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Using Legendre spectral element method with Quasi-linearization method for solving Bratu's problem

Mahmoud Lotfi*
Department of Applied Mathematics,
University of Kurdistan, Sanandaj, Iran.
E-mail: m.lotfi@sci.uok.ac.ir and lotfi.mahmud@gmail.com

Amjad Alipanah
Department of Applied Mathematics,
University of Kurdistan, Sanandaj, Iran.
E-mail: A.Alipanah@uok.ac.ir

Abstract This work presented here is the solution of the one-dimensional Bratus problem. The nonlinear Bratus problem is first linearised using the quasi-linearization method and then solved by the spectral element method. We use the Legendre polynomials for interpolation. Finally we show the results with a numerical example.

Keywords. Bratu's problem, Quasi-linearization, Spectral element method, Legendre polynomials.

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1. Introduction

1.1. **problem definition.** The Bratu's problem which was set by Bratu [8], in 1914 comes in different forms. The most generalized Bratu's problem is the so-called Liouville-Bratu-Gelfand equation [18, 28] which has the form

$$\nabla^2 u(\mathbf{x}) = -\lambda e^{u(\mathbf{x})}, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n,$$

$$u = 0, \qquad \mathbf{x} \in \partial\Omega,$$

where the constant $\lambda > 0$ is a physical parameter and $\partial\Omega$ is the boundary of Ω. In this paper, we restrict ourselves to the Bratu's problem in the one-dimension given by

$$\begin{aligned} &u''(x) = -\lambda e^{u(x)}, \quad 0 \le x \le 1, \\ &u(0) = u(1) = 0. \end{aligned} \tag{1.1}$$

From literature [23, 2], the analytical solution of equation (1.1) is given by

$$y(x) = -2\ln\left[\frac{\cosh\left(\left(x-\frac{1}{2}\right)\frac{\omega}{2}\right)}{\cosh\left(\frac{\omega}{4}\right)}\right],$$

$*$ Corresponding author.

where ω is the solution of the equation $\omega = \sqrt{2\lambda}\cosh\left(\frac{\omega}{4}\right)$. The Bratu's problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$, respectively, where the critical value λ_c satisfies the equation $l = \frac{l}{4}\sqrt{2\lambda_c}\sinh\left(\frac{l}{4}\omega\right)$. According to Boyd [7], $\lambda_c = 3.513830719$.

The Bratu's problem is worth investigating due to its several applications in both science and engineering. Some of the applications of the two-point boundary value problem for Bratus equation include the Chandrasekhar model of the expansion of the universe [20]. The Bratu's problem arises in the electrospinning process for the production of ultra-fine polymer fibers [15]. Apart from these physical applications, the Bratu's problem has been used as a benchmark for non-linear solvers. In particular, Motsa and Sibanda [27] tested and proved the accuracy and validity of the modified quasi-linearization method using the Bratu's problem. A lot of work has been done by researchers to find the numerical solution of the Bratu's problem in one dimension. Ben-Romdhane and Temimi [6] proposed a new iterative finite difference scheme based on the Newton-Raphson-Kantrovich approximation method to solve the classical Bratu's problem. Mohsen [23] used the non-standard finite difference method to treat the one-dimensional Bratu's problem. Other numerical techniques which were used to solve the Bratu's problem include the shooting method [1], finite element method [9], homotopy analysis method [13] and the Laplace Adomian decomposition method [16].

1.2. **A summary of the spectral element method.** A spectral element method (SEM) combine the advantages and disadvantages of Galerkin spectral methods with those of finite element methods by a simple application of the spectral method per element. One of the advantages of this method is the high accuracy and stable solving algorithm with a small number of elements under a wide range of conditions [29].

Finite element method (FEM) was proposed for the first time in 1943 by Richard Courant [12]. He solve the Poisson equation based on minimizing piecewise linear approximations on finite subdomains.

Spectral method is a conventional method for solving partial differential equations, which was first introduced by the Navier for elastic sheet problems in 1825. In spectral method approximate the solution on the one general domain.

In 1984, Patera with the division of domains, applied a spectral method to a greater number of subdomains. He proposed the spectral element method by combining the spectral method and the FEM [24]. Patera in his innovative method, use the Chebyshev polynomials as the interpolation basis function. Legendre spectral element method (LSEM) were developed by Maday and Patera [22]. The use of the Lagrangian interpolation conjugate with the Gauss-Legendre-Lobatto quadrature leads to a matrix of mass with a diameter structure [4]. The diagonal mass matrix is a very important property of the LSEM and is different from the Chebyshev spectral element method [26]. Chen et al., in [10], used the LSEM to solve the constrained optimal control problem. An alternating direction implicit (ADI) LSEM for the two-dimensional Schrodinger equation is developed in [34], and the optimal H^1 error estimate for the linear case is given. The aim of [14] is the Lagrange-Galerkin SEM for solving the two-dimensional shallow water equations. Authors of [33] have considered the numerical approximation of the acoustic wave equation by the SEM based on Gauss-Lobatto-Legendre quadrature formulas, and finite difference Newmark's explicit time advancing schemes. A modified set of basis functions for use with SEMs is presented in [30] for solving a mixed elliptic boundary value problem. These basis functions are constructed so that the axial conditions along a plane or axis of symmetry are satisfied identically. A numerical SEM for the computation of fluid flows governed by the incompressible Euler equations in a complex geometry is presented in [32]. Zhuang and Chen used this method to solve the biharmonic equations [35]. In [19], authors used the SEM with least-square formulation for parabolic interface problems. Ai et al., used fully diagonalized LSEMs using Sobolev orthogonal/biorthogonal basis functions for solving second order elliptic boundary value problems [3]. A Legendre spectral element formulation of an improved time-splitting method is developed for the natural convection heat transfer problem in a square cavity by Wang and Qin [31]. Lotfi and Alipanah in [21], study the LSEM for solving the sine-Gordon equation in one dimension. The stability and convergence analysis of the method is also done.

1.3. **The main aim of this article.** The main contribution of this article is to introduce an efficient numerical method for Bratu's problem in one dimension. First linearised the nonlinear Bratu's problem by using the quasi-linearization method and then solved by the LSEM. In section 2, we first obtain linear form of the Eq. (1.1) using the quasi-linearization method. In section 3, the Legendre polynomials and the associated SEM are given, and discrete form of the problem is obtained using the Legendre SEM and its matrices form is calculated. In Section 4, we show the efficiency of the method by solving a numerical example.

2. Quasi-linearization method

Our main method is a combination of two numerical methods, quasi-linearization method (QLM) and the LSEM. The QLM which is Newton-Raphson based, was originally proposed by [5]. It is used to linearize the non-linear differential equation into an iterative sequence of linear differential equations. The resulting system of equations is solved using the LSEM.
Let us consider an n^{th} order nonlinear differential equation of the form

$$\mathbf{F}\left[\mathbf{u}\left(x\right)\right]=0, \qquad x\in\left[a,b\right], \tag{2.1}$$

where x is an independent variable and $\mathbf{u}\left(x\right)=\left(u,u',...,u^{(n)}\right)$ is a vector of solutions of (2.1). As in [11] it is assumed that $\mathbf{z}=\left(z,z',...,z^{(n)}\right)$is an approximate solution of (2.1) which is sufficiently close to the true solution u. Assuming that all the partial derivatives of $\mathbf{F}$ exists, applying Taylors theorem we get

$$\mathbf{F}\left[\mathbf{u}\right]=\mathbf{F}\left(\mathbf{z}\right)+\nabla\mathbf{F}\left(\mathbf{z}\right).\left(\mathbf{u}-\mathbf{z}\right)+\left(higher\ \ order\ \ terms\right). \tag{2.2}$$

Upon ignoring higher order terms equation (2.2) becomes

$$\nabla\mathbf{F}\left(\mathbf{z}\right).\mathbf{u}=\nabla\mathbf{F}\left(\mathbf{z}\right).\mathbf{z}\ \ -\mathbf{F}\left(\mathbf{z}\right) \tag{2.3}$$

The solution from (2.3) will not be, generally, the exact solution of (2.1) because of the discarded higher order terms. We will use the initial approximate solution $\mathbf{z}$ as a calculated solution to iteratively compute the new solution $\mathbf{u}$. With this in mind, denote $\mathbf{z}$ and $\mathbf{u}$ by $\mathbf{u}_s$ and $\mathbf{u}_{s+1}$ respectively to get the iterative formula

$$\nabla \mathbf{F}\left(\mathbf{u}_s\right).\mathbf{u}_{s+1} = \nabla \mathbf{F}\left(\mathbf{u}_s\right).\mathbf{u}_s - \mathbf{F}\left(\mathbf{u}_s\right) \tag{2.4}$$

where $s = 0, 1, 2, ...$

3. LSEM

3.1. **Legendre polynomials.** The Nth-degree Legendre polynomial $\mathcal{L}_N\left(\theta\right)$, is a solution of the second-order differential equation

$$\left(\left(\theta^2 - 1\right)\mathcal{L}'_N\left(\theta\right)\right)' - N\left(N+1\right)\mathcal{L}_N\left(\theta\right) = 0$$

In the normalized form of $\mathcal{L}_N\left(\theta\right)$ we have $\mathcal{L}_N\left(1\right) = 1$, which can be calculated as follows

$$\mathcal{L}_N\left(\theta\right) = 2^{-N}\sum_{i=0}^{\left[\frac{N}{2}\right]}\left(-1\right)^i\left(\begin{array}{c} N \\ i \end{array}\right)\left(\begin{array}{c} 2N-2i \\ N \end{array}\right)\theta^{N-2i}$$

where $[x]$ denotes the integer part of x. For each pair of Legendre polynomial of degrees N and M, the following orthogonality property applies

$$\int_{-1}^{1}\mathcal{L}_N\left(\theta\right)\mathcal{L}_M\left(\theta\right)d\theta = \frac{2}{2N+1}\delta_{NM},$$

where δ_{NM} is Kroneckers delta. The Nth-degree Lobatto polynomial, $\mathcal{LO}_N$, derives from the $(N+1)$-degree Legendre polynomial, $\mathcal{L}_{N+1}$, as

$$\mathcal{LO}_N\left(\theta\right) = \mathcal{L}'_{N+1}\left(\theta\right).$$

Legendre and Lobatto polynomials can be calculated using the recursive relations [25]

$$\mathcal{L}_{N+1}\left(\theta\right) = \tfrac{2N+1}{N+1}\theta\mathcal{L}_N\left(\theta\right) - \tfrac{N}{N+1}\mathcal{L}_{N-1}\left(\theta\right),$$

$$\mathcal{LO}_{N-1}\left(\theta\right) = \tfrac{N(N+1)}{2N+1}\tfrac{\left(\mathcal{L}_{N-1}(\theta) - \mathcal{L}_{N+1}(\theta)\right)}{1-\theta^2}.$$

3.2. **LSEM.** In the LSEM, we first divide the domain Ω into N_e non-overlapping subdomains Ω_e,

$$\bar{\Omega} = \bigcup_{e=1}^{N_e}\bar{\Omega}_e, \quad \bigcap_{e=1}^{N_e}\Omega_e = \phi.$$

Basis functions are considered as the Lagrangian interpolation polynomials defined at Gauss-Lobatto integration points on each element. If $N_e = 1$ we obtain a spectral Galerkin method of order $N-1$. If $N = 1$ or $N = 2$ a standard Galerkin FEM is obtained based on linear and quadratic elements respectively. Convergence is either obtained by increasing the degree of the polynomials or by increasing the number of

elements N_e.
Now on each element Ω_e we define the approximate solution of order N as

$$u^e(x) = \sum_{j=0}^{N} u_j^e \varphi_j(x), \qquad 1 \le e \le N_e, \tag{3.1}$$

where φ_j is the j^{th} Lagrange polynomial of order N on the Gauss-Legendre-Lobatto points $\{\theta_i\}_{i=0}^{N}$ [17]

$$\varphi_j(\theta) = \frac{1}{N(N+1)\mathcal{L}_N(\theta_j)} \frac{\left(\theta^2 - 1\right)\mathcal{LO}_{N-1}(\theta)}{\theta - \theta_j}, \ 0 \le j \le N, \ -1 \le \theta \le 1.$$

To convert the $[-1, 1]$ to eth element and its inverse, we use the following mapping functions

$$x(\theta) = \frac{(x_e - x_{e-1})\theta}{2} + \frac{x_e + x_{e-1}}{2}, \qquad -1 \le \theta \le 1,$$

$$\theta(x) = \frac{2x - (x_e + x_{e-1})}{x_e - x_{e-1}}, \qquad x_{e-1} \le x \le x_e,$$

where x_e and x_{e-1} are the endpoints of e^{th} element. The stiffness and mass matrices on each element are calculated as follows

$$S_{ij}^e = \int_{x_{e-1}}^{x_e} \varphi'_i(x)\varphi'_j(x)\,dx = \frac{2}{h_e}\int_{-1}^{1} \varphi'_i(\theta)\varphi'_j(\xi)\,d\theta,$$

$$M_{ij}^e = \int_{x_{e-1}}^{x_e} \varphi_i(x)\varphi_j(x)\,dx = \frac{h_e}{2}\int_{-1}^{1} \varphi_i(\theta)\varphi_j(\theta)\,d\theta,$$

where

$$h_e = x_e - x_{e-1}.$$

By using the Gauss quadrature we obtain [25]

$$S_{ij}^e = \frac{2}{h_e}\sum_{k=0}^{N} d_{ik} d_{jk} w_k,$$

$$M_{ij}^e = \frac{h_e}{2}\delta_{ij} w_i,$$

where

$$w_k = \frac{2}{N(N+1)\left[\mathcal{L}_N(t_k)\right]^2}, \qquad 0 \le k \le N,$$

and

$$d_{ik} = \frac{\mathcal{L}_N(\theta_k)}{\mathcal{L}_N(\theta_i)} \frac{1}{\theta_k - \theta_i}, \qquad i \ne k,$$

$$d_{ii} = \frac{\mathcal{LO}_{N-1}(\theta_i)}{2\mathcal{L}_N(\theta_i)}.$$

3.3. **Application to the Bratu's problem.** The Bratu's problem (1.1) can be transformed to a linear differential problem using the QLM. Equation (1.1) is of second order, thus we have

$$\mathbf{F}(u, u', u'') = u''(x) + \lambda e^{u(x)}.$$

Substituting into (2.4) we get the iterative scheme

$$\begin{aligned} &u''_{s+1}(x) + \lambda e^{u_s(x)} u_{s+1}(x) = \lambda e^{u_s(x)} (u_s(x) - 1), \\ &u_{s+1}(0) = u_{s+1}(1) = 0, \end{aligned} \tag{3.2}$$

where $s = 0, 1, 2, \ldots$. Equation (3.2) can be used to compute $u_{s+1}(x)$ provided $u_s(x)$ is known. In particular, the initial approximation $u_0(x)$ must be specified so that we compute $u_1(x)$. Once $u_1(x)$ is known, we compute $u_2(x)$ using equation (3.2) and so on. Also, $u_0(x)$ must satisfy boundary conditions.
The weak form of the Equation (3.2) is obtained as follows
For each element Ω_e, find $u^e \in U^h$, such that

$$\begin{aligned} &-\int_{\Omega_e} u^e_{s+1,x} v_x dx + \lambda \int_{\Omega_e} e^{{u_s}^e} u_{s+1} v dx \\ &= \lambda \int_{\Omega_e} e^{{u_s}^e} (u_s - 1) v dx, \qquad \forall v \in U^h, \leq e \leq N_e. \end{aligned}$$

The first integral on the left hand side, is obtained by integration by parts. Now, if we consider the test function v to be the k^{th} Lagrange's function of order N and use the equation (3.1), we have

$$\begin{aligned} &-\sum_{j=0}^{N} u^e_{j,s+1} \left(\int_{\Omega_e} {\varphi'}_j {\varphi'}_k dx \right) + \lambda \sum_{j=0}^{N} u^e_{j,s+1} \left(e^{u^e_{j,s}} \int_{\Omega_e} \varphi_j \varphi_k dx \right) \\ &\qquad = \lambda \sum_{j=0}^{N} e^{u^e_{j,s}} \left(u^e_{j,s} - 1 \right) \left(\int_{\Omega_e} \varphi_j \varphi_k dx \right). \end{aligned} \tag{3.3}$$

The right hand side of equation (3.3) is obtained using the following equation

$$e^{u^e}(u^e - 1) \cong \sum_{j=0}^{N} e^{u^e_j} \left(u^e_j - 1 \right) \varphi_j.$$

The matrix form of the semi-discrete form equation (3.3) will be as follows

$$-S^e U^e_{s+1} + \lambda M^e e^{U^e_s} U^e_{s+1} = \lambda M^e e^{U^e_s} (U^e_s - E^e), \tag{3.4}$$

Where $E^e = [1, 1, ..., 1]^T$. The vector U^e contains the approximate solution of the order N on the element Ω_e, M^e is a local diagonal mass matrix and S^e is a local stiffness matrix on the element Ω_e.
In order to obtain a discrete form on the general domain, we must assemble the local matrices M^e and S^e and obtain the general matrices M and S [25]. So the equation (3.4) will be as follows

$$-SU_{s+1} + \lambda M e^{U_s} U_{s+1} = \lambda M e^{U_s} (U_s - E), \tag{3.5}$$

in which U is the vector of the approximate solution on the general domain Ω.

4. Numerical results

In this section, we consider the numerical example to validate the proposed scheme.The accuracy of the scheme is verified by L_2 and L_∞ norms calculating and root mean square errors.
We set

$$L_\infty err \equiv \|u - U\|_\infty,$$

$$RMSerr = \frac{L_2 err}{N_{e,N}+1},$$

$$u^e(x) = \sum_{j=0}^{N} u_j^e \varphi_j(x), \qquad 1 \le e \le N_e,$$

Where $N_{e,N}$ is the all nodes of the domain and U_n is the vector of nodal values of the numerical solution corresponding to the discretization parameters N, N_e and k at time t_n, and, for each continuous function f

$$\|f\|_2 = \sqrt{\sum_{r=1}^{N_{e,N}} f^2(x_r)},$$

$$\|f\|_\infty = \max_{1\le r\le N_{e,N}} |f(x_r)|.$$

The linear system (3.2) is solved using the proposed scheme for $\lambda = 1, 2$. We solve this problem with several values of N and N_e. Table 1 shows the errors of proposed scheme with several values of N and $N_e = 20$. Figure 1 show graph of exact solution and approximate solution with $N = 4$ and $N_e = 20$.

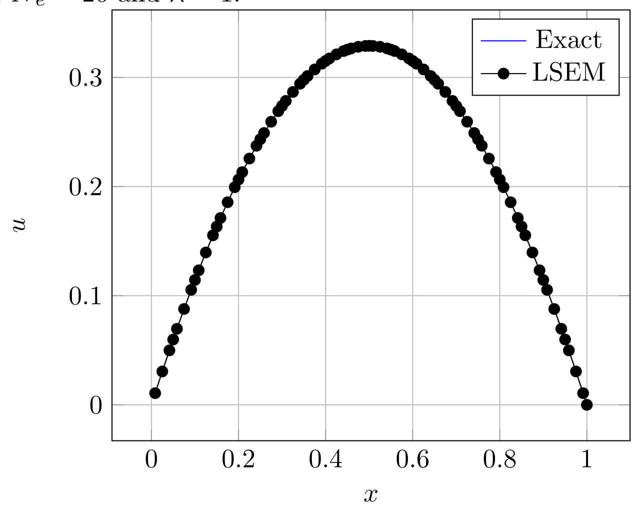
Table 1. Numerical results for Bratu's problem with $N_e = 20$.

	$\lambda = 2$			$\lambda = 1$		
N	$L_2 err$	$L_\infty err$	$RMSerr$	$L_2 err$	$L_\infty err$	$RMSerr$
1	1.3053e-02	1.6500e-02	1.1023e-03	2.1529e-02	2.2097e-02	1.1028e-03
2	1.0018e-02	7.9776e-03	3.0531e-04	1.3130e-02	8.3774e-03	3.2024e-04
3	2.2023e-03	9.9553e-04	3.6104e-05	2.4071e-03	1.0268e-03	3.9460e-05
4	3.9778e-04	1.9421e-04	4.9109e-06	6.4771e-04	3.4862e-04	7.9964e-06
5	2.5862e-04	1.3858e-04	3.5507e-06	1.0154e-04	5.3085e-05	1.0053e-06
6	1.5050e-04	1.0004e-04	2.8967e-06	3.5765e-05	1.5567e-05	2.9558e-07
7	3.7733e-05	4.2734e-05	1.0061e-06	3.5359e-06	1.0017e-05	1.5078e-07

5. Conclusion

The spectral methodss are useful tools for solving ordinary and partial differential equations. Also, the incorporation of the finite element method with the spectral polynomials i.e. the use of the spectral polynomials as a new shape function in the finite element method is very efficient for obtaining a numerical algorithm with high

Figure 1. Exact and numerical results for Bratu's problem with $N = 4$, $N_e = 20$ and $\lambda = 1$.



accuracy. In this article, we constructed a LSEM for the solution of the Bratu's problem. We used the LSEM for discretizing the spatial space. Also we used a quasi-linearization method for linearised the problem. Finally, using the test problems, we demonstrated that the algorithm is efficient for obtaining approximation solutions of Bratu's problem.

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