On eigenvalues of generalized shift linear vector isomorphisms

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Abstract
Our main aim is to compute eigenvalues of generalized shift isomorphism \( \sigma_\varphi : V^\Gamma \to V^\Gamma \) with \( \sigma_\varphi ((x_\alpha)_{\alpha \in \Gamma}) = (x_{\varphi(\alpha)})_{\alpha \in \Gamma} \) for arbitrary set \( \Gamma \) and \( \varphi : \Gamma \to \Gamma \) arbitrary bijection where \( V \) is a vector space (over field \( F \)).

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1. Introduction

Let’s mention that one–sided shift \( \{1, \ldots, k\}^N \to \{1, \ldots, k\}^N \) and two–sided shift \( \{1, \ldots, k\}^\mathbb{Z} \to \{1, \ldots, k\}^\mathbb{Z} \) are two well-known operators in dynamical systems, ergodic theory [5], etc. For arbitrary set \( X \) with at least two elements, nonempty set \( \Gamma \) and \( \varphi : \Gamma \to \Gamma \) generalized shift \( \sigma_\varphi : X^\Gamma \to X^\Gamma \) has been introduced for the first time in [2] as a generalization of one–sided and two–sided shifts. It’s evident that if \( X \) has group structure, then \( \sigma_\varphi : X^\Gamma \to X^\Gamma \) is group homomorphism (see e.g., [1]) and for topological space \( X \), \( \sigma_\varphi : X^\Gamma \to X^\Gamma \) is continuous, where \( X^\Gamma \) is equipped with product topology, see e.g. [3].

Convention. In the following text, suppose \( V(\neq 0) \) is a linear vector space over field \( F \), \( \Gamma \) is an arbitrary set with at least two elements and \( \varphi : \Gamma \to \Gamma \) is an arbitrary map. Generalized shift \( \sigma_\varphi : V^\Gamma \to V^\Gamma \) with \( \sigma_\varphi ((x_\alpha)_{\alpha \in \Gamma}) = (x_{\varphi(\alpha)})_{\alpha \in \Gamma} \) is a linear vector space homomorphism since for \( (x_\alpha)_{\alpha \in \Gamma}, (y_\alpha)_{\alpha \in \Gamma} \in V^\Gamma \) and \( r \in F \), let \( z_\alpha = x_\alpha + r y_\alpha \), now we have:

\[
\sigma_\varphi((x_\alpha)_{\alpha \in \Gamma} + r(y_\alpha)_{\alpha \in \Gamma}) = \sigma_\varphi((z_\alpha)_{\alpha \in \Gamma}) = (z_{\varphi(\alpha)})_{\alpha \in \Gamma} = (x_{\varphi(\alpha)})_{\alpha \in \Gamma} + r(y_{\varphi(\alpha)})_{\alpha \in \Gamma} = \sigma_\varphi((x_\alpha)_{\alpha \in \Gamma}) + r\sigma_\varphi((y_\alpha)_{\alpha \in \Gamma}).
\]

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For $\alpha, \beta \in \Gamma$, let $\alpha \sim_{\varphi} \beta$ or briefly $\alpha \sim \beta$ if there exist $n, m > 0$ with $\varphi^n(\alpha) = \varphi^m(\beta)$. Clearly $\sim_{\varphi}$ is an equivalence relation on $\Gamma$. For $\alpha \in \Gamma$ by $\frac{\alpha}{\sim_{\varphi}}$ we mean the equivalence class of $\alpha$ in $\sim_{\varphi}$. In other words $\frac{\alpha}{\sim_{\varphi}} = \bigcup\{\varphi^n(\alpha) : n \in \mathbb{Z}\}$. In particular one may consider $\varphi|_{\frac{\alpha}{\sim_{\varphi}}} : \frac{\alpha}{\sim_{\varphi}} \to \frac{\alpha}{\sim_{\varphi}}$.

2. Eigenvalues of generalized shifts

In this section we compute eigenvalues of homomorphism $\sigma_{\varphi} : V^{\Gamma} \to V^{\Gamma}$.

Lemma 2.1 (Decomposition Lemma). For $r \in F$, $r$ is an eigenvalue of $\sigma_{\varphi} : V^{\Gamma} \to V^{\Gamma}$ if and only if there exists $\alpha \in \Gamma$ such that $r$ is an eigenvalue of $\sigma_{\varphi}|_{\frac{\alpha}{\sim_{\varphi}}} : V^{\frac{\alpha}{\sim_{\varphi}}} \to V^{\frac{\alpha}{\sim_{\varphi}}}$. I.e.,

$$ \text{Eigen}(\sigma_{\varphi}, V^{\Gamma}) = \bigcup_{\alpha \in \Gamma} \text{Eigen}(\sigma_{\varphi}|_{\frac{\alpha}{\sim_{\varphi}}}, V^{\frac{\alpha}{\sim_{\varphi}}}) $$

where for vector space $W$ and homomorphism $h : W \to W$, $\text{Eigen}(h, W)$ denotes the collection of all eigenvalues of $h : W \to W$.

Proof. For $\alpha \in \Gamma$, suppose $r \in \text{Eigen}(\sigma_{\varphi}|_{\frac{\alpha}{\sim_{\varphi}}}, V^{\frac{\alpha}{\sim_{\varphi}}})$. Choose $(x_{\lambda})_{\lambda \in \frac{\alpha}{\sim_{\varphi}}} \in V^{\frac{\alpha}{\sim_{\varphi}}} \setminus \{0\}$ with $\varphi|_{\frac{\alpha}{\sim_{\varphi}}}((x_{\lambda})_{\lambda \in \frac{\alpha}{\sim_{\varphi}}}) = r(x_{\lambda})_{\lambda \in \frac{\alpha}{\sim_{\varphi}}}$. For $\lambda \in \Gamma \setminus \frac{\alpha}{\sim_{\varphi}}$ let $x_{\lambda} := 0$. For all $\lambda \in \Gamma \setminus \frac{\alpha}{\sim_{\varphi}}$ we have $\varphi(\lambda) \in \Gamma \setminus \frac{\alpha}{\sim_{\varphi}}$ which leads to

$$ x_{\lambda} = x_{\varphi(\lambda)} = r x_{\lambda}, \quad (\lambda \in \Gamma \setminus \frac{\alpha}{\sim_{\varphi}}). \quad (*) $$

Using $(*)$ and $(x_{\varphi(\lambda)})_{\lambda \in \frac{\alpha}{\sim_{\varphi}}} = (r x_{\lambda})_{\lambda \in \frac{\alpha}{\sim_{\varphi}}}$ we have $\sigma_{\varphi}((x_{\lambda})_{\lambda \in \Gamma}) = r(x_{\lambda})_{\lambda \in \Gamma}$. Moreover $(x_{\lambda})_{\lambda \in \Gamma} \neq 0$ since $(x_{\lambda})_{\lambda \in \frac{\alpha}{\sim_{\varphi}}} \neq 0$. Hence $r$ is an eigenvalue of $\sigma_{\varphi} : V^{\Gamma} \to V^{\Gamma}$.

Conversely, if $r \in F$ an eigenvalue of $\sigma_{\varphi} : V^{\Gamma} \to V^{\Gamma}$, there exist $(x_{\lambda})_{\lambda \in \Gamma} \in V^{\Gamma} \setminus \{0\}$ such that $\sigma_{\varphi}((x_{\lambda})_{\lambda \in \Gamma}) = r(x_{\lambda})_{\lambda \in \Gamma}$. Choose $\alpha \in \Gamma$ such that $x_{\alpha} \neq 0$. Using $\sigma_{\varphi}|_{\frac{\alpha}{\sim_{\varphi}}}((x_{\lambda})_{\lambda \in \frac{\alpha}{\sim_{\varphi}}}) = r(x_{\lambda})_{\lambda \in \frac{\alpha}{\sim_{\varphi}}}$, we have $r \in \text{Eigen}(\sigma_{\varphi}|_{\frac{\alpha}{\sim_{\varphi}}}, V^{\frac{\alpha}{\sim_{\varphi}}})$ which completes the proof. \qed

Remark 2.2. We have the following statements [2]:

- the mapping $\sigma_{\varphi} : V^{\Gamma} \to V^{\Gamma}$ is injective if and only if $\varphi : \Gamma \to \Gamma$ is surjective;
- the mapping $\sigma_{\varphi} : V^{\Gamma} \to V^{\Gamma}$ is bijective if and only if $\varphi : \Gamma \to \Gamma$ is injective.

Corollary 2.3. By Remark 2.2, 0 is an eigenvalue of $\sigma_{\varphi} : V^{\Gamma} \to V^{\Gamma}$ if and only if $\varphi : \Gamma \to \Gamma$ is not surjective.
**Definition 2.4.** For $\alpha, \beta \in \Gamma$ let:

$$
m(\alpha, \beta) = \begin{cases} 
\min\{n > 0 : \varphi^n(\alpha) = \beta\}, & \exists n > 0(\varphi^n(\alpha) = \beta), \\
0, & \text{otherwise}.
\end{cases}
$$

It’s evident that for distinct $\alpha, \beta \in \Gamma$ we have $m(\alpha, \beta) + m(\beta, \alpha) > 0$ if and only if $\beta \in \sim \varphi \alpha$.

**Definition 2.5.** For $\alpha \in \Gamma$ we say:

- $\alpha$ is a periodic point of $\varphi$ if there exists $n \geq 1$ with $\varphi^n(\alpha) = \alpha$,
- $\alpha$ is a quasi–periodic point of $\varphi$ if there exist $n > m \geq 1$ with $\varphi^n(\alpha) = \varphi^m(\alpha)$.

By $W(\Gamma, \varphi)$ we mean the collection of all non–quasi–periodic points of $\varphi : \Gamma \to \Gamma$. Consequently $Q(\Gamma, \varphi)$ denotes the set of all quasi–periodic points of $\varphi : \Gamma \to \Gamma$ and by $P(\Gamma, \varphi)$ we mean the collection of all periodic points of $\varphi : \Gamma \to \Gamma$.

**Remark 2.6.** For $\alpha \in \Gamma$ we have:

- $\alpha \in Q(\Gamma, \varphi)$ if and only if $\frac{\alpha}{\sim \varphi} \subseteq Q(\Gamma, \varphi)$.
- $\alpha \in W(\Gamma, \varphi)$ if and only if $\frac{\alpha}{\sim \varphi} \subseteq W(\Gamma, \varphi)$.

In particular by $W(\Gamma, \varphi) \cap Q(\Gamma, \varphi) = \emptyset$ and $W(\Gamma, \varphi) \cup Q(\Gamma, \varphi) = \Gamma$, hence if $\sim \varphi = \Gamma \times \Gamma$, then $\Gamma = W(\Gamma, \varphi)$ or $\Gamma = Q(\Gamma, \varphi)$.

**Proof.** Consider $\alpha, \beta \in \Gamma$ with $\beta \in \frac{\alpha}{\sim \varphi}$, so there exist $n, m > 1$ such that $\varphi^n(\alpha) = \varphi^m(\beta)$. Now we have the following cases:

- Case 1: $\alpha \in Q(\Gamma, \varphi)$. In this case there exists $p > q \geq 1$ with $\varphi^p(\alpha) = \varphi^q(\alpha)$, thus:

  $$
  \varphi^{p+q}(\beta) = \varphi^q(\alpha) = \varphi^p(\alpha) = \varphi^{p+q}(\alpha),
  $$

  and $\beta \in Q(\Gamma, \varphi)$.

- Case 2: $\alpha \in W(\Gamma, \varphi)$. If $\beta \notin W(\Gamma, \varphi) = \Gamma \setminus Q(\Gamma, \varphi)$, then $\beta \in Q(\Gamma, \varphi)$ and using Case 1 we have $\alpha \in Q(\Gamma, \varphi)$ which leads to contradiction $\alpha \in Q(\Gamma, \varphi) \cap W(\Gamma, \varphi) = \emptyset$. Hence $\beta \in W(\Gamma, \varphi)$.

**Lemma 2.7.** If $\varphi : \Gamma \to \Gamma$ is bijective and $\sim \varphi = \Gamma \times \Gamma$ with $W(\Gamma, \varphi) \neq \emptyset$, then $F \setminus \{0\} = \text{Eigen}(\sigma_{\varphi}, V^T)$.

**Proof.** Suppose $\varphi : \Gamma \to \Gamma$ is bijective and $\sim \varphi = \Gamma \times \Gamma$ with $W(\Gamma, \varphi) \neq \emptyset$. By Corollary 2.3 we have $0 \notin \text{Eigen}(\sigma_{\varphi}, V^T)$ and $\text{Eigen}(\sigma_{\varphi}, V^T) \subseteq F \setminus \{0\}$.

Choose $\theta \in W(\Gamma, \varphi) \setminus \{0\}$, $x \in V \setminus \{0\}$ and $r \in F \setminus \{0\}$. Since $\sim \varphi = \Gamma \times \Gamma$ we have $\Gamma = \frac{\theta}{\sim \varphi}$, moreover by Remark 2.6 we have $\frac{\theta}{\sim \varphi} \subseteq W(\Gamma, \varphi)$. As a matter of fact $\Gamma = \{\varphi^n(\theta) : n \in \mathbb{Z}\}$. Let:

$$
x_{\alpha} = \begin{cases} 
m^{m(\theta, \alpha)}x, & m(\theta, \alpha) > 0, \\
m^{-m(\alpha, \theta)}x, & m(\alpha, \theta) > 0, \\
x, & \alpha = \theta,
\end{cases}
$$
Lemma 2.10. which completes the proof. □

α

Conversely, for Choose

σ

exist

n > m

2.

Suppose

2.

Use (1).

Proof.

By Corollary 2.3 we have \( \text{Eigen}(\sigma, V^T) \subseteq F \setminus \{0\} \). Choose \( \theta \in W(\Gamma, \varphi) \), then \( \varphi|_\theta : \varphi \sim \theta \sim \) is bijective and \( \theta \in W(\varphi|_\theta) \), thus by Lemma 2.8 we have \( F \setminus \{0\} = \text{Eigen}(\sigma|_\theta, V^\theta) \). Using Lemma 2.1 we have \( F \setminus \{0\} = \text{Eigen}(\sigma|_\theta, V^\theta) \subseteq \text{Eigen}(\sigma, V^T) \) which completes the proof. □

Lemma 2.8. If \( \varphi : \Gamma \to \Gamma \) is bijective with \( W(\Gamma, \varphi) \neq \emptyset \), then \( F \setminus \{0\} = \text{Eigen}(\sigma, V^T) \).

Proof. By Corollary 2.3 we have \( \text{Eigen}(\sigma, V^T) \subseteq F \setminus \{0\} \). Choose \( \theta \in W(\Gamma, \varphi) \), then \( \varphi|_\theta : \varphi \sim \theta \sim \) is bijective and \( \theta \in W(\varphi|_\theta) \), thus by Lemma 2.8 we have \( F \setminus \{0\} = \text{Eigen}(\sigma|_\theta, V^\theta) \). Using Lemma 2.1 we have \( F \setminus \{0\} = \text{Eigen}(\sigma|_\theta, V^\theta) \subseteq \text{Eigen}(\sigma, V^T) \) which completes the proof. □

Lemma 2.9. If \( \varphi : \Gamma \to \Gamma \) is bijective and \( \sim_\varphi = \Gamma \times \Gamma \) with \( W(\Gamma, \varphi) = \emptyset \), then (where for \( r \in F \setminus \{0\} \) by \( o(r) \) we mean order of \( r \) in commutative multiplying group \( F \setminus \{0\} \) and for finite set \( A \), \( |A| \) denotes cardinality of \( A \):

1. \( P(\Gamma, \varphi) = \Gamma \) is finite,
2. for all \( \alpha \in \Gamma \) we have \( \{\varphi^n(\alpha) : 0 \leq n < |\Gamma|\} = \Gamma(= \{\varphi^n(\alpha) : n \geq 0\}) \) and \( \varphi|_\Gamma(\alpha) = \alpha \),
3. \( \text{Eigen}(\sigma, V^T) = \{r \in F \setminus \{0\} : o(r) \) is finite and \( o(r) \) divides \( |\Gamma|\} \).

Proof. 1. Since \( W(\Gamma, \varphi) = \emptyset \), we have \( Q(\Gamma, \varphi) = \Gamma \) and for \( \alpha \in \Gamma = Q(\Gamma, \varphi) \) there exist \( n \geq 1 \) with \( \varphi^n(\alpha) = \varphi^m(\alpha), \) thus \( \varphi^{n-m}(\alpha) = \alpha \) and \( \alpha \in P(\Gamma, \varphi). \)
2. Use (1).
3. Suppose \( r \in \text{Eigen}(\sigma, V^T) \), by Corollary 2.3 we have \( r \neq 0 \). There exists nonzero \( (x_\alpha)_{\alpha \in \Gamma} \in V^T \) with \( \sigma(\alpha)(x_\alpha)_{\alpha \in \Gamma} = r(x_\alpha)_{\alpha \in \Gamma} \). By (2) we have \( \varphi|_\Gamma(\alpha) = \alpha \) for all \( \alpha \in \Gamma \), thus

\[
\sigma|_\Gamma((x_\alpha)_{\alpha \in \Gamma}) = (\varphi|_\Gamma)(x_\alpha)_{\alpha \in \Gamma} = (x_\alpha)_{\alpha \in \Gamma}.
\]

On the other hand \( \sigma|_\Gamma((x_\alpha)_{\alpha \in \Gamma}) = r|_\Gamma(x_\alpha)_{\alpha \in \Gamma} = (r|_\Gamma)x_\alpha)_{\alpha \in \Gamma} \). Hence:

\[
\forall \alpha \in \Gamma \ (r|_\Gamma)x_\alpha = x_\alpha.
\]

Choose \( \beta \in \Gamma \) with \( x_\beta \neq 0 \). By \( r|_\Gamma x_\beta = x_\beta \) we have \( r|_\Gamma = 1 \) and \( o(r) \) divides \( |\Gamma| \).

Conversely, for \( r \in F \setminus \{0\} \) if \( o(r) \) divides \( |\Gamma| \), then choose fix \( x \in V \setminus \{0\} \) and \( \theta \in \Gamma \). For \( \alpha \in \Gamma \) let \( x_\alpha = r^{m(\theta, \alpha)}x \). Using \( \sigma(\alpha)(x_\alpha)_{\alpha \in \Gamma} = r(x_\alpha)_{\alpha \in \Gamma} \) we have \( r \in \text{Eigen}(\sigma, V^T) \) which completes the proof. □

Lemma 2.10. For \( \varphi : \Gamma \to \Gamma \) if \( P(\Gamma, \varphi) = \Gamma \), then:

- \( \varphi : \Gamma \to \Gamma \) is bijective,
- \( \text{Eigen}(\sigma, V^T) = \{r \in F \setminus \{0\} : o(r) \) is finite and there exists \( \alpha \in \Gamma \) such that \( o(r) \) divides \( |\alpha|\} \).

Proof. Use Lemmas 2.1 and 2.9. □
Theorem 2.11 (Main Theorem). For isomorphism $\sigma : V^\Gamma \to V^\Gamma$ (so $\varphi : \Gamma \to \Gamma$ is bijective) we have:

$$\text{Eigen}(\sigma, V^\Gamma) = \begin{cases} F \setminus \{0\}, & W(\varphi, \Gamma) \neq \emptyset, \\ \{r \in F \setminus \{0\} : \exists \theta \in P(\varphi, \Gamma)(o(r) | m(\theta, \theta))\}, & W(\varphi, \Gamma) = \emptyset. \end{cases}$$

In particular $\text{Eigen}(\sigma, V^\Gamma)$ does not depend on $V$.

Proof. Note that for periodic point $\alpha$ of $\varphi$ we have $m(\alpha, \alpha) = |\{\varphi^n(\alpha) : n \geq 0\}|$. Also by bijection of $\varphi$ we have $Q(\varphi, \Gamma) = P(\varphi, \Gamma)$. Use Lemmas 2.8 and 2.10 to complete the proof. \qed

Corollary 2.12. For isomorphism $\sigma : V^\Gamma \to V^\Gamma$ and $r, s \in F \setminus \{0\}$ with $r \in \text{Eigen}(\sigma, V^\Gamma)$ if $o(s)|o(r)$, then $s \in \text{Eigen}(\sigma, V^\Gamma)$.

Proof. If $r \in \text{Eigen}(\sigma, V^\Gamma)$, then by Theorem 2.11 there exists $\alpha \in \Gamma$ with $o(r)|m(\alpha, \alpha)$ thus $o(s)|m(\alpha, \alpha)$ too, which leads to $s \in \text{Eigen}(\sigma, V^\Gamma)$. \qed

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References


