legendre-Galerkin Algorithm for Solving Time Fractional Telegraph Equation*, Romanian Journal of Physics (2018) 63, 107.