Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 7, No. 3, 2019, pp. 359-369

Application of the invariant subspace method in conjunction with the fractional Sumudu's transform to a nonlinear conformable time-fractional dispersive equation of the fifth-order

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Abstract During the past years, a wide range of distinct approaches has been exerted to solve the nonlinear fractional differential equations (NLFDEs). In this paper, the invariant subspace method (ISM) in conjunction with the fractional Sumudu's transform (FST) in the conformable context is formally adopted to deal with a nonlinear conformable time-fractional dispersive equation of the fifth-order. As an outcome, a new exact solution of the model is procured, corroborating the exceptional performance of the hybrid scheme.

Keywords. Fifth-order time-fractional dispersive equation, Conformable context, Invariant subspace method, Fractional Sumudu's transform, A new exact solution.

2010 Mathematics Subject Classification. 34A08, 83C15.

1. Introduction

The investigation of exact solutions is one of the hottest topics in mathematical physics; since a lot of information is provided using the exact solutions. In the last years, several various schemes [5, 10–12, 19–21, 25, 26, 28, 33, 39, 41, 45] have been used to solve the nonlinear fractional differential equations. Recently, a systematic approach called the invariant subspace method [16–18, 34, 35, 38, 40, 44] has received significant attention among academic scholars. For instance, Sahadevan and Prakash [40] utilized the ISM to extract the exact solutions of time-fractional

Received: 19 April 2018 ; Accepted: 22 September 2018.
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Hunter−Saxton equation, time-fractional coupled nonlinear diffusion system, time-fractional coupled Boussinesq equation, and time-fractional Whitman−Broer−Kaup system in the Caputo sense and Hashemi [18] adopted the ISM along with the conformable fractional Laplace transform to retrieve the exact solutions of time-fractional thin-film, Hunter−Saxton and dispersive equations in the conformable sense. For further information check references [2–4, 6–8, 22–24, 31, 42].

The Sumudu's transform is another famous method that was first established by Watugala [43] to deal with the problems in engineering. The Sumudu's transform of a function like $f(t)$ is given by [43]

$$G(u) = S[f(t); u] = \int_0^\infty e^{-t} f(ut)dt,$$

provided that the integral exists for some u. The Sumudu's transform consists of many interesting properties which have been pointed out by Watugala in [43]. Due to the super importance of the integral transforms [9, 14, 27, 30, 32], the current paper aims to utilize the ISM in conjunction with the FST in the conformable context for handling a nonlinear conformable time-fractional dispersive equation of the fifth-order. The conformable fractional calculus and some of its features will be reviewed below.

Definition 1.1. For a function like $f(t)$ defined for $t \geq 0$, the αth order of the conformable fractional derivative is given as [29]

$${}_tT_\alpha(f(t)) = \frac{d^\alpha f(t)}{dt^\alpha} = \lim_{\tau \to 0} \frac{f(t + \tau t^{1-\alpha}) - f(t)}{\tau}, \qquad \alpha \in (0, 1],\ t > 0,$$

and ${}_tT_\alpha(f(0)) = \lim_{t \to 0^+} {}_tT_\alpha(f(t))$.

Definition 1.2. For $f : [a, \infty[\longrightarrow \mathbb{R}$, $a \geq 0$, the conformable fractional integral of f is expressed by [29]

$$I^a_\alpha(f(t)) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

in which $\alpha \in (0, 1]$.

The conformable fractional derivative provides a series of interesting features which have been presented in [1, 15, 29].

Theorem 1.3. *If $f(t)$ and $g(t)$ are α-differentiable for $t > 0$ when $\alpha \in (0, 1]$, then*

i. ${}_tT_\alpha(af(t) + bg(t)) = a\,{}_tT_\alpha(f(t)) + b\,{}_tT_\alpha(g(t)),\ \forall a, b \in \mathbb{R}$.

ii. ${}_tT_\alpha(t^\beta) = \beta t^{\beta-\alpha},\ \forall \beta \in \mathbb{R}$.

iii. ${}_tT_\alpha(f(t)g(t)) = g(t)\,{}_tT_\alpha(f(t)) + f(t)\,{}_tT_\alpha(g(t))$.

iv. ${}_tT_\alpha\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)\,{}_tT_\alpha(f(t)) - f(t)\,{}_tT_\alpha(g(t))}{g(t)^2}$.

v. ${}_tT_\alpha(f(t)) = t^{1-\alpha} \frac{df(t)}{dt}$.

Theorem 1.4. *If $f(t)$ and $g(t)$ are differentiable and $f(t)$ is also α-differentiable, then*

$${}_tT_\alpha(fog(t)) = t^{1-\alpha} g'(t) f'(g(t)).$$

The outline of the present article is as follows: In the second section, the ISM is described in detail. In the third section, the FST in the conformable context and its features are introduced. In the fourth section, the ISM along with the FST in the conformable context is exerted to solve a nonlinear conformable time-fractional dispersive equation of the fifth-order. Ultimately, the last section summarizes the results of the current work.

2. Invariant subspace method

Suppose that a nonlinear conformable time-fractional PDE can be written as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \Xi(u(x,t)), \qquad \alpha \in (0,1], \tag{2.1}$$

in which Ξ is a nonlinear differential operator with respect to the variable x.

Definition 2.1. The finite-dimensional linear space

$$W_n = span\{w_1(x), w_2(x), \cdots, w_n(x)\}$$

is an invariant subspace with respect to (2.1), if $\Xi(W_n) \subseteq W_n$.

Theorem 2.2. *If $W_n = span\{w_1(x), w_2(x), \cdots, w_n(x)\}$ is an invariant subspace with respect to (2.1), then, there exist the functions $\psi_1, \psi_2, \cdots, \psi_n$ such that*

$$\Xi\Big[\sum_{i=1}^{n} \lambda_i w_i(x)\Big] = \sum_{i=1}^{n} \psi_i(\lambda_1, \lambda_2, \cdots, \lambda_n) w_i(x), \qquad \lambda_i \in \mathbb{R},\ i = 1, \cdots, n.$$

Furthermore

$$u(x,t) = \sum_{i=1}^{n} \lambda_i(t) w_i(x),$$

is the solution of Eq. (2.1), if the coefficients $\lambda_i(t)$ gratify the following system of conformable FDEs

$$_tT_\alpha(\lambda_i(t)) = \psi_i(\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)), \qquad i = 1, \cdots, n.$$

Proof. See [18]. □

3. Fractional Sumudu's transform in the conformable context and its features

In this section, the FST in the conformable context and its features are introduced. For this purpose, let's first define the FST in the conformable context.

Definition 3.1. For a function like $f(t) : [0, \infty[\to \mathbb{R}$, the αth order of the FST in the conformable context is given as

$$S_\alpha[f(t); u] = \int_0^\infty e^{-\frac{1}{\alpha}t^\alpha} f(ut) d_\alpha t = \int_0^\infty e^{-\frac{1}{\alpha}t^\alpha} f(ut) t^{\alpha-1} dt,$$

when it is finite.

The conformable fractional Sumudu's transform (CFST) of some elementary functions has been given below.

i. $S_\alpha\Big[\sin(\frac{a}{\alpha}t^\alpha);u\Big] = \frac{au^\alpha}{1+(au^\alpha)^2}$.

ii. $S_\alpha\Big[\cos(\frac{a}{\alpha}t^\alpha);u\Big] = \frac{1}{1+(au^\alpha)^2}$.

Proof. (i):

$$\begin{aligned} S_\alpha\Big[\sin(\frac{a}{\alpha}t^\alpha);u\Big] &= \int_0^\infty e^{-\frac{1}{\alpha}t^\alpha}\sin(\frac{a}{\alpha}(ut)^\alpha)t^{\alpha-1}dt \quad \Big(x=\frac{1}{\alpha}t^\alpha\Big) \\ &= \int_0^\infty e^{-x}\sin(au^\alpha x)dx \\ &= S\Big[\sin(at);u^\alpha\Big] \\ &= \frac{au^\alpha}{1+(au^\alpha)^2}, \end{aligned}$$

which completes the proof.

(ii):

$$\begin{aligned} S_\alpha\Big[\cos(\frac{a}{\alpha}t^\alpha);u\Big] &= \int_0^\infty e^{-\frac{1}{\alpha}t^\alpha}\cos(\frac{a}{\alpha}(ut)^\alpha)t^{\alpha-1}dt \quad (x=\frac{1}{\alpha}t^\alpha) \\ &= \int_0^\infty e^{-x}\cos(au^\alpha x)dx \\ &= S\Big[\cos(at);u^\alpha\Big] \\ &= \frac{1}{1+(au^\alpha)^2}, \end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let $S_\alpha[f(t);u]$ and $S_\alpha[g(t);u]$ exist. Then*

$$S_\alpha[(c_1f+c_2g)(t);u] = c_1S_\alpha[f(t);u]+c_2S_\alpha[g(t);u].$$

Proof.

$$\begin{aligned} S_\alpha\left[(c_1f+c_2g)(t);u\right] &= \int_0^\infty e^{-\frac{1}{\alpha}t^\alpha}\left(c_1f+c_2g\right)(ut)\,t^{\alpha-1}dt \\ &= c_1\int_0^\infty e^{-\frac{1}{\alpha}t^\alpha}f\,(ut)t^{\alpha-1}dt \\ &+ c_2\int_0^\infty e^{-\frac{1}{\alpha}t^\alpha}g\,(ut)t^{\alpha-1}dt \\ &= c_1S_\alpha\left[f(t);u\right]+c_2S_\alpha\left[g(t);u\right], \end{aligned}$$

which completes the proof. It is clear that the CFST is a linear operator. □

Theorem 3.3. *Let $f(t):[0,\infty[\longrightarrow\mathbb{R}$ be α-differentiable and $S_\alpha[{}_tT_\alpha(f(t));u]$ exists. Then*

$$S_\alpha\left[{}_tT_\alpha\left(f(t)\right);u\right] = \frac{S_\alpha\left[f(t);u\right]}{u^\alpha}-\frac{f(0)}{u^\alpha}.$$

Proof.

$$
\begin{aligned}
S_\alpha\left[{}_tT_\alpha\left(f(t)\right);u\right] &= \int_0^\infty e^{-\frac{1}{\alpha}t^\alpha}\,{}_tT_\alpha\left(f(ut)\right)t^{\alpha-1}dt\\
&= u^{1-\alpha}\int_0^\infty e^{-\frac{1}{\alpha}t^\alpha}f'(ut)dt \qquad \text{integration by parts}\\
&= u^{-\alpha}\left[\int_0^\infty e^{-\frac{1}{\alpha}t^\alpha}f(ut)t^{\alpha-1}dt - f(0)\right]\\
&= \frac{S_\alpha\left[f(t);u\right]-f(0)}{u^\alpha}.
\end{aligned}
$$

□

If $f(t)$ is n times α-differentiable and $S_\alpha\left[{}_tT_\alpha^{(n)}\left(f(t)\right);u\right]$ exists, then it can be readily demonstrated that

$$
S_\alpha\left[{}_tT_\alpha^{(n)}\left(f(t)\right);u\right] = \frac{S_\alpha\left[f(t);u\right]-f\left(0\right)}{u^{n\alpha}} - \frac{{}_tT_\alpha\left(f(0)\right)}{u^{(n-1)\alpha}} - \cdots - \frac{{}_tT_\alpha^{(n-1)}\left(f(0)\right)}{u^\alpha}.
$$

4. Nonlinear conformable time-fractional dispersive equation and its new exact solution

Consider the following nonlinear conformable time-fractional dispersive equation of the fifth-order [17, 18]

$$
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \lambda\frac{\partial^5 u^2(x,t)}{\partial x^5} + \mu\frac{\partial^3 u^2(x,t)}{\partial x^3} + \eta\frac{\partial u^2(x,t)}{\partial x}, \quad \alpha\in(0,1]. \tag{4.1}
$$

As shown in [17, 18], the invariant subspace for the Eq. (4.1) is

$$
W_3 = span\left\{1,\cos\left(x\right),\sin\left(x\right)\right\},
$$

if $16\lambda-4\mu+\eta=0$. To review this assertion, suppose that

$$
E = \lambda_1+\lambda_2\cos\left(x\right)+\lambda_3\sin\left(x\right),
$$

and so

$$
\begin{aligned}
\Xi\left(E\right) &= 4\left(16\lambda-4\mu+\eta\right)\lambda_2\lambda_3\cos^2\left(x\right)\\
&+\left(\left(-2\left(16\lambda-4\mu+\eta\right)\lambda_2^2+2\left(16\lambda-4\mu+\eta\right)\lambda_3^2\right)\sin\left(x\right)\right.\\
&+2\left(\lambda-\mu+\eta\right)\lambda_1\lambda_3\right)\cos\left(x\right)\\
&-2\left(\lambda-\mu+\eta\right)\lambda_1\lambda_2\sin\left(x\right)-2\left(16\lambda-4\mu+\eta\right)\lambda_2\lambda_3.
\end{aligned}
$$

Now, by considering $16\lambda-4\mu+\eta=0$, we find

$$
\Xi\left(E\right) = 2\left(\lambda-\mu+\eta\right)\lambda_1\lambda_3\cos\left(x\right)-2\left(\lambda-\mu+\eta\right)\lambda_1\lambda_2\sin\left(x\right),
$$

which recommends the solution of Eq. (4.1) can be written as

$$
u\left(x,t\right) = \lambda_1(t)+\lambda_2(t)\cos\left(x\right)+\lambda_3(t)\sin\left(x\right). \tag{4.2}
$$

By inserting (4.2) in (4.1) and through some operations, we find

$$
\begin{aligned}
&{}_tT_\alpha\left(\lambda_1(t)\right)=0,\\
&{}_tT_\alpha\left(\lambda_2(t)\right)=\gamma\lambda_1(t)\lambda_3(t),\\
&{}_tT_\alpha\left(\lambda_3(t)\right)=-\gamma\lambda_1(t)\lambda_2(t),
\end{aligned}
$$

in which $\gamma = 2(\lambda - \mu + \eta)$. Solving the equation ${}_tT_\alpha\left(\lambda_1(t)\right) = 0$ results in $\lambda_1(t) = d_0$ and thus

$$
\begin{aligned}
&{}_tT_\alpha\left(\lambda_2(t)\right)=\gamma d_0\lambda_3(t),\\
&{}_tT_\alpha\left(\lambda_3(t)\right)=-\gamma d_0\lambda_2(t).
\end{aligned}
\tag{4.3}
$$

By differentiating the first equation, we acquire

$$
{}_tT_\alpha({}_tT_\alpha\left(\lambda_2(t)\right))=\gamma d_0{}_tT_\alpha(\lambda_3(t)).
$$

Now, it is obvious that above equation can be presented as

$$
{}_tT_\alpha({}_tT_\alpha\left(\lambda_2(t)\right))=-\gamma^2 d_0^2\lambda_2(t).
$$

Using the CFST, yields

$$
S_\alpha\left[{}_tT_\alpha\left({}_tT_\alpha(\lambda_2(t))\right);u\right]=-\gamma^2 d_0^2 S_\alpha\left[\lambda_2(t);u\right],
$$

and therefore

$$
\frac{S_\alpha\left[{}_tT_\alpha\left(\lambda_2(t)\right);u\right]}{u^\alpha}-\frac{{}_tT_\alpha\left(\lambda_2(0)\right)}{u^\alpha}=-\gamma^2 d_0^2 S_\alpha\left[\lambda_2(t);u\right],
$$

where ${}_tT_\alpha\left(\lambda_2(0)\right) = \gamma d_0 d_2 \left(\lambda_3(0) = d_2\right)$. In a similar manner, the equation

$$
\frac{S_\alpha\left[{}_tT_\alpha\left(\lambda_2(t)\right);u\right]}{u^\alpha}=\frac{\gamma d_0 d_2}{u^\alpha}-\gamma^2 d_0^2 S_\alpha\left[\lambda_2(t);u\right],
$$

can be written as

$$
\frac{S_\alpha\left[\lambda_2(t);u\right]-\underbrace{\lambda_2\left(0\right)}_{d_1}}{u^{2\alpha}}=\frac{\gamma d_0 d_2}{u^\alpha}-\gamma^2 d_0^2 S_\alpha\left[\lambda_2(t);u\right].
$$

It is clear that

$$
S_\alpha\left[\lambda_2(t);u\right]=\frac{\gamma d_0 d_2 u^\alpha}{1+\left(\gamma d_0 u^\alpha\right)^2}+\frac{d_1}{1+\left(\gamma d_0 u^\alpha\right)^2}.
$$

Now, by means of the inverse CFST, we retrieve

$$
\lambda_2\left(t\right)=d_1\cos\left(\gamma d_0\frac{t^\alpha}{\alpha}\right)+d_2\sin\left(\gamma d_0\frac{t^\alpha}{\alpha}\right).
\tag{4.4}
$$

Setting (4.4) in (4.3) leads to

$$
{}_tT_\alpha\left(\lambda_3(t)\right)=-\gamma d_0\left(d_1\cos\left(\gamma d_0\frac{t^\alpha}{\alpha}\right)+d_2\sin\left(\gamma d_0\frac{t^\alpha}{\alpha}\right)\right).
$$

Using the CFST, results in

$$
S_\alpha\left[{}_tT_\alpha\left(\lambda_3(t)\right);u\right]=-\gamma d_0 S_\alpha\left[d_1\cos\left(\gamma d_0\frac{t^\alpha}{\alpha}\right)+d_2\sin\left(\gamma d_0\frac{t^\alpha}{\alpha}\right);u\right],
$$

and consequently

$$\frac{S_\alpha\left[\lambda_3(t);u\right]-\underbrace{\lambda_3(0)}_{d_2}}{u^\alpha}=-\frac{\gamma d_0 d_1}{1+\left(\gamma d_0 u^\alpha\right)^2}-\frac{\gamma^2 d_0^2 d_2 u^\alpha}{1+\left(\gamma d_0 u^\alpha\right)^2}.$$

It is obvious that

$$S_\alpha\left[\lambda_3(t);u\right]=-\frac{\gamma d_0 d_1 u^\alpha}{1+\left(\gamma d_0 u^\alpha\right)^2}+\frac{d_2}{1+\left(\gamma d_0 u^\alpha\right)^2}.$$

Now, by means of the inverse CFST, we gain

$$\lambda_3\left(t\right)=-d_1\sin\left(\gamma d_0\frac{t^\alpha}{\alpha}\right)+d_2\cos\left(\gamma d_0\frac{t^\alpha}{\alpha}\right).$$

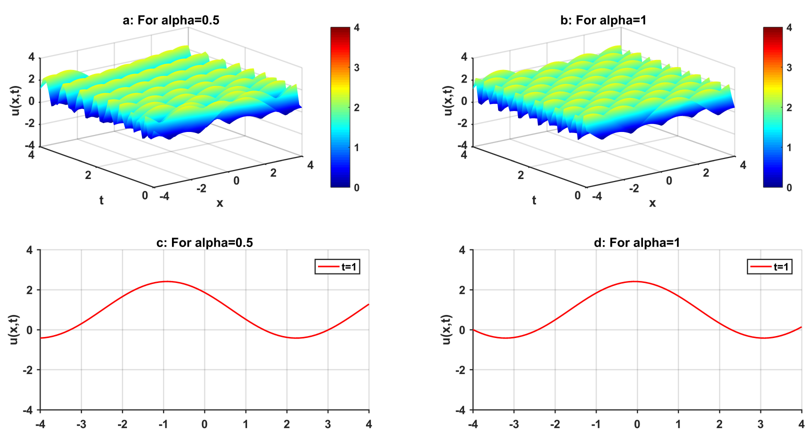
Hence, the following exact solution to the nonlinear conformable time-fractional dispersive equation of the fifth-order is acquired

$$\begin{aligned}u\left(x,t\right)=d_0&+\left(d_1\cos\left(\gamma d_0\frac{t^\alpha}{\alpha}\right)+d_2\sin\left(\gamma d_0\frac{t^\alpha}{\alpha}\right)\right)\cos\left(x\right)\\&+\left(-d_1\sin\left(\gamma d_0\frac{t^\alpha}{\alpha}\right)+d_2\cos\left(\gamma d_0\frac{t^\alpha}{\alpha}\right)\right)\sin\left(x\right),\end{aligned}$$

which is the corrected form of the solution reported in [18].

The new exact solution derived through the present hybrid scheme has been plotted for different values of α in the Figure 1.

FIGURE 1. The new exact solution derived through the present hybrid scheme for different values of α when $\lambda = 1$, $\mu = 2$, $\eta = -8$ and $d_0 = d_1 = d_2 = 1$.



Note: Although there is another kind of exact solutions to a wide range of $(2+1)$-dimensional differential equations called the lump solutions [13, 36, 37], it should be mentioned that extracting such solutions to (2+1)-dimensional fractional differential equations through the presented method is not applicable.

5. Conclusion

A nonlinear time-fractional dispersive equation of the fifth-order with the conformable derivative was analytically solved in the current work. The invariant subspace method along with the fractional Sumudu's transform in the conformable context has been applied for the first time successfully to handle the intended aim. The present study reveals that the current hybrid method provides a new and effective systematic technique to deal with the conformable time-fractional differential equations in mathematical physics. It is worth noting that the validity of the reported result was checked by substituting it into the Eq. (4.1).

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