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Impulsive initial value problems for a class of implicit fractional differential equations

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Abstract In this article we consider, impulsive initial value problems for a class of implicit fractional differential equations involving the Caputo fractional derivative of order $\beta \in (1, 2]$. The solutions of this nonlinear equation are analyzed by establishing sufficient conditions for existence and uniqueness using Banach's contraction mapping principle and the Schaefer's fixed point theorem. In addition, using the Banach contraction principle, we establish uniqueness result. To demonstrate main results two examples are presented.

Keywords. Initial value problem; Implicit fractional differential equations; Impulses; Fixed point theorem.
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1. Introduction

The theory of fractional calculus has received remarkable consideration from the researchers of widespread fields of science and engineering due its applicability of describing the hereditary and memory effect of numerous processes arise in mathematical modeling of many nonlinear phenomena which includes aerodynamics, thermodynamics, diffusion processes, control theory, blood flow phenomena, electromagnetic and many more, see the monographs [2, 13, 14, 15, 17]. The analysis of these types of initial and boundry value problems involving fractional differential operators plays significant role in finding applications to many realistic problems. Several numerical and analytical methods are suggested to find exact and approximate solutions of nonlinear fractional differential equations [8, 9, 11, 18, 19, 20].
Moreover, the study of impulsive fractional differential equations has turned out to be significant object of research as of late due to its extensive applications to numerous physical phenomena in nature like the impact of mechanical systems, dynamical systems with automatic regulations, electromechanical systems having relaxation oscillations, the function of pendulum clock, and so forth are governed by impulsive

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fractional differential equations [3, 16, 21, 24]. These equations gives natural description of the processes of development subjected to unexpected changes and irregular jumps in their states and present a realistic outline for describing several mathematical models. Many mathematicians contributed in development of existence, uniqueness and stability results of solution for fractional differential equations with impulses [1, 4, 5, 12, 22, 23].

In [7], Benchohra and Slimani considered the impulsive initial value problem to find existence and uniqueness criteria of solutions,

$$\begin{aligned} {}^cD^\beta u(\mathbf{t}) &= f(\mathbf{t}, u(\mathbf{t})), \quad \mathbf{t} \in \Omega = [0, T], \\ &\mathbf{t} \neq \mathbf{t}_j, \quad j = 1, 2, \cdots, m, \quad \beta \in (0, 1], \\ \Delta u|_{\mathbf{t}=\mathbf{t}_j} &= I_j(u(\mathbf{t}_j^-)), \quad j = 1, 2, \cdots, m \\ u(0) &= u_0, \end{aligned}$$

where ${}^cD^\beta$ is the Caputo fractional derivative, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous function, $u(0) = u_0 \in \mathbb{R}$, $I_j : \mathbb{R} \to \mathbb{R}, \ j = 1, 2, \cdots, m$, and $0 = \mathbf{t}_0 < \mathbf{t}_1 < \mathbf{t}_2 < \cdots < \mathbf{t}_j < \cdots < \mathbf{t}_m < \mathbf{t}_{m+1} = T$.

In [6], Benchohra and Lazreg have discussed fractional order impulsive differential equations

$$\begin{aligned} {}^cD^\beta u(\mathbf{t}) &= f(\mathbf{t}, u(\mathbf{t}), D^\beta u(\mathbf{t})) \quad \mathbf{t} \in \Omega = [0, T], \\ &\mathbf{t} \neq \mathbf{t}_j, \quad j = 1, 2, \cdots, m, \quad \beta \in (0, 1], \\ \Delta u|_{\mathbf{t}=\mathbf{t}_j} &= I_j(u(\mathbf{t}_j^-)), \quad j = 1, 2, \cdots, m \\ u(0) &= u_0, \end{aligned}$$

where ${}^cD^\beta$ is the Caputo fractional derivative, $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $u(0) = u_0 \in \mathbb{R}$, $I_j : \mathbb{R} \to \mathbb{R}, \ j = 1, 2, \cdots, m$, and $0 = \mathbf{t}_0 < \mathbf{t}_1 < \mathbf{t}_2 < \cdots < \mathbf{t}_j < \cdots < \mathbf{t}_m < \mathbf{t}_{m+1} = T$.

Motivated by some ongoing work on impulsive fractional differential equation, in this paper, we explore the existence and uniqueness results for the implicit fractional differential equations with impulses:

$$\begin{aligned} {}^cD^\beta u(\mathbf{t}) &= f(\mathbf{t}, u(\mathbf{t}), D^\beta u(\mathbf{t})), \quad \mathbf{t} \in \Omega = [0, T], \quad (1.1) \\ &\mathbf{t} \neq \mathbf{t}_j, \quad j = 1, 2, \cdots, m, \ \beta \in (1, 2] \\ \Delta u|_{\mathbf{t}=\mathbf{t}_j} &= I_j(u(\mathbf{t}_j^-)), \quad j = 1, 2, \cdots, m \quad (1.2) \\ \Delta u'|_{\mathbf{t}=\mathbf{t}_j} &= \bar{I}_j(u(\mathbf{t}_j^-)), \ j = 1, 2, \cdots, m \quad (1.3) \\ u(0) &= u_0, \ u'(0) = u_1, \quad (1.4) \end{aligned}$$

where ${}^cD^\beta$ signifies the Caputo fractional derivative, $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function, $u(0) = u_0, \ u'(0) = u_1 \in \mathbb{R}$, $I_j, \ \bar{I}_j : \mathbb{R} \to \mathbb{R}, \ j = 1, 2, \cdots, m$, and $0 = \mathbf{t}_0 < \mathbf{t}_1 < \mathbf{t}_2 < \cdots < \mathbf{t}_j < \cdots < \mathbf{t}_m < \mathbf{t}_{m+1} = T$, $\Delta u|_{\mathbf{t}=\mathbf{t}_j} = u(\mathbf{t}_j^+) - u(\mathbf{t}_j^-)$, $u(\mathbf{t}_j^+) = \lim_{\zeta \to 0^+} u(\mathbf{t}_j + \zeta)$ and $u(\mathbf{t}_j^-) = \lim_{\zeta \to 0^-} u(\mathbf{t}_j + \zeta)$ shows the right and left limits of $u(\mathbf{t})$ at

$\mathbf{t} = \mathbf{t}_j,\ j = 1, 2, \cdots, m.$

The rest of this paper is sorted out as follows. In Section 2, some helpful fundamental definitions and introductory results which will be required to demonstrate our principle hypotheses are displayed. In Section 3, we center around the verification of main results of existence and uniqueness criteria by utilizing Schaefer's fixed point theorem. In Section 4, two examples are introduced to represent the fundamental outcomes.

2. Preliminaries

Throughout this article we denote $C(\Omega, \mathbb{R})$ as the Banach space of all continuous functions with the norm $\|u\|_\infty = \sup\{|u(\mathbf{t})| : \mathbf{t} \in \Omega\}$.

Further, we introduce the Banach space

$$PC(\Omega, \mathbb{R}) = \Big\{ u : \Omega \to \mathbb{R}/u \in C((\mathbf{t}_i, \mathbf{t}_{i+1}], \mathbb{R}), i = 0, 1, \cdots, m \text{ in addtion}$$
$$\text{there exists } u(\mathbf{t}_i^-) \text{and } u(\mathbf{t}_i^+), i = 1, 2, \cdots, m \text{ with } u(\mathbf{t}_i^-) = u(\mathbf{t}_i^+) \Big\}.$$

$PC(\Omega, \mathbb{R})$ is the Banach space with the norm $\|u\|_{PC} = \sup_{\mathbf{t}\in\Omega} |u(\mathbf{t})|$.

Set $\Omega' := [0, T] \setminus \{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_m\}$.

Definition 2.1. ([14]) The Riemann-Liouville fractional integral operator of order $\beta \in \mathbb{R}_+$ for the function $f \in L^1([a, b], \mathbb{R}_+)$ is defined as,

$$I_a^\beta f(\mathbf{t}) = \frac{1}{\Gamma(\beta)} \int_a^{\mathbf{t}} (\mathbf{t} - x)^{\beta-1} f(x) dx, \quad \mathbf{t} > a.$$

Definition 2.2. ([14]) The Caputo fractional derivative operator for the function $f \in L^1([a, b], \mathbb{R}_+)$, is defined for order $\beta \geq 0$ by

$$D_{\mathbf{t}}^\beta f(\mathbf{t}) = \frac{1}{\Gamma(n-\beta)} \int_a^{\mathbf{t}} (\mathbf{t} - x)^{n-\beta-1} \frac{d^n}{dx^n} f(x) dx, \quad \mathbf{t} > a,$$

where $n - 1 < \beta \leq n \in \mathbb{N} \cup \{0\}$.

Proposition 2.3. [14] *For* $\alpha,\ \beta > 0$. *We have*
(*i*) $I^\beta : L^1([0, T], \mathbb{R}_+) \to L^1([0, T], \mathbb{R}_+)$, *and if* $f \in L^1([0, T], \mathbb{R}_+)$, *then*

$$I^\alpha I^\beta f(\mathbf{t}) = I^\beta I^\alpha f(\mathbf{t}) = I^{\alpha+\beta} f(\mathbf{t}).$$

(*ii*) *If* $f \in L^P([0, T],\ \mathbb{R}_+)$, $1 \leq p \leq +\infty$, *then* $\|I^\beta f\|_{L_p} \leq \frac{T^\beta}{\Gamma(\beta+1)} \|f\|_{L_p}$.
(*iii*) $\lim_{\beta\to n} I^\beta f(\mathbf{t}) = I^n f(\mathbf{t}),\ n = 1, 2, \cdots$. *uniformly.*

We state the following theorems

Theorem 2.4. *([10]) For non-empty closed subset* U *of a Banach space* X*, any contraction mapping* F *of* U *into itself has a unique fixed point.*

Theorem 2.5. [10] *Let X be a Banach space, and assume that $F : X \to X$ is a completely continuous operator with the set S is bounded, where*

$$S = \{u \in X | u = \theta F u, \text{ for some } \theta \in (0,1)\}$$

then F has fixed point.

The above theorem is known as Schaefer's fixed point theorem. Motivated from [5], for further analysis we consider the following hypothesis are satisfied.

(H1) Consider the function $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.
(H2) There exists constants $A > 0$ and $0 < B < 1$, such that

$$|f(\mathbf{t}, x_1, x_2) - f(\mathbf{t}, y_1, y_2)| \leq A|x_1 - y_1| + B|x_2 - y_2|,$$

$$\forall \mathbf{t} \in \Omega \text{ and } x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

(H3) For all $x,\ y \in \mathbb{R}$, and $\mathbf{t} \in \Omega$, there exists constants $\kappa,\ \bar{\kappa}\ > 0$ such that

$$|I_j(x) - I_j(y)| \leq \kappa|x - y|,$$
and
$$|\bar{I}_j(x) - \bar{I}_j(y)| \leq \bar{\kappa}|x - y|, \quad j = 1, 2, \cdots, m.$$

3. Main results

Definition 3.1. A function $u \in PC(\Omega, \mathbb{R})$ is said to be a solution of (1.1)-(1.4) with its β-derivative if u satisfies the equation ${}^cD^\beta u(\mathbf{t}) = f(\mathbf{t}, u(\mathbf{t}))$ on Ω, and the conditions

$$\Delta u|_{\mathbf{t}=\mathbf{t}_j} = I_j(u(\mathbf{t}_j^-)),$$

$$\Delta u'|_{\mathbf{t}=\mathbf{t}_j} = \bar{I}_j(u(\mathbf{t}_j^-)),$$

$$u(0) = u_0,\ u'(0) = u_1,$$

are satisfied for $j = 1, 2, \cdots, m$.

Lemma 3.2. [25] *The differential equation*

$$ {}^cD^\beta p(\mathbf{t}) = 0, \quad \beta > 0, \tag{3.1}$$

has solutions $p(\mathbf{t}) = k_0 + k_1\mathbf{t} + k_2\mathbf{t}^2 + \cdots + k_n\mathbf{t}^{n-1},\ n = [\beta] + 1,\ k_i \in \mathbb{R},\ i = 0, 1, 2, \cdots, n.$

Lemma 3.3. [25]. *Let $\beta > 0$, then*

$$I^\beta D^\beta p(\mathbf{t}) = p(\mathbf{t}) + k_0 + k_1\mathbf{t} + k_2\mathbf{t}^2 + \cdots + k_n\mathbf{t}^{n-1}, \tag{3.2}$$

for $n = [\beta] + 1$ and $k_i \in \mathbb{R},\ i = 0, 1, 2, \cdots, n.$

Lemma 3.4. [1] *Let* p:$\Omega \to \mathbb{R}$ *is continuous and* $1 < \beta \leq 2$. *A function* u *is a solution of the impulsive fractional integral equation*

$$u(\mathbf{t}) = \begin{cases} u_0 + u_1\mathbf{t} + \frac{1}{\Gamma(\beta)} \int_0^{\mathbf{t}} (\mathbf{t} - \tau)^{\beta-1} p(\tau) d\tau \;\; for \;\; \mathbf{t} \in [0, \mathbf{t}_1], \\ u_0 + u_1\mathbf{t} + \frac{1}{\Gamma(\beta)} \sum_{k=1}^{j} \int_{\mathbf{t}_{k-1}}^{\mathbf{t}_k} (\mathbf{t}_k - \tau)^{\beta-1} p(\tau) d\tau + \\ + \frac{1}{\Gamma(\beta-1)} \sum_{k=1}^{j} (\mathbf{t} - \mathbf{t}_k) \int_{\mathbf{t}_{k-1}}^{\mathbf{t}_k} (\mathbf{t}_k - \tau)^{\beta-2} p(\tau) d\tau + \\ + \frac{1}{\Gamma(\beta)} \int_{\mathbf{t}_k}^{\mathbf{t}} (\mathbf{t} - \tau)^{\beta-1} p(\tau) d\tau + \\ + \sum_{k=1}^{j} I_k(u(\mathbf{t}_k^-)) + \sum_{k=1}^{j} (\mathbf{t} - \mathbf{t}_k) \bar{I}_k(u(\mathbf{t}_k^-)), \\ for \;\; \mathbf{t} \in (\mathbf{t}_j, \mathbf{t}_{j+1}], \;\; j = 1, 2, \cdots, m, \end{cases} \tag{3.3}$$

if and only if u *is a solution of the impulsive fractional IVP*

$$^{c}D^{\beta} u(\mathbf{t}) = p(\mathbf{t}) \;\; \mathbf{t} \in \Omega' = [0, T], \tag{3.4}$$
$$\mathbf{t} \neq \mathbf{t}_j, \quad \beta \in (1, 2]$$
$$\Delta u|_{\mathbf{t}=\mathbf{t}_j} = I_j(u(\mathbf{t}_j^-)), \tag{3.5}$$
$$\Delta u'|_{\mathbf{t}=\mathbf{t}_j} = \bar{I}_j(u(\mathbf{t}_j^-)), \;\; j = 1, 2, \cdots, m \tag{3.6}$$
$$u(0) = u_0 \;\; u'(0) = u_1. \tag{3.7}$$

Theorem 3.5. *Suppose that (H1), (H2) and (H3) hold. If*

$$\frac{(m(1+\beta)+1)AT^{\beta}}{(1-B)\Gamma(\beta+1)} + m(\kappa + T\bar{\kappa}) < 1, \tag{3.8}$$

then impulsive fractional IVP (1.1)-(1.4) possesses a unique solution on Ω.

Proof. We define the operator H by rewriting the equations (1.1)-(1.4) as a fixed point problem given by $\digamma : PC(\Omega, \mathbb{R}) \to PC(\Omega, \mathbb{R})$ as

$$\begin{aligned} \digamma(u)(\mathbf{t}) = u_0 + u_1\mathbf{t} &+ \frac{1}{\Gamma(\beta)} \sum_{0<\mathbf{t}_j<\mathbf{t}} \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j} (\mathbf{t}_j - \tau)^{\beta-1} f(\tau, u(\tau), D^{\beta}u(\tau)) d\tau \\ &+ \frac{1}{\Gamma(\beta-1)} \sum_{0<\mathbf{t}_j<\mathbf{t}} (\mathbf{t} - \mathbf{t}_j) \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j} (\mathbf{t}_j - \tau)^{\beta-2} f(\tau, u(\tau), D^{\beta}u(\tau)) d\tau \\ &+ \frac{1}{\Gamma(\beta)} \int_{\mathbf{t}_j}^{\mathbf{t}} (\mathbf{t} - \tau)^{\beta-1} f(\tau, u(\tau), D^{\beta}u(\tau)) d\tau \\ &+ \sum_{0<\mathbf{t}_j<\mathbf{t}} I_j(u(\mathbf{t}_j^-)) + \sum_{0<\mathbf{t}_j<\mathbf{t}} (\mathbf{t} - \mathbf{t}_j) \bar{I}_j(u(\mathbf{t}_j^-)). \end{aligned} \tag{3.9}$$

We define $\sigma,\ \xi \in C(\Omega,\mathbb{R})$ by $\sigma(\mathbf{t}) = D^{\beta}u(\mathbf{t})$ and $\xi(\mathbf{t}) = D^{\beta}v(\mathbf{t})$, hence we get $\sigma(\mathbf{t}) = f(\mathbf{t}, u(\mathbf{t}), \sigma(\mathbf{t}))$ and $\xi(\mathbf{t}) = f(\mathbf{t}, v(\mathbf{t}), \xi(\mathbf{t}))$.
It is obvious that the solutions of the equations (1.1)-(1.4) are the fixed points of the operator $\mathcal{F}$. Next, We prove that $\mathcal{F}$ is a contraction.
Let $u, v \in PC(\Omega,\mathbb{R})$, for every $\mathbf{t} \in \Omega$ we get,

$$\begin{aligned}
&|\ \mathcal{F}(u)(\mathbf{t}) - \mathcal{F}(v)(\mathbf{t})\ | \le \frac{1}{\Gamma(\beta)} \sum_{0<\mathbf{t}_j<\mathbf{t}} \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j} (\mathbf{t}_j - \tau)^{\beta-1}\ |\ \sigma(\tau) - \xi(\tau)\ |\ d\tau \\
&+ \frac{1}{\Gamma(\beta-1)} \sum_{0<\mathbf{t}_j<\mathbf{t}} (\mathbf{t} - \mathbf{t}_j) \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j} (\mathbf{t}_j - \tau)^{\beta-2}\ |\ \sigma(\tau) - \xi(\tau)\ |\ d\tau \\
&+ \frac{1}{\Gamma(\beta)} \int_{\mathbf{t}_j}^{\mathbf{t}} (\mathbf{t} - \tau)^{\beta-1}\ |\ \sigma(\tau) - \xi(\tau)\ |\ d\tau \\
&+ \sum_{0<\mathbf{t}_j<\mathbf{t}} |I_j(u(\mathbf{t}_j^-)) - I_j(v(\mathbf{t}_j^-))| + \sum_{0<\mathbf{t}_j<\mathbf{t}} (\mathbf{t} - \mathbf{t}_j)|\bar{I}_j(u(\mathbf{t}_j^-)) - \bar{I}_j(v(\mathbf{t}_j^-))|.
\end{aligned}$$

By (H2) we have,

$$\begin{aligned}
|\ \sigma(\mathbf{t}) - \xi(\mathbf{t})\ | &= |f(\mathbf{t}, u(\mathbf{t}), \sigma(\mathbf{t})) - f(\mathbf{t}, v(\mathbf{t}), \xi(\mathbf{t}))| \\
&\le A|u(\mathbf{t}) - v(\mathbf{t})| + B|\sigma(\mathbf{t}) - \xi(\mathbf{t})|,
\end{aligned}$$

hence we get

$$|\ \sigma(\mathbf{t}) - \xi(\mathbf{t})\ | \le \frac{A}{1-B}|u(\mathbf{t}) - v(\mathbf{t})|.$$

For $t \in \Omega$,

$$\begin{aligned}
&|\ \mathcal{F}(u)(\mathbf{t}) - \mathcal{F}(v)(\mathbf{t})\ | \le \\
&\le \frac{A}{(1-B)\Gamma(\beta)} \sum_{0<\mathbf{t}_j<\mathbf{t}} \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j} (\mathbf{t}_j - \tau)^{\beta-1}\ |\ u(\tau) - v(\tau)\ |\ d\tau + \\
&+ \frac{A}{(1-B)\Gamma(\beta-1)} \sum_{0<\mathbf{t}_j<\mathbf{t}} (\mathbf{t} - \mathbf{t}_j) \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j} (\mathbf{t}_j - \tau)^{\beta-2}\ |\ u(\tau) - v(\tau)\ |\ d\tau + \\
&+ \frac{A}{(1-B)\Gamma(\beta)} \int_{\mathbf{t}_j}^{\mathbf{t}} (\mathbf{t} - \tau)^{\beta-1}\ |\ u(\tau) - v(\tau)\ |\ d\tau + \\
&+ \kappa \sum_{0<\mathbf{t}_j<t} |(u(\mathbf{t}_j^-)) - (v(\mathbf{t}_j^-))| + \bar{\kappa} \sum_{0<\mathbf{t}_j<\mathbf{t}} (\mathbf{t} - \mathbf{t}_j)|(u(\mathbf{t}_j^-)) - (v(\mathbf{t}_j^-))| \\
&\le \frac{mAT^{\beta}}{(1-B)\Gamma(\beta+1)}\|u - v\|_{PC} + \frac{mAT^{\beta}}{(1-B)\Gamma(\beta)}\|u - v\|_{PC} + \\
&+ \frac{AT^{\beta}}{(1-B)\Gamma(\beta+1)}\|u - v\|_{PC} + m\kappa\|u - v\|_{PC} + m\bar{\kappa}T\|u - v\|_{PC} \\
&= \left[\frac{mAT^{\beta}}{(1-B)\Gamma(\beta+1)} + \frac{mAT^{\beta}}{(1-B)\Gamma(\beta)} + \frac{AT^{\beta}}{(1-B)\Gamma(\beta+1)} + m(\kappa + \bar{\kappa}T)\right]\|u - v\|_{PC}.
\end{aligned}$$

Thus

$$| \mathcal{F}(u)(\mathbf{t}) - \mathcal{F}(v)(\mathbf{t}) |\leq \left[\frac{(m(1+\beta)+1)AT^{\beta}}{(1-B)\Gamma(\beta+1)} + m(\kappa + T\bar{\kappa})\right] \|u-v\|_{PC}.$$

Consequently, by (3.8) $\mathcal{F}$ is a contraction. Hence $\mathcal{F}$ has a fixed point by Banach fixed point theorem. □

Theorem 3.6. *Suppose that (H1), (H2), (H3) and following assumptions hold*

(H4) *There exist constants* $r, h, q \in C(\Omega, \mathbb{R}_+)$ *with* $q* = \sup_{\mathbf{t}\in\Omega} q(\mathbf{t}) < 1$ *such that*

$$|f(\mathbf{t}, u, v)| \leq r(\mathbf{t}) + h(\mathbf{t})|u| + q(\mathbf{t})|v|, \quad \forall\ \mathbf{t} \in \Omega \text{ and } u, v \in \mathbb{R}.$$

(H5) *The functions* $I_j, \bar{I}_j : \mathbb{R} \to \mathbb{R}$ *are continuous for some constants* $\eta, \zeta, \bar{\eta}, \bar{\zeta} > 0$, $\forall u \in \mathbb{R}\ \ j = 1, 2, \cdots, m$, *such as*

$$|I_j(u)| \leq \eta|u| + \zeta \quad \text{and} \quad |\bar{I}_j(u)| \leq \bar{\eta}|u| + \bar{\zeta},$$

$$\frac{(m + m\beta + 1)h^* T^{\beta}}{(1-q^*)\Gamma(\beta+1)} + m\eta + mT\bar{\eta} < 1.$$

Then IVP (1.1)-(1.4) has at least one solution in Ω.

Proof. We prove this result in four steps by using the Schaefer's fixed point theorem. **Step 1**. We show that $\mathcal{F}$ is continuous for a sequence u_n such that $u_n \to u$ in $PC(\Omega, \mathbb{R})$. For any $t \in \Omega$,

$$\begin{aligned}
&| \mathcal{F}(u_n)(\mathbf{t}) - \mathcal{F}(u)(\mathbf{t}) |\leq \\
&\leq \frac{1}{\Gamma(\beta)} \sum_{0<\mathbf{t}_j<t} \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j} (\mathbf{t}_j - \tau)^{\beta-1} \mid \sigma_n(\tau) - \sigma(\tau) \mid d\tau + \\
&+ \frac{1}{\Gamma(\beta-1)} \sum_{0<\mathbf{t}_j<\mathbf{t}} (\mathbf{t} - \mathbf{t}_j) \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j} (\mathbf{t}_j - \tau)^{\beta-2} \mid \sigma_n(\tau) - \sigma(\tau) \mid d\tau + \\
&+ \frac{1}{\Gamma(\beta)} \int_{\mathbf{t}_j}^{\mathbf{t}} (\mathbf{t} - \tau)^{\beta-1} \mid \sigma_n(\tau) - \sigma(\tau) \mid d\tau + \\
&+ \sum_{0<\mathbf{t}_j<\mathbf{t}} |I_j(u_n(\mathbf{t}_j^-)) - I_j(u(\mathbf{t}_j^-))| + \sum_{0<\mathbf{t}_j<\mathbf{t}} (\mathbf{t} - \mathbf{t}_j)|\bar{I}_j(u_n(\mathbf{t}_j^-)) - \bar{I}_j(u(\mathbf{t}_j^-))|,
\end{aligned}$$

where $\sigma_n and\ \sigma \in C(\Omega, \mathbb{R})$ and given by

$$\sigma_n(\mathbf{t}) = f(\mathbf{t}, u_n(\mathbf{t}), \sigma_n(\mathbf{t}))\ \sigma(\mathbf{t}) = f(\mathbf{t}, u(\mathbf{t}), \sigma(\mathbf{t})).$$

By (H2) we get

$$\begin{aligned}
| \sigma_n(\mathbf{t}) - \sigma(\mathbf{t}) | &= |f(\mathbf{t}, u_n(\mathbf{t}), \sigma_n(\mathbf{t})) - f(\mathbf{t}, u(\mathbf{t}), \sigma(\mathbf{t}))| \\
&\leq A|u_n(\mathbf{t}) - u(\mathbf{t})| + B|\sigma_n(\mathbf{t}) - \sigma(\mathbf{t})|,
\end{aligned}$$

Hence we get

$$| \sigma_n(\mathbf{t}) - \sigma(\mathbf{t}) |\leq \frac{A}{1-B}|u_n(\mathbf{t}) - u(\mathbf{t})|.$$

Since $u_n \to u$, we get $\sigma_n(\mathbf{t}) \to \sigma(\mathbf{t})$ as $n \to \infty$ for each $\mathbf{t} \in \Omega$. For $\delta > 0$ and for any $t \in \Omega$, we get $|\sigma_n(\mathbf{t})| \leq \delta$ and $|\sigma(\mathbf{t})| \leq \delta$. Therefore, we have

$$\begin{aligned}(\mathbf{t}-\tau)^{\beta-1}|\sigma_n(\tau)-\sigma(\tau)| &\leq (\mathbf{t}-\tau)^{\beta-1}[|\sigma_n(\tau)|+|\sigma(\tau)|]\\ &\leq 2\delta(\mathbf{t}-\tau)^{\beta-1}\end{aligned}$$

and

$$\begin{aligned}(\mathbf{t}_k-\tau)^{\beta-1}|\sigma_n(\tau)-\sigma(\tau)| &\leq (\mathbf{t}_k-\tau)^{\beta-1}[|\sigma_n(\tau)|+|\sigma(\tau)|]\\ &\leq 2\delta(\mathbf{t}_k-\tau)^{\beta-1}.\end{aligned}$$

For each $\mathbf{t} \in \Omega$, the functions $\tau \to 2\delta(\mathbf{t}-\tau)^{\beta-1}$and $\tau \to 2\delta(\mathbf{t}_k-\tau)^{\beta-1}$ are integrable on $[0,\mathbf{t}]$, hence equation (3.7) and the Lebesgue dominated convergence theorem imply that

$$|\ \mathcal{F}(u_n)(\mathbf{t}) - \mathcal{F}(u)(\mathbf{t})\ | \to 0 \ as \ n \to \infty$$

and hence

$$\|\mathcal{F}(u_n)(\mathbf{t}) - \mathcal{F}(u)(\mathbf{t})\| \to 0 \ as \ n \to \infty$$

consequently, H is continuous.

Step 2. Next we prove boundedness of $\mathcal{F}$ in $PC(\Omega,\mathbb{R})$. Indeed, we show that for any $R > 0$, there is some positive constant K such that for every $u \in B_R = \{u \in PC(\Omega,\mathbb{R}) : \|u\|_{PC} \leq R\}$, we have $\|\mathcal{F}(u)\|_{PC} \leq K$. For each $\mathbf{t} \in \Omega$ we have,

$$\begin{aligned}|\mathcal{F}(u)(\mathbf{t})| &\leq |u_0| + Tu_1 + \frac{1}{\Gamma(\beta)}\sum_{0<\mathbf{t}_j<\mathbf{t}}\int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j}(\mathbf{t}_j-\tau)^{\beta-1}|\sigma(\tau)|d\tau\\ &+ \frac{1}{\Gamma(\beta-1)}\sum_{0<\mathbf{t}_j<\mathbf{t}}(\mathbf{t}-\mathbf{t}_j)\int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j}(\mathbf{t}_j-\tau)^{\beta-2}|\sigma(\tau)|d\tau\\ &+ \frac{1}{\Gamma(\beta)}\int_{\mathbf{t}_j}^{\mathbf{t}}(\mathbf{t}-\tau)^{\beta-1}|\sigma(\tau)|d\tau\\ &+ \sum_{0<\mathbf{t}_j<\mathbf{t}}|I_j(u(\mathbf{t}_j^-))| + \sum_{0<\mathbf{t}_j<\mathbf{t}}(\mathbf{t}-\mathbf{t}_j)|\bar{I}_j(u(\mathbf{t}_j^-))|, \end{aligned}\tag{3.10}$$

where $\sigma \in C(\Omega,\mathbb{R})$ is such that

$$\sigma(\mathbf{t}) = f(\mathbf{t}, u(\mathbf{t}), \sigma(\mathbf{t})).$$

By (H4) for each $\mathbf{t} \in \Omega$,

$$\begin{aligned}|\sigma(\mathbf{t})| &= |f(\mathbf{t},u(\mathbf{t}),\sigma(\mathbf{t}))|\\ &\leq r(\mathbf{t}) + h(\mathbf{t})|u(\mathbf{t})| + q(\mathbf{t})|\sigma(\mathbf{t})|\\ &\leq r(\mathbf{t}) + h(\mathbf{t})R + q(\mathbf{t})|\sigma(\mathbf{t})|\\ &\leq r^* + h^*R + q^*|\sigma(\mathbf{t})|,\end{aligned}$$

where $r^* = \sup_{\mathbf{t}\in\Omega} r^*(\mathbf{t})$, and $h^* = \sup_{\mathbf{t}\in\Omega} h^*(\mathbf{t})$. Then

$$|\sigma(\mathbf{t})| \leq \frac{r^* + h^* R}{1 - q^*} := L.$$

Thus equation (3.10) gives

$$\begin{aligned} |\digamma(u)(\mathbf{t})| &\leq |u_0| + T|u_1| + \frac{mLT^\beta}{\Gamma(\beta+1)} + \frac{mLT^\beta}{\Gamma(\beta)} + \frac{mT^\beta}{\Gamma(\beta+1)} + \\ &\quad + m(\eta|u| + \zeta) + mT(\bar{\eta}|u| + \bar{\zeta}) \\ &\leq |u_0| + T|u_1| + \frac{mLT^\beta}{\Gamma(\beta+1)} + \frac{mLT^\beta}{\Gamma(\beta)} + \frac{LT^\beta}{\Gamma(\beta+1)} + \\ &\quad + m(\eta R + \zeta) + mT(\bar{\eta} R + \bar{\zeta}), \end{aligned}$$

hence

$$\begin{aligned} \|\digamma(u)\|_{PC} &\leq |u_0| + T|u_1| + \left[\frac{(m + m\beta + 1)T^\beta L}{\Gamma(\beta+1)}\right] + \\ &\quad + m[(\eta R + \zeta) + T(\bar{\eta} R + \bar{\zeta})] := K. \end{aligned}$$

Step 3: Here we prove $\digamma$ maps bounded sets of $PC(\Omega, \mathbb{R})$ into equicontinuous sets.

Let $x_1, x_2 \in \Omega$, $x_1 < x_2$, B_R be a bounded set of $PC(\Omega, \mathbb{R})$ and take $u \in B_R$. This gives

$$\begin{aligned} &|\ \digamma(u)(x_2) - \digamma(u)(x_1)\ | \leq \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^{x_1} \left|(x_2 - \tau)^{\beta-1} - (x_1 - \tau)^{\beta-1}\right| |\sigma(\tau)| d\tau \\ &+ \frac{1}{\Gamma(\beta)} \int_{x_1}^{x_2} \left|(x_2 - \tau)^{\beta-1}\right| |\sigma(\tau)| d\tau \\ &+ \frac{1}{\Gamma(\beta-1)} \sum_{0<\mathbf{t}_j<x_2-x_1} (x_2 - \mathbf{t}_j) \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_i} (\mathbf{t}_j - \tau)^{\beta-2} |\sigma(\tau)| d\tau \\ &+ \frac{1}{\Gamma(\beta-1)} \sum_{0<\mathbf{t}_j<x_1} (x_2 - x_1) \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_i} (\mathbf{t}_j - \tau)^{\beta-2} |\sigma(\tau)| d\tau \\ &+ \int_{x_1}^{x_2} (x_2 - \tau)^{\beta-1} |\sigma(\tau)| d\tau + \int_{\mathbf{t}_j}^{x_1} \left|(x_2 - \tau)^{\beta-1} - (x_1 - \tau)^{\beta-1}\right| |\sigma(\tau)| d\tau \\ &+ \sum_{0<\mathbf{t}_j<x_2-x_1} \left|I_j(u(t_j^-))\right| + \sum_{0<\mathbf{t}_j<x_2-x_1} (x_2 - \mathbf{t}_j) \left|\bar{I}_j(u(\mathbf{t}_j^-))\right| \\ &+ (x_2 - x_1) \sum_{0<\mathbf{t}_j<x_1} \left|\bar{I}_j(u(\mathbf{t}_j^-))\right|, \end{aligned}$$

where $\sigma \in C(\Omega, \mathbb{R})$ is such that

$$\sigma(\mathbf{t}) = f(\mathbf{t}, u(\mathbf{t}), \sigma(\mathbf{t})).$$

Whenever $x_1 \to x_2$, the RHS of the inequality approaches to zero, consequently by using Arzelá-Ascoli theorem and above steps 1 to 3 the map $\digamma : PC(\Omega, \mathbb{R}) \to PC(\Omega, \mathbb{R})$ is completely continuous.

Step 4: A priori bounds.

In this step we prove that the set
$D = \{u \in PC(\Omega, \mathbb{R}) | u = \theta\digamma(u), \text{for some } \theta \in (0,1)\}$ is bounded. Consider $u \in D$, where $u = \theta\digamma(u)$ for some $\theta \in (0,1)$. Then for every $\mathbf{t} \in \Omega$, we have

$$
\begin{aligned}
u(\mathbf{t}) &= \theta u_0 + \theta T|u_1| + \frac{\theta}{\Gamma(\beta)} \sum_{0<\mathbf{t}_j<\mathbf{t}} \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j} (\mathbf{t}_j - \tau)^{\beta-1} f(\tau, u(\tau), D^\beta u(\tau)) d\tau \\
&+ \frac{\theta}{\Gamma(\beta-1)} \sum_{0<\mathbf{t}_j<\mathbf{t}} (\mathbf{t} - \mathbf{t}_j) \int_{\mathbf{t}_{j-1}}^{\mathbf{t}_j} (\mathbf{t}_j - \tau)^{\beta-2} f(\tau, u(\tau), D^\beta u(\tau)) d\tau \\
&+ \frac{\theta}{\Gamma(\beta)} \int_{\mathbf{t}_j}^{\mathbf{t}} (\mathbf{t} - \tau)^{\beta-1} f(\tau, u(\tau), D^\beta u(\tau)) d\tau \\
&+ \theta \sum_{0<\mathbf{t}_j<\mathbf{t}} I_j(u(\mathbf{t}_j^-)) + \theta \sum_{0<\mathbf{t}_j<\mathbf{t}} (\mathbf{t} - \mathbf{t}_j) \bar{I}_j(u(\mathbf{t}_j^-)). \qquad (3.11)
\end{aligned}
$$

By (H4) for each $\mathbf{t} \in \Omega$, we get

$$
\begin{aligned}
|\sigma(\mathbf{t})| &= |f(\mathbf{t}, u(\mathbf{t}), \sigma(\mathbf{t}))| \\
&\leq r(\mathbf{t}) + h(\mathbf{t})|u(\mathbf{t})| + q(\mathbf{t})|\sigma(\mathbf{t})| \\
&\leq r^* + h^*|u(\mathbf{t})| + q^*|\sigma(\mathbf{t})|,
\end{aligned}
$$

where $r^* = \sup_{\mathbf{t} \in \Omega} r^*(\mathbf{t})$, and $h^* = \sup_{\mathbf{t} \in \Omega} h^*(t)$. Then

$$
|\sigma(\mathbf{t})| \leq \frac{r^* + h^*|u(\mathbf{t})|}{1 - q^*}.
$$

By (H4) and (H5), for every t$\in \Omega$, we have

$$
\begin{aligned}
|u(\mathbf{t})| \leq |u_0| + T|u_1| &+ \frac{mT^\beta}{\Gamma(\beta+1)} \left(\frac{r^* + h^*|u(\mathbf{t})|}{1 - q^*} \right) \\
&+ \frac{mT^\beta}{\Gamma(\beta)} \left(\frac{r^* + h^*|u(\mathbf{t})|}{1 - q^*} \right) + \frac{T^\beta}{\Gamma(\beta+1)} \left(\frac{r^* + h^*|u(\mathbf{t})|}{1 - q^*} \right) + m(\eta|u(\mathbf{t})| \\
&+ m(\eta|u(\mathbf{t})| + \zeta) + mT(\bar{\eta}|u(\mathbf{t})| + \bar{\zeta}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|u(\mathbf{t})\|_{PC} \leq |u_0| + T|u_1| &+ \frac{mT^\beta}{\Gamma(\beta+1)} \left(\frac{r^* + h^*\|u(\mathbf{t})\|_{PC}}{1 - q^*} \right) + \\
&+ \frac{mT^\beta}{\Gamma(\beta)} \left(\frac{r^* + h^*\|u(\mathbf{t})\|_{PC}}{1 - q^*} \right) + \frac{T^\beta}{\Gamma(\beta+1)} \left(\frac{r^* + h^*\|u(\mathbf{t})\|_{PC}}{1 - q^*} \right) + \\
&+ m(\eta\|u(\mathbf{t})\|_{PC} + \zeta) + mT(\bar{\eta}\|u(\mathbf{t})\|_{PC} + \bar{\zeta}).
\end{aligned}
$$

This gives

$$\|u(\mathbf{t})\|_{PC} \leq \frac{|u_0| + T|u_1| + \frac{(m+m\beta+1)r^* T^\beta}{(1-q^*)\Gamma(\beta+1)} + m(\zeta + T\bar{\zeta})}{1 - \frac{(m+m\beta+1)h^* T^\beta}{(1-q^*)\Gamma(\beta+1)} - m(\eta + T\bar{\eta})} : R.$$

This gives the set D is bounded and consequently H has a fixed point. □

4. Illustration

To demonstrate the applicability of Theorems 3.1 and 3.2, we present two examples.

Example 4.1. Consider the following fractional initial value problem with impulses:

$$^cD^\beta u(\mathbf{t}) = \frac{1}{9e^{t+2}(1 + |u(\mathbf{t})| + |^cD^\beta u(\mathbf{t})|)}, \tag{4.1}$$

$$\Delta u|_{\mathbf{t}=\frac{1}{2}} = \frac{|u(\frac{1}{2}^-)|}{30 + |u(\frac{1}{2}^-)|}, \tag{4.2}$$

$$\Delta u'|_{\mathbf{t}=\frac{1}{2}} = \frac{|u(\frac{1}{2}^-)|}{50 + |u(\frac{1}{2}^-)|}, \tag{4.3}$$

$$u(0) = u'(0) = 0, \tag{4.4}$$

for $\mathbf{t} \in \Omega = [0,1],\ \mathbf{t} \neq \frac{1}{2},\ 1 < \beta \leq 2.$

Set

$$f(\mathbf{t}, x, y) = \frac{1}{9e^{\mathbf{t}+2}(1 + x_1 + x_2)}, \quad \mathbf{t} \in \Omega = [0,1],\ x_1, x_2 \in [0, \infty),$$

$$I_k(x) = \frac{x_1}{30 + x_1},$$

$$\bar{I}_k(x) = \frac{x_1}{50 + x_1}\ .$$

Let $x_1, x_2, \bar{x_1}, \bar{x_2} \in [0, \infty)$ and $\mathbf{t} \in \Omega$. Then we have

$$\begin{aligned}
|f(\mathbf{t}, x_1, x_2) - f(\mathbf{t}, \bar{x_1}, \bar{x_2})| &= \frac{1}{9e^{t+2}} \left| \frac{1}{1 + x_1 + x_2} - \frac{1}{1 + \bar{x_1} + \bar{x_2}} \right| \\
&= \frac{|x_1 - \bar{x_1}| + |x_2 - \bar{x_2}|}{9e^{\mathbf{t}+2}(1 + x_1 + x_2)(1 + \bar{x_1} + \bar{x_2})} \\
&\leq \frac{|x_1 - \bar{x_1}| + |x_2 - \bar{x_2}|}{9e^{\mathbf{t}+2}} \\
&\leq \frac{1}{9e^2}(|x_1 - \bar{x_1}| + |x_2 - \bar{x_2}|).
\end{aligned}$$

Hence the condition (H2) holds with $A = B = \frac{1}{9e^2}$. Also we have

$$\begin{aligned}
|I_k(x_1) - I_k(x_2)| &= \left| \frac{x_1}{30 + x_1} - \frac{x_2}{30 + x_2} \right| = \frac{30|x_1 - x_2|}{(30 + x_1)(30 + x_2)} \\
&\leq \frac{1}{30}|x_1 - x_2|
\end{aligned}$$

and

$$|\bar{I}_k(x_1) - \bar{I}_k(x_2)| \leq \frac{1}{50}|x_1 - x_2|,$$

therefore the condition (H3) verified with $\kappa = \frac{1}{30}$ and $\bar{\kappa} = \frac{1}{50}$. Indeed, we will see that the condition (3.8) is fulfilled with $T = 1$ and $m = 1$.

$$\begin{aligned}\frac{(m(1+\beta)+1)AT^{\beta}}{(1-B)\Gamma(\beta+1)} + m(\kappa + T\bar{\kappa}) &= \left(\frac{(\beta+2)\frac{1}{9e^2}}{(1-\frac{1}{9e^2})\Gamma(\beta+1)} + \frac{1}{30} + \frac{1}{50}\right)\\ &= \left(\frac{\beta+2}{(9e^2-1)\Gamma(\beta+1)} + \frac{1}{30} + \frac{1}{50}\right)\\ &< 0.09913,\end{aligned}$$

which is satisfied for any $\beta \in (1, 2]$. Hence by Theorem 3.1 the Eqs. (4.1)-(4.4) has a unique solution on Ω.

Example 4.2. Consider following fractional initial value problem with impulses:

$$^{c}D^{\beta}u(\mathbf{t}) = \frac{3 + |u(\mathbf{t})| + |^{c}D^{\beta}u(\mathbf{t})|}{81e^{\mathbf{t}+3}(1 + |u(\mathbf{t})| + |^{c}D^{\beta}u(\mathbf{t})|)}, \tag{4.5}$$

$$\Delta u|_{\mathbf{t}=\frac{1}{2}} = \frac{|u(\frac{1}{3}^{-})|}{7 + |u(\frac{1}{3}^{-})|}, \tag{4.6}$$

$$\Delta u'|_{\mathbf{t}=\frac{1}{2}} = \frac{|u(\frac{1}{3}^{-})|}{11 + |u(\frac{1}{3}^{-})|}, \tag{4.7}$$

$$u(0) = u'(0) = 0, \tag{4.8}$$

for $\mathbf{t} \in \Omega = [0, 1]$, $\mathbf{t} \neq \frac{1}{2}$, $1 < \beta \leq 2$.

Set

$$f(\mathbf{t}, x_1, x_2) = \frac{3 + x_1 + x_2}{81e^{\mathbf{t}+3}(1 + x_1 + x_2)}, \ \mathbf{t} \in \Omega = [0, 1], \ x_1, x_2 \in [0, \infty),$$

obviously the function f is mutually continuous. For any $x_1, x_2, \bar{x_1}, \bar{x_2} \in [0, \infty)$ and $\mathbf{t} \in \Omega$,

$$|f(\mathbf{t}, x_1, x_2) - f(\mathbf{t}, \bar{x_1}, \bar{x_2})| \leq \frac{1}{81e^3}(|x_1 - \bar{x_1}| + |x_2 - \bar{x_2}|).$$

Hence, condition (H2) is satisfied with $A = B = \frac{1}{81e^3}$. For each $\mathbf{t} \in [0, 1]$,

$$|f(\mathbf{t}, x_1, x_2)| \leq \frac{1}{81e^3}(3 + x_1 + x_2).$$

Thus condition (H4) is satisfied with $r(\mathbf{t}) = \frac{1}{27e^{t+3}}$ and $h(\mathbf{t}) = q(\mathbf{t}) = \frac{1}{81e^{\mathbf{t}+3}}$. Let

$$I_k(x_1) = \frac{x_1}{7 + x_1}, \quad \bar{I}_k(x_1) = \frac{x_1}{11 + x_1},$$

therefore for each $x_1 \in [0, \infty)$, $|I_k(x_1)| = \frac{1}{7}x_1 + 1$, and $|\bar{I}_k(x_1)| = \frac{1}{11}x_1 + 1$.
Thus condition (H5) is satisfied with $\eta = \frac{1}{7}$, $\zeta = 1, \bar{\eta} = \frac{1}{11}$, $\bar{\zeta} = 1$. Hence the

condition

$$\begin{aligned}\frac{(m+m\beta+1)h^*T^\beta}{(1-q^*)\Gamma(\beta+1)}+m\eta+mT\bar{\eta} &= \left(\frac{(\beta+2)\frac{1}{81e^3}}{(1-\frac{1}{81e^3})\Gamma(\beta+1)}+\frac{1}{7}+\frac{1}{11}\right)\\ &= \left(\frac{\beta+2}{(81e^3-1)\Gamma(\beta+1)}+\frac{1}{7}+\frac{1}{11}\right)\\ &< 0.2356,\end{aligned}$$

is satisfied for any $\beta \in (1, 2]$. Hence by Theorem 3.2 the Eqs. (4.5)-(4.8) has a unique solution on Ω.

5. Conclusion

In this article, sufficient condition for the existence and impulsive initial value problems for a class of implicit fractional differential equations involving the Caputo fractional derivative were obtained using Banach's contraction mapping principle and the Schaefer's fixed-point theorem. Obtained results were illustrated using two initial value problems of fractional order. We conclude that the present method to analyse impulsive implicit fractional differential equations is most reliable and efficient, moreover it can be used in various differential equations of fractional order arising in mathematical modelling of several real-life problems.

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