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Dynamics of a predator-prey system with prey refuge

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Abstract In this paper, we investigate the dynamical complexities of a prey predator model prey refuge providing additional food to predator. We determine dynamical behaviors of the equilibria of this system and characterize codimension 1 and codimension 2 bifurcations of the system analytically. Hopf bifurcation conditions are derived analytically. We especially approximate a family of limit cycles emanating from a Hopf point. The analytical results are in well agreement with the numerical simulation results. Our bifurcation analysis indicates that the system exhibits numerous types of bifurcation phenomena, including Hopf, and Bogdanov-Takens bifurcations.

Keywords. Hopf bifurcation, Bogdanov-Takens bifurcation, Dynamical behavior, Limit cycles.

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1. Introduction

The fundamental goals of theoretical ecology are to understand the interactions among individual organisms and interactions with the environment to determine the distribution of populations and the structure of communities [4]. Mathematical modelling in ecology has received great attention since the pioneer work of Lotka [19] and Volterra [31]. The first Mathematical model for prey-predator was developed independently by Aefred James Lotka [19] and Vito Volterra [31]. Motivated by the work of Lotka [19] and Volterra [31] many food chain models are formulated and analyzed [10, 24, 36]. The dynamic predation behavior of a predator in an ecological system depends strongly on functional response. The most common functional response in a prey-predator system is the famous Holling type-II and modified of Holling type-II response. It is important to note that predators functional response approaches to a constant value as the prey population increases. Prey refugia are areas occupied by

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prey which potentially minimize their rate of encounter with predators [29]. For an example, in a wolf-ungulate system, ungulates may seek refuge by migrating to areas outside the core territories of wolves (migration) or survive longer outside the wolves a core use areas (mortality)[23]. Prey refuge has many potential impacts on prey-predator dynamics, including promoting stability, creating dynamic fluctuations or producing instability in an ecosystem [17, 26, 27]. The presence of refuge in attacking rate of predator may cause the extinction of predator in prey-predator system. The prey-predator interaction often exhibits spatial refugee which afford the prey some degree of protection from predation and reduces the chance of extinction due to predation. Therefore, the existence of prey refuges can have important effects on the coexistence of prey and predator population. According to Taylor [29], refuges are of three following types: (a) provide permanent spatial protection for a small subset of the prey population, (b) provide temporary spatial protection, and (c) provide a temporal refuge in numbers by decrease the risk of predation by increasing the abundance of vulnerable prey. Understanding the effects of prey refuge, we introduce a refuge parameter c' into the attacking rate as $c(1-c')$. Incorporating the term of prey refuge, the functional response becomes $c' \in [0,1]$. Incorporating the term of prey refuge, the functional response is $f(N) = \frac{c(1-c')e_1N}{a+h_1e_1N}$, where N is total number of prey individuals. The term c represents the attack rate of predator on prey and the parameter a represents the half saturation constant. Now, if h_1 and e_1 are two constants representing handling time of the predator per prey item and ability of the predator to detect the prey respectively. The additional food can reduce the predation pressure on prey population. This additional food is an important component of most predators diet, although they receive less attention than basal prey. In scientific literature, these foods are fundamentally shape the life histories of many predator species. The role of alternative prey (additional food) in sustaining predator populations has been reported in laboratory studies and as well as in theoretical studies [12, 13, 30]. [28] reported the dynamics of prey-predator system in presence of additional food for predator and discussed the effect of quality and quantity of additional food. The functional response in presence of additional food for predator takes the form $f(N) = \frac{c(1-c')e_1N}{a+h_2e_2A+h_1e_1N}$ where h_2 be the handling time of additional food biomass (A) and e_2 represents the ability to detect the additional food. We now propose a prey-predator model with logistic growth rate and prey refuge in presence of additional food for predator. The model takes the following form:

$$\begin{cases} \dfrac{dN}{dT} = rN(1-\dfrac{N}{K}) - \dfrac{c(1-c')e_1NP}{a+h_2e_2A+h_1e_1N}, \\ \dfrac{dP}{dT} = \dfrac{b[(1-c')e_1N+e_2A]P}{a+h_2e_2A+h_1e_1N} - mP, \end{cases} \tag{1.1}$$

where N, P denote the biomass of prey and predator respectively. The initial population level is taken as $N(0) \geq 0$, $P(0) \geq 0$. The parameters r, K respectively represent the intrinsic growth rate and environmental carrying capacity of the prey. Parameter b denotes the conversion efficiency of prey into predator and m represents death rate of predator. Defining $c_1 = \frac{c}{h_1}$, $\alpha = \frac{h_2}{h_1}$, $\eta = \frac{e_2}{e_1}$, $b_1 = \frac{b}{h_1}$, $a_1 \frac{a}{e_1h_1}$, system (1.1) can be written as:

$$\begin{cases} \dfrac{dN}{dt} = rN(1-\dfrac{N}{K}) - \dfrac{c_1(1-c')NP}{a_1+\alpha\eta A} + N, \\ \dfrac{dP}{dt} = \dfrac{b_1[(1-c')N+\eta A]P}{a_1+\alpha\eta A+N} - mP, \end{cases} \tag{1.2}$$

To reduce the number of parameters, we non-dimensionalized the system (1.2) using following transformations $x = \frac{N}{a_1}$, $y = \frac{c_1 P}{r a_1}$, $t = rT$. Then system (1.2) reduces to the following form:

$$\begin{cases} \dfrac{dx}{dt} = x(1-\dfrac{x}{\gamma}) - \dfrac{(1-c')xy}{1+\alpha\xi+x}, \\ \dfrac{dy}{dt} = \dfrac{\beta[(1-c')x+\xi]y}{1+\alpha\xi+x} - \delta y, \end{cases} \tag{1.3}$$

where, $\gamma = \frac{K}{a_1}$, $\alpha = \frac{h_2}{h_1}$, $\xi = \frac{\eta A}{a_1}$, $\beta = \frac{b_1}{r}$, $\delta = \frac{m}{r}$. The terms α and ξ are the parameters which respectively characterize as quality and quantity of additional food. The term $\alpha = \frac{h_2}{h_1}$ is directly proportional to the handling time of the additional food and is known as quality of the additional food. The relation $\xi = \frac{\eta A}{a_1}$ represents that ξ is directly proportional to the biomass of additional food (A) and thus ξ is a representative of quantity of additional food which is supplied to predator [28]. In particular, the model (1.3) reduces to the model of Srinivasu et al. [28] in absence of prey refuge $c' = 0$. This system first was studied in [32], in which some local bifurcations were studied analytically. The proposed models usually depend on some parameters and are studied by bifurcation method. The objective is to maximize the monetary social benefit as well as conservation of the ecosystem. More and more complex bifurcation phenomena are discovered and studied in predator-prey models, see, for example, Hopf bifurcations of codimension 1 in [15, 16, 34], cusp bifurcation of codimension 2 in [14, 25, 34, 35], Bogdanov-Takens bifurcation of codimension 3 (cusp case) in [17, 40], Bogdanov-Takens bifurcation of codimension 3 (focus case) in [35], Bogdanov-Takens bifurcation of codimension 3 (saddle case) in [7], Hopf bifurcation of codimension 2 in [14, 17, 36], etc. In this work we will use the term complexity to describe the ecological complexity found in nature as well as the of dynamical complexity of the models. In recent years, a significant number of the published papers on the mathematical Continuous and discrete time models of biology, physics, engineering,... discussed the systems of differential equations and the associated numerical methods. Mathematical models on prey predator systems create a major interest during the last few decades. Study of such system with discrete models and continuous models can be found in [2, 5, 8, 15, 16, 18, 33, 37, 38, 39]. The main aim of this paper is to study the pattern of bifurcation that takes place as we vary some of the model parameters. We specially focus on the biological implications of the found bifurcations. Most importantly we show that the Hopf bifurcation plays, for various reasons, a crucial role. Ecological systems are complex because of the diversity of biological species as well as the complex nature of their interactions. We study the existence of a supercritical Hopf bifurcation in a small neighbourhood of E_* and a Bogdanov-Takens bifurcation in a small neighbourhood of E_* in system

(1.3). In this paper, we rely heavily on advanced continuation and bifurcation techniques implemented in the software package MATCONT [1, 6, 9] to obtain results that cannot be obtained analytically. MATCONT is a dynamical toolbox based on numerical continuation technique which is a well-understood subject [1, 3, 11]. This software computes a solution curve of equation $F(x) = 0$ for the system of the form $\frac{dx}{dt} = f(x, \alpha)$with $x \in R^n, f(x, \alpha) \in R^n$ and a vector of parameters where equilibria, limit points, limit cycles, etcetera can be computed. MATCONT is compatible with the standard MATLAB ODE of differential equations. General descriptions, functionalities and the dynamical algorithms implemented in MATCONT can be found in [1, 6, 9]. Numerical bifurcation analysis techniques are very powerful and efficient in physics, biology, engineering, and economics [21, 22, 33].

This paper is organized as follows: In section 2, we consider the mathematical model and discussed some basic dynamical results and existence of equilibrium and stability of equilibrium. In section 3, Hopf bifurcation and Bogdanov-Takens bifurcation of the interior equilibrium point of the model system is discussed. Numerical simulation results are included to support our analytical results in section 4. The paper concludes with a brief discussion and we summarize our results in section 5.

2. Equilibria and basic dynamical results

The system (1.3) has trivial equilibrium point $E_0 = (0, 0)$, axial equilibrium point $E_1 = (\gamma, 0)$ and co-existing equilibrium point $E_* = (x_*, y_*)$ where $x_* = \frac{\delta+(\alpha\delta-\beta)\xi}{\beta(1-c')-\delta}$ and $y_* = (\frac{1+\alpha\xi+x_*}{(1-c')})(1 - \frac{x_*}{\gamma})$. The Jacobian matrix of system (1.3) at (x, y) takes the form:

$$J = \begin{pmatrix} 1 - \frac{2x}{\gamma} - \frac{(1-c')(1+\alpha\xi)y}{(1+\alpha\xi+x)^2} & -\frac{(1-c')x}{1+\alpha\xi+x} \\ \frac{\beta y[(1-c')(1+\alpha\xi)-\xi]}{(1+\alpha\xi+x)^2} & \frac{\beta[(1-c')x+\xi]}{1+\alpha\xi+x} - \delta \end{pmatrix}.$$

At trivial equilibrium point $(0, 0)$, the Jacobian matrix is given by

$$J = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\beta\xi}{1+\alpha\xi} - \delta \end{pmatrix}.$$

It has always one positive eigenvalue 1 and other eigenvalue $\frac{\beta\xi}{1+\alpha\xi} - \delta$.

Lemma 2.1. *The equilibrium $E_0 = (0, 0)$ of system (1.3) is always a saddle point if $\frac{\beta\xi}{1+\alpha\xi} < \delta$, and the equilibrium E_0 is unstable if $\frac{\beta\xi}{1+\alpha\xi} > \delta$.*

At the axial equilibrium point $(\gamma, 0)$, the Jacobian matrix is given by

$$J = \begin{pmatrix} -1 & -\frac{(1-c')\gamma}{1+\alpha\xi+\gamma} \\ 0 & \frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma} - \delta \end{pmatrix}.$$

One negative eigenvalue is -1 and other eigenvalue is $\frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma} - \delta$.

Lemma 2.2. *Equilibrium point $E_\gamma = (\gamma, 0)$ is saddle for $\frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma} > \delta$ and stable for $\frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma} < \delta$.*

At the co-existing equilibrium point $(\frac{\delta+(\alpha\delta-\beta)\xi}{\beta(1-c')-\delta}, (\frac{1+\alpha\xi+x_*}{(1-c')})(1-\frac{x_*}{\gamma}))$, the Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{x_*}{\gamma}(\frac{\gamma-x_*}{1+\alpha\xi+x_*}-1) & -\frac{(1-c')x_*}{1+\alpha\xi+x_*} \\ \frac{\beta y_*[(1-c')(1+\alpha\xi)-\xi]}{(1+\alpha\xi+x_*)^2} & 0 \end{pmatrix}.$$

Theorem 2.3. *When $1 > c'$ and $\frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma} = \delta$, the degenerate equilibrium point $E_\gamma = (\gamma, 0)$ of system (1.3) is a saddle-node.*

Proof. Clearly, when $\frac{\beta[(1-c')\gamma+\xi]}{1+\alpha\xi+\gamma} = \delta$, at the equilibrium $E_\gamma = (\gamma, 0)$, the Jacobian matrix has a zero eigenvalue. We translate the equilibrium point $E_\gamma = (\gamma, 0)$ at the origin by means of the translation $(u, v) = (x-\gamma, y)$. Then the system (1.3) becomes:

$$\begin{cases} \dfrac{du}{dt} = r(u+\gamma)(1-\dfrac{u+\gamma}{\gamma}) - \dfrac{(1-c')(u+\gamma)v}{1+\alpha\xi+u+\gamma}, \\ \dfrac{dv}{dt} = \dfrac{\beta[(1-c')(u+\gamma)+\xi]v}{1+\alpha\xi+u+\gamma} - \delta v. \end{cases} \tag{2.1}$$

The Jacobian matrix of the system (2.1) is diagonalizable with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 0$ and the corresponding eigenvectors $\Omega_1 = (1, 0)^T$ and $\Omega_2 = (-\frac{(1-c'\gamma)}{1+\alpha\xi+\gamma}, 1)^T$. Thus, with the change of coordinates:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -\frac{(1-c'\gamma)}{1+\alpha\xi+\gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

system (2.1) reduces to:

$$\begin{cases} \dfrac{dX}{dt} = -X + O_2(X, Y), \\ \dfrac{dY}{dt} = -\delta Y^2 + O_3(X, Y), \end{cases} \tag{2.2}$$

where $O_2(X, Y)$ and $O_3(X, Y)$ begin with second and third order terms in X and Y, respectively. If a change of coordinates is performed to bring the system onto the center manifold, the term$-\delta Y^2$ does not change, then the equilibrium point $E_\gamma = (\gamma, 0)$ is a saddle-nod singularity. □

3. Bifurcations

The critical parameter value at which qualitative change of dynamics occur is called bifurcation point. Qualitatively different dynamical behavior may appear in the model with the variation of model parameters. To identify the possible qualitatively different dynamical behavior with the variation of parameters c', β, ξ, δ, α we do bifurcation analysis of the system (1.3) with respect to c', β, ξ, δ, α.

3.1. The Bogdanov-Takens (or double zero) bifurcation analysis.

Theorem 3.1. *If we choose δ and c' as bifurcation parameters, then system (1.3) undergoes Bogdanov-Takens bifurcation in a small neighborhood of E_* (see Figure 16).*

Proof. Consider the following system

$$\begin{cases} \dfrac{dx}{dt} = x(1-\dfrac{x}{\gamma}) - \dfrac{(1-(c'+\lambda_1))xy}{1+\alpha\xi+x} = f(x,y), \\ \dfrac{dy}{dt} = \dfrac{\beta[(1-c')x+\xi]y}{1+\alpha\xi+x} - (\delta+\lambda_2)y = g(x,y), \end{cases} \tag{3.1}$$

where (λ_1, λ_2) is a parameter vector in a small neighbourhood of $(0,0)$. In this case, with the help of the transformation $x = x_1 + x_*$, $y = y_1 + y_*$, $c' = c'_* + \lambda_1$ and $\delta = \delta_* + \lambda_2$. System (3.1) can be written as

$$\begin{cases} \dfrac{dx_1}{dt} = p_0(\lambda) + a_1(\lambda)x_1 + b_1(\lambda)x_2 + p_{11}x_1^2 + p_{12}(\lambda)x_1x_2 + p_{22}(\lambda)x_2^2 + O(\|x\|^3), \\ \dfrac{dx_2}{dt} = q_0(\lambda) + c_1(\lambda)x_1 + d_1(\lambda)x_2 + q_{11}(\lambda)x_1^2 + q_{12}(\lambda)x_1x_2 + q_{22}(\lambda)x_2^2 + O(\|x\|^3), \end{cases} \tag{3.2}$$

where $a_1(\lambda) = \frac{\partial f}{\partial x}|_{(x_*,y_*)}$, $b_1(\lambda) = \frac{\partial f}{\partial y}|_{(x_*,y_*)}$, $c_1(\lambda) = \frac{\partial g}{\partial x}|_{(x_*,y_*)}$, $d_1(\lambda) = \frac{\partial g}{\partial x}|_{(x_*,y_*)}$, $p_{11}(\lambda) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}|_{(x_*,y_*)}$, $p_{12}(\lambda) = \frac{\partial^2 f}{\partial x \partial y}|_{(x_*,y_*)}$, $p_{22}(\lambda) = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}|_{(x_*,y_*)}$, $q_{11}(\lambda) = \frac{1}{2}\frac{\partial^2 g}{\partial x^2}|_{(x_*,y_*)}$, $q_{12}(\lambda) = \frac{\partial^2 g}{\partial x \partial y}|_{(x_*,y_*)}$, $q_{22}(\lambda)\frac{1}{2}\frac{\partial^2 f}{\partial y^2}|_{(x_*,y_*)}$. We then have,
$p_0(\lambda) = \lambda_1 x_* y_*$, $q_0(\lambda) = -\lambda_2 y_*$, $a_1 = \frac{x_*}{\gamma}(\frac{\gamma - x_*}{1+\alpha\xi+x_*} - 1)$, $b_1 = -\frac{(1-c')x_*}{1+\alpha\xi+x_*}$,
$c_1 = \frac{\beta y_*[(1-c')(1+\alpha\xi)-\xi]}{(1+\alpha\xi+x_*)^2}$, $d_1 = 0$, $p_{11} = \frac{\gamma - x_*}{2\gamma(1+\alpha\xi+x_*)} - \frac{1+\alpha\xi+\gamma}{2(1+\alpha\xi+x_*)^2} - \frac{1}{2\gamma}$,
$p_{12} = \frac{(1-c')(1+\alpha\xi+2x_*)}{(1+\alpha\xi+x_*)^2}$, $p_{22} = 0$, $q_{11} = -\frac{\beta y_*}{(1+\alpha\xi+x_*)^3}$, $q_{12} = 0$, $q_{12} = 0$.

Making the affine transformation

$$y_1 = x_1, \ y_2 = a_1x_1 + b_1x_2.$$

We have

$$\begin{cases} \dfrac{dy_1}{dt} = p_0(\lambda) + y_2 + \alpha_{11}(\lambda)y_1^2 + \alpha_{12}(\lambda)y_1y_2 + \alpha_{22}(\lambda)y_2^2 + O(\|y\|), \\ \dfrac{dy_2}{dt} = q\prime_0(\lambda) + c_2(\lambda)y_1 + d_2(\lambda)y_2 + \beta_{11}(\lambda)y_1^2 + \beta_{12}y_1y_2 + \beta_{22}y_2^2 + O(\|y\|), \end{cases} \tag{3.3}$$

where
$q_0\prime(\lambda) = p_0a_1 + b_1q_0$, $c_2 = b_1c_1 - a_1d_1$, $d_2 = a_1 + d_1$,
$\alpha_{11} = \frac{p_{22}a_1^2}{b_1} - \frac{p_{12}a_1}{b_1} + p_{11}$, $\alpha_{12} = -\frac{2p_{22}a_1}{b_1^2} + \frac{p_{12}}{b_2}$, $\alpha_{22} = \frac{p_{22}}{b_1^2}$,
$\beta_{11} = b_1q_{11} + a_2(p_{11} - q_{12}) - \frac{a_1^2(p_{12}-q_{22})}{b_1} + \frac{p_{22}a_1^3}{b_1^2}$,
$\beta_{12} = -(2\frac{p_{22}a_1^2}{b_1^2} - \frac{a_1(p_{12}-q_{22})}{b_1} - q_{12})$, $\beta_{22} = \frac{p_{22}a_1}{b_1^2} + \frac{q_{22}}{b_2}$.

The functions $q_0\prime(\lambda)$, α_{kl}, β_{kl}, c_2, d_2 are smooth functions of λ. We have $q\prime_0(\lambda^*)$=0. Now, we consider the following transformation

$$z_1 = y_1,\ z_2 = p_0(\lambda) + y_2 + \alpha_{11}(\lambda)y_1^2 + \alpha_{12}(\lambda)y_1y_2 + \alpha_{22}y_2^2 + O(\|y\|).$$

This transformation brings (3.3) into the following

$$\begin{cases} \dfrac{dz_1}{dt} = z_2, \\ \dfrac{dz_2}{dt} = g_{00}(\lambda) + g_{10}(\lambda)z_1 + g_{01}(\lambda)z_2 + g_{20}(\lambda)z_1^2 + g_{11}(\lambda)z_1z_2 + g_{02}(\lambda)z_2^2 + O(\|z\|^3), \end{cases} \tag{3.4}$$

where $g_{00}(0) = 0$, and $z = (z_1, z_2)$. Furthermore, we also have
$g_{00}(\lambda) = q_0\prime(\lambda) - p_0(\lambda)d_2(\lambda) + \ldots,$
$g_{10}(\lambda) = c_2(\lambda) + \alpha_{12}(\lambda)q_0\prime(\lambda) - \beta_{12}(\lambda)p_0(\lambda) + \ldots,$
$g_{01}(\lambda) = d_2(\lambda) + 2\alpha_{22}(\lambda)q_0\prime(\lambda) - \alpha_{12}(\lambda)p_0(\lambda) - 2\beta_{22}(\lambda)p_0(\lambda) + \ldots,$
$g_{20}(\lambda) = \beta_{11}(\lambda) - \alpha_{11}(\lambda)d_2(\lambda) + c_2(\lambda)\alpha_{12}(\lambda) + \ldots,$
$g_{02}(\lambda) = \alpha_{12}(\lambda) + \beta_{22}(\lambda) - \alpha_{22}(\lambda)d_1(\lambda) + \ldots,$
$g_{11}(\lambda) = \beta_{12}(\lambda) + 2\alpha_{11}(\lambda) + 2\alpha_{22}(\lambda)c_2(\lambda) - \alpha_{12}(\lambda)d_2(\lambda) + \ldots.$

Correspondingly,

$$g_{00}(\lambda^*) = 0,\ g_{10}(\lambda^*) = c_2(\lambda^*),\ g_{01}(\lambda^*) = d_2(\lambda^*),$$
$$g_{20}(\lambda^*) \neq 0,\ g_{02}(\lambda^*) \neq 0,\ g_{11}(\lambda^*) \neq 0.$$

Again, we can write (3.4) as of the following form:

$$\begin{cases} \dfrac{dz_1}{dt} = z_2, \\ \dfrac{dz_2}{dt} = (g_{00}(\lambda) + g_{10}(\lambda)z_1 + g_{20}(\lambda)z_1^2 + O(\|z\|^3)) \\ +(g_{01}(\lambda)z_2 + (g_{11}(\lambda)z_1 + O(\|z\|^2))z_2 + (g_{02}(\lambda) + O(\|z\|))z_2^2 \\ = \mu(z_1, \lambda) + \nu(z_1, \lambda)z_2 + \Phi(z, \lambda)z_2^2, \end{cases} \tag{3.5}$$

where μ, ν, Φ are smooth functions and satisfy the following

$$\mu(0, \lambda^*) = g_{00}(\lambda^*) = 0,\ \frac{\partial\mu}{\partial z_1}|_{(0,\lambda^*)} = g_{10}(\lambda^*) = c_2(\lambda^*) = \rho_1 \neq 0,$$
$$\nu(0, \lambda^*) = g_{01}(\lambda^*) = \rho_2 \neq 0.$$

There exists a C^∞ function z_1 defined in a small neighbourhood of $\lambda = \lambda^*$ such that $\phi(\lambda^*) = 0$, $\nu(\phi, \lambda) = 0$ for any $\lambda \in N(\lambda^*)$

$$z_1 = u_1 + \phi(\lambda),\ z_2 = u_2.$$

The above transformation brings the system (3.5) to the following system

$$\begin{cases} \dfrac{du_1}{dt} = u_2, \\ \dfrac{du_2}{dt} = (h_{00}(\lambda) + h_{10}(\lambda)u_1 + h_{20}(\lambda)u_1^2 + O(\|u_1\|^3)) \\ +(h_{01}(\lambda)u_2 + (h_{11}(\lambda)u_1 + O(\|u\|^2))u_2 + (h_{02}(\lambda) + O(\|u\|))u_2^2 \\ = \overline{\mu}(u_1,\lambda) + \overline{\nu}(u_1,\lambda)u_2 + \overline{\Phi}(u,\lambda)u_2^2, \end{cases} \tag{3.6}$$

where $u = (u_1, u_2)$,

$$h_{00} = g_{00} + g_{10}\phi + ..., \; h_{10} = g_{10} + 2g_{20}\phi + ...,$$
$$h_{20} = g_{20} + ..., \; h_{01} = g_{01} + g_{11}\phi + ...,$$
$$h_{11} = g_{11} + ..., \; h_{02} = g_{02} + ...$$

The coefficient of u_2 term on the RHS of the second equations of (3.6) is given by

$$h_{01} = \overline{\nu}(0,\lambda) = g_{01} + g_{11}\phi + O(\|\phi\|^2) = [d_2 + 2\alpha_{22}q_0\prime - \alpha_{12}p_0 - 2\beta_{22}p_0 + ...] \\ +[\beta_{12} + 2\alpha_{11} + \alpha_{22}c_2 - \alpha_{12}d_2 + ...]\phi.$$

Thus we have the following

$$h_{01}(0,\lambda) = g_{01}(\lambda^*) \neq 0.$$

Let for $\lambda \in N(\lambda^*)$, $\phi(\lambda) \in M$. Then in the region$M$, $\phi(\lambda)$ can be approximated by

$$\phi(\lambda) \approx -\frac{g_{01}(\lambda)}{\rho_2}.$$

Thus, (3.6) reduces to the following

$$\begin{cases} \dfrac{du_1}{dt} = u_2, \\ \dfrac{du_2}{dt} = h_{00}(\lambda) + h_{10}(\lambda)u_1 + h_{20}(\lambda)u_1^2 + h_{11}(\lambda)u_1u_2 + h_{02}(\lambda)u_2^2 + O(\|u\|)^3. \end{cases} \tag{3.7}$$

We now introduce a new time scale, defined by $dt = (1 + \psi u_1)d\tau$, where $\psi = \psi(\lambda)$ is a smooth function to be defined later. With this transformation, (3.7) reduces to

$$\begin{cases} \dfrac{du_1}{d\tau} = u_2(1 + \psi u_1), \\ \dfrac{du_2}{d\tau} = h_{00} + (h_{10} + h_{00}\psi)u_1 + (h_{20} + h_{10}\psi)u_1^2 + h_{11}u_1u_2 + h_{02}u_2^2 + O(\|u\|)^3. \end{cases} \tag{3.8}$$

Assume

$$\nu_1 = u_1, \; \nu_2 = u_2(1 + \psi u_1).$$

Then we obtain,

$$\begin{cases} \dfrac{d\nu_1}{d\tau} = \nu_2, \\ \dfrac{d\nu_2}{d\tau} = l_{00}(\lambda) + l_{10}(\lambda)\nu_1 + l_{20}(\lambda)\nu_1^2 + l_{11}(\lambda)\nu_1\nu_2 + l_{02}(\lambda)\nu_2^2 + O(\|\nu\|)^3, \end{cases} \tag{3.9}$$

where

$$\begin{aligned} l_{00}(\lambda) &= h_{00},\ l_{10}(\lambda) = h_{10} + 2h_{00}\psi(\lambda), \\ l_{20} &= h_{20} + 2h_{00}\psi(\lambda) + h_{00}(\lambda)\psi(\lambda)^2, \\ l_{11}(\lambda) &= h_{11}(\lambda),\ l_{02}(\lambda) = h_{02} + \psi(\lambda). \end{aligned}$$

Now, we take $\psi(\lambda) = -h_{02}(\lambda)$ in order to get rid of ν_2^2-term. Then we have

$$\begin{cases} \dfrac{d\nu_1}{d\tau} = \nu_2, \\ \dfrac{d\nu_2}{d\tau} = \beta_1(\lambda) + \beta_2(\lambda)\nu_1 + \eta(\lambda)\nu_1^2 + \zeta(\lambda)\nu_1\nu_2 + O(\|\nu\|)^3, \end{cases} \tag{3.10}$$

where $v = (v_1, v_2)$,

$$\begin{aligned} \beta_1(\lambda) = h_{00}(\lambda),\ \beta_2(\lambda) &= h_{10}(\lambda) - 2h_{00}(\lambda)h_{02}(\lambda), \\ \eta(\lambda) = h_{20}(\lambda) - 2h_{10}(\lambda)h_{02}(\lambda) &+ h_{02}^2(\lambda)h_{00}(\lambda) \neq 0, \\ \zeta(\lambda) &= h_{11}(\lambda) \neq 0. \end{aligned}$$

Now we introduce a new time scale given by

$$t = |\frac{\eta(\lambda)}{\zeta(\lambda)}|\tau.$$

With the new stable variables $\xi_1 = \frac{\eta(\lambda)}{\zeta^2(\lambda)}\nu_1$ and $\xi_2 = \frac{\eta^2(\lambda)}{\zeta^3(\lambda)}\nu_2$ such that $s = sign\frac{\eta(\lambda)}{\zeta(\lambda)} = sign\frac{\eta(\lambda^*)}{\zeta(\lambda^*)} = \frac{\rho_2}{g_{20}(\lambda^*)} = \pm 1$. This yields (3.10) into the form

$$\begin{cases} \dfrac{d\xi_1}{d\tau} = \xi_2, \\ \dfrac{d\xi_2}{d\tau} = \mu_1 + \mu_2\xi_1 + \xi_1^2 + s\xi_1\xi_2 + O(\|\xi\|)^3, \end{cases} \tag{3.11}$$

where

$$\mu_1(\lambda) = \frac{\eta(\lambda)}{\zeta^2(\lambda)}\beta_1(\lambda),\ \mu_2(\lambda) = \frac{\eta(\lambda)}{\zeta^2(\lambda)}\beta_2(\lambda).$$

The system (3.11) is locally topologically equivalent near the origin for small $\|\mu\|$ to the system

$$\begin{cases} \dfrac{d\xi_1}{d\tau} = \xi_2, \\ \dfrac{d\xi_2}{d\tau} = \mu_1 + \mu_2\xi_1 + \xi_1^2 + s\xi_1\xi_2, \end{cases} \tag{3.12}$$

where $s = \pm 1$. We have obtained the generic normal from of the Bogdanov-Takens bifurcation for the system (3.12)

$$rank(\frac{\partial(\mu_1, \mu_2)}{\partial \lambda})_{\lambda=\lambda^*} = 2,$$

$$J = \begin{vmatrix} \frac{\partial \mu_1}{\partial \lambda_2} & \frac{\partial \mu_1}{\partial \lambda_1} \\ \frac{\partial \mu_2}{\partial \lambda_2} & \frac{\partial \mu_2}{\partial \lambda_1} \end{vmatrix} \neq 0.$$

□

3.2. **Hopf bifurcation and local stability.** We study the existence of a Hopf bifurcation and stability in a small neighborhood of E_* when parameters α, β, c' vary. In this section, we consider system (1.3). It is easy to see that the determinant of $J|_{E_*}$ is positive if $c' < 1$.

Theorem 3.2. *Define* $c'_* = 1 - \frac{\delta(\gamma+2)+2\xi(\alpha\delta-\beta)}{\gamma\beta-\beta(1+\alpha\xi)}$ *then the following statements hold:*
(a) *If* $c' > c'_*$*, the positive equilibrium* E_* *is locally asymptotically stable.*
(b) *If* $c' < c'_*$*, the positive equilibrium* E_* *is unstable.*
(c) *If* $c' = c'_*$*, then a Hopf bifurcation occurs around the positive equilibrium* E_**.*

Proof. (a), (b) The characteristic equation is given by

$$\lambda^2 - tr(J)\lambda + det(J) = 0,$$

with

$$det(J|_{E_*}) = \frac{\beta(1-c')[(1-c')(1+\alpha\xi) - \xi]x_* y_*}{(1+\alpha\xi + x_*)^3} > 0,$$

and

$$tr(J) = \frac{x_*}{\gamma}(\frac{\gamma - x_*}{1+\alpha\xi + x_*} - 1.$$

Then the solutions of the characteristic equation give

$$\lambda_{1,2} = \frac{1}{2} tr(J|_{E_*}) \pm \sqrt{(tr(J|_{E_*}))^2 - 4det(J|_{E_*})}.$$

By analyzing the distribution of roots of characteristic equation, we obtain the following results. We note that if $det(J|_{E_*}) > 0$ then E_* is locally asymptotically stable if $tr(J|_{E*}) < 0$, that is, if

$$c' > 1 - \frac{\delta(\gamma+2) + 2\xi(\alpha\delta - \beta)}{\gamma\beta - \beta(1+\alpha\xi)},$$

and unstable if $tr(J) > 0$

$$c' < 1 - \frac{\delta(\gamma+2) + 2\xi(\alpha\delta - \beta)}{\gamma\beta - \beta(1+\alpha\xi)}.$$

(c) By defining $c' = 1 - \frac{\delta(\gamma+2)+2\xi(\alpha\delta-\beta)}{\gamma\beta-\beta(1+\alpha\xi)}$, when $c' = c'_*$, $tr(J|_{E_*}) = 0$ and the characteristic equation has a pair of imaginary eigenvalues $\lambda = \pm i\sqrt{det(J|_{E_*})}$.
Let $\lambda(c') = p(c') \pm iq(c')$ be the roots characteristic equation when c' is near c'_*, then $p(c') = \frac{1}{2}tr(J|_{E_*})$. In order to ensure the changes of stability through non-degenerate Hopf bifurcation, we need to verify the transversality condition for Hopf bifurcation. Obviously,

$$\frac{dp}{dc'} = \frac{d}{dc'}Re(\lambda(c'))|_{c'=c'_*} \neq 0.$$

□

Theorem 3.3. *Define* $\alpha^* = \frac{\gamma - 2x_* - 1}{\xi}$*, then the following statements hold:*
(a) *If* $\alpha > \alpha^*$*, the positive equilibrium* E_* *is locally asymptotically stable.*
(b) *If* $\alpha < \alpha^*$*, the positive equilibrium* E_* *is unstable.*
(c) *If* $\alpha = \alpha^*$*, then a Hopf bifurcation occurs around the positive equilibrium* E_**.*

Proof. The proof is similar to that of the Theorem (3.2). Let $\lambda(\alpha) = v(\alpha) \pm w(\alpha)$ be the roots characteristic equation when a is near α^*, then $v(\alpha) = \frac{1}{2}tr(J|_{E_*}) = -\frac{1}{2}\frac{\xi(\gamma - x_*)x_*}{\gamma(1+\alpha\xi+x_*)}$,if $\gamma > x_*$ then

$$\frac{dv}{d\alpha} = \frac{d}{d\alpha}Re(\lambda(\alpha))|_{\alpha=\alpha^*} < 0.$$

If $\gamma < x_*$ then

$$\frac{dv}{d\alpha} = \frac{d}{d\alpha}Re(\lambda(\alpha))|_{\alpha=\alpha^*} > 0.$$

□

It is easy to see that existence of a Hopf bifurcation and stability in a small neighbourhood of E_* when parameter β vary.

4. Numerical simulation

In this section we present computer simulation of some solutions of the system (1.3). Beside verification of our analytical findings, these numerical solutions are very important from practical point of view. The phase portraits were calculated with ode45 of MATLAB. This is done by calculating the solutions forward and backward in time for initial values located on a equally spaced grid in the first quadrant. We use the following symbols in the bifurcation diagrams of this paper: H: Hopf bifurcations of an equilibrium point, Lpc: for the tangent bifurcations of limit cycles, BT: for the Bogdanov-Takens.

4.1. **Continuation Curve of Equilibrium Point (one-parameter bifurcation diagram).** Analytical studies can never be completed without numerical verification of the derived results. The main aim of this section is to study the pattern of bifurcation that takes place as we vary the parameters β, c'. This is actually done by studying the change in the eigenvalue of the Jacobian matrix and also following the continuation algorithm. To start with, we consider a set of fixed point initial solution, $(x, y) = (1.6, 2.08)$, corresponding to a parameter set of values, $\gamma = 4, c' = 0.21, \xi = 0.2, \beta = 0.15, \delta = 0.08, \alpha = 0.6$. We note that this set of parameters are chosen such that the continuation will start from a stable region of parameters. The characteristics of Hopf point, the limit cycle and the general bifurcation may be explored. To compute curve of equilibrium from the equilibrium point we take parameter $c' = 0.21$ as the free parameter with fixed $\gamma = 4, \xi = 0.2, \beta = 0.15, \delta = 0.08, \alpha = 0.6$. it is evident that the system has a Hopf point in a neighbourhood of E_*, as predicted by the theory, with purely imaginary eigenvalues $\pm i0.122066$, in a small neighbourhood of E_*. For this Hopf point the first Lyapunov coefficient is in Table 1 indicating a supercritical Hopf bifurcation. It being negative implies that a stable limit cycle bifurcates from the equilibrium when this loses stability. From Figures 1, 2, 6, 12 and 14 it is evident that the system has a Hopf point at:

label= H, x= (1.440000 2.024567 0.190741),
First Lyapunov coefficient= -9.110524e-002,
label= BP, x= (4.000000 -0.000000 0.367333).

To compute curve of equilibrium from the equilibrium point we take $\beta = 0.15$ as the free parameter with fixed $\gamma = 4, c' = 0.21, \xi = 0.2, \delta = 0.08, \alpha = 0.6$. It is evident that the system has two Hopf point in a small neighbourhood of E_* with purely imaginary eigenvalues $\pm i0.000858$, $\pm i0.121427$. For this Hopf point the first Lyapunov coefficient ℓ_1 is in Table 1 indicating a supercritical Hopf bifurcation. From Figures 3, 4, 5, 11 and 13 it is evident that the system has a Hopf point

label= H, x= (1.440002 2.073924 0.153110),
First Lyapunov coefficient= -9.087072e-002,
label= BP, x= (0.000000 1.417722 0.448000),
label= H, x= (0.000003 1.417724 0.447996),
First Lyapunov coefficient= -9.148776e-001.

To compute curve of equilibrium from the equilibrium point we take $\alpha = 0.39$ as the free parameter with fixed $\gamma = 4, c' = 0.21, \xi = 0.2, \delta = 0.08, \beta = 0.15$. It is evident that the system has a Hopf point in a small neighbourhood of E_* with purely imaginary eigenvalues $\pm i0.118582$. For this Hopf point the first Lyapunov coefficient ℓ_1 is in Table 1 indicating a supercritical Hopf bifurcation. From Figures 7 and 8 it is evident that the system has a Hopf point

label= H, x= (1.460958 2.040107 0.390430),
First Lyapunov coefficient= -9.0218824e-002.

By selecting Hopf point in the one-parameter bifurcation diagram of the equilibrium as initial point, we can plot the limit cycles and bifurcations of limit cycles starting from the Hopf point (see Figures 11 and 12).

When we take $\beta = 0.15$ as the free parameter:
Limit point cycle (period = 5.174496e+001, parameter = 1.531102e-001),
Normal form coefficient = -1.160727e-001.

4.2. **Two-parameter bifurcation diagram.** By selecting limit point in the one-parameter bifurcation diagram of the equilibrium as initial point, and taking β, c' as the free parameter (see Figure 9).
label= BT, x= (1.440000 9.175039 0.821429 0.448000 0.000000),
(a,b)= (-3.842550e-010, 2.812500e-001).

Taking ξ, c' as the free parameter (see Figure 10).
label= BT, x= (1.264706 3.507104 0.466666 0.784313 0.000000),
(a,b)= (1.123532e-008, -2.311829e-001).

Taking ξ, α as the free parameter (see Figure 15).
label= BT, x= (0.000000 5.063361 1.406256 2.133363 0.000000),
(a,b)= (-2.70825e-018, -1.974973e-001).

Taking δ, c' as the free parameter (see Figure 16).
label= BT, x= (1.440000 9.175038 0.821429 0.026786 0.000000),
(a,b)= (-2.950715e-010, 2.812500e-001).

Table 1. One-parameter bifurcation points and eigenvalues.

lable	eigenvalues	ℓ_1	free parameter
H	$\lambda_{1,2} = 7.4404e-009 \pm i0.122066$	$\ell_1 = -9.110524e-002$	c'
H	$\lambda_{1,2} = 9.93684e-007 \pm i0.000858$	$\ell_1 = -9.148776e-001$	β
H	$\lambda_{1,2} = -2.14975e-007 \pm i0.121427$	$\ell_1 = -9.087072e-001$	β
H	$\lambda_{1,2} = -1.44344e-007 \pm i0.118582$	$\ell_1 = -9.218824e-002$	α

We notice that the numerical bifurcation diagram and numerical phase portraits confirm the results established by the theory.

Table 2. Two-parameter bifurcation points, limit cycles and eigenvalues.

lable	eigenvalues	free parameter
BT	$\lambda_{1,2} = -3.22138e-015 \pm i2.74842e-005$	c', δ
BT	$\lambda_{1,2} = -2.14898e-014 \pm i0.000175$	c', ξ
BT	$\lambda_{1,2} = \pm 0.00384514$	c', β
BT	$\lambda_{1,2} = 1.59595e016$	α, ξ
LPC	$\mu_{1,2} = 0.999914, 1$	β

Table 3. Sensitivity of parameter α, β, c' on population (x), (y).

variable (x)	variable (y)	(α)	(β)	(c')	status
1.452338	1.772509	–	–	0.180741	unstable Figures 1, 14
1.485092	2.072778	–	–	0.190741	stable Figure 2
$8.9650e-007$	1.419166	–	0.447995	–	stable Figure 4
$3.9229e-006$	1.050957	–	0.437996	–	unstable Figure 3
1.023190	1.820431	0.29043	–	–	unstable Figure 8
1.462115	2.057857	0.39043	–	–	stable Figure 8

Figure 1. Trajectories of system (1.3), when $c' = 0.180741$. E_* is locally asymptotically unstable.

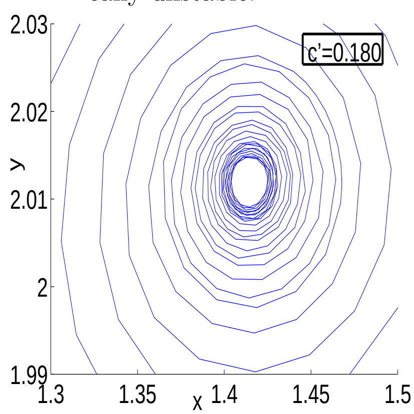
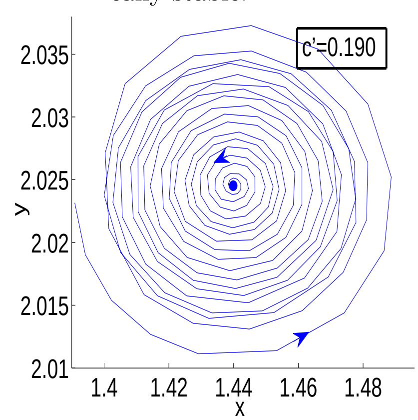


Figure 2. Trajectories of system (1.3), when $c' = 0.190741$. E_* is locally asymptotically stable.



5. Conclusion

In this paper, we have studied the dynamics of a predator-prey system with harvesting of the predator guided by its population. We obtain conditions that affect the persistence of the system. Local asymptotic stability of various equilibrium solutions

FIGURE 3. Trajectories of system (1.3), when $\beta = 0.437996$.

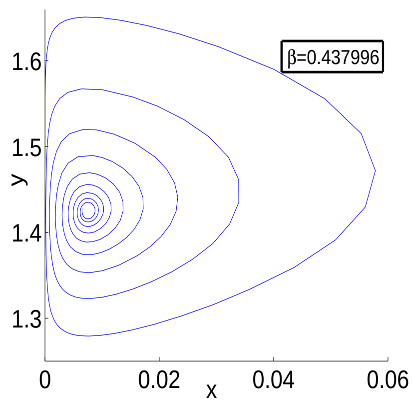
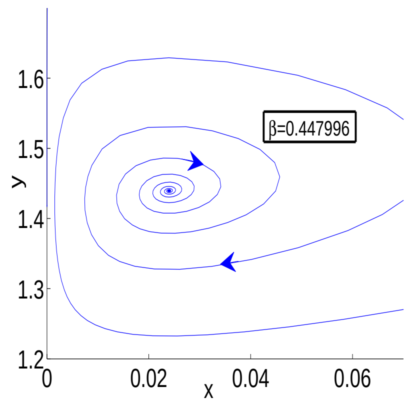


FIGURE 4. Trajectories of system (1.3), when $\beta = 0.447996$.



is explored to understand the dynamics of the model system. Based on the normal form theory, we find that it is a cusp point of co-dimension 2 so that Bogdanov-Takens bifurcation may occur. In contrast, by the introduction of prey refuge providing additional food to predator, our proposed model (1.3), can exhibit much richer behaviors, i.e., numerous bifurcations may occur including the, Hopf,and Bogdanov-Takens bifurcations. The detected bifurcations have biological implications. At a computed supercritical Hopf bifurcation a stable limit cycle is born that gives rise to periodic behavior of the populations. In fact, if the predator death rate is smaller than the bifurcation parameter at a supercritical Hopf, both predator and prey coexist in the steady state, but if the predator death rate exceeds this value of bifurcation parameter then both predator and prey still coexist and their densities vary periodically.

In summary, our analysis of the dynamics of the dynamics system can assist decision-making in a variety of field management to improve implementation of harvesting strategies.

FIGURE 5. Bifurcation diagram of the equilibrium in a small neighbourhood of E_* with the variation of the parameter β undergoing a supercritical Hopf bifurcation.

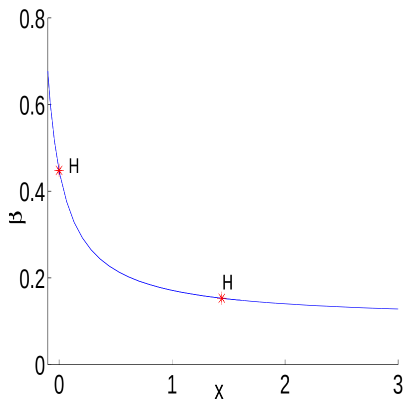


FIGURE 6. Bifurcation diagram of the equilibrium in a small neighbourhood of E_* with the variation of the parameter c' undergoing a supercritical Hopf bifurcation.

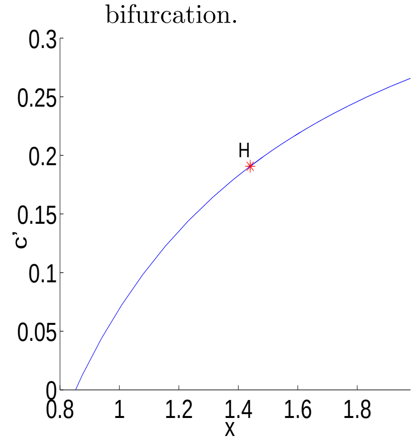


FIGURE 7. Continuation curves of equilibrium with the variation of the parameter α.

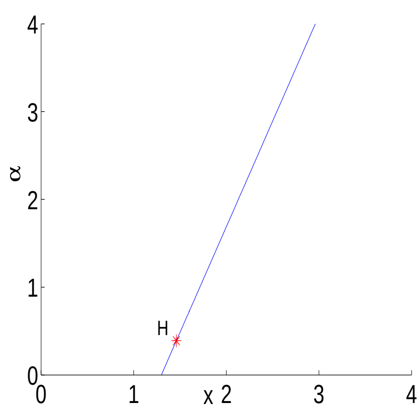
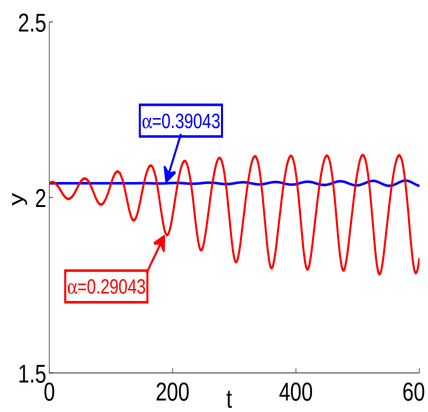


FIGURE 8. Periodic solution for system (1.3) with the variation of α at E_*.



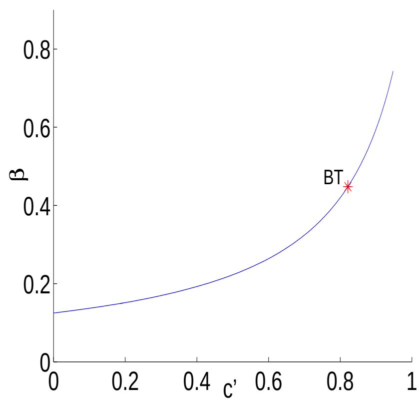


FIGURE 9. Hopf curve in model (1.3), two-parameter bifurcation diagram when we take β, c' as the free parameter with fixed $\xi, \delta, \gamma, \alpha$.

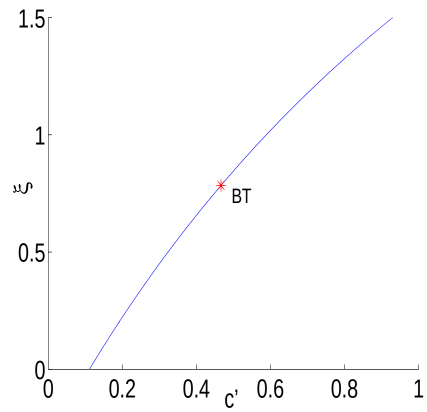


FIGURE 10. Hopf curve in model (1.3), two-parameter bifurcation diagram when we take ξ, c' as the free parameter with fixed $\beta, \delta, \gamma, \alpha$.

FIGURE 11. The family of limit cycles and bifurcations of limit cycles starting from the Hopf point with the variation of the parameter $\beta = 0.447996$.

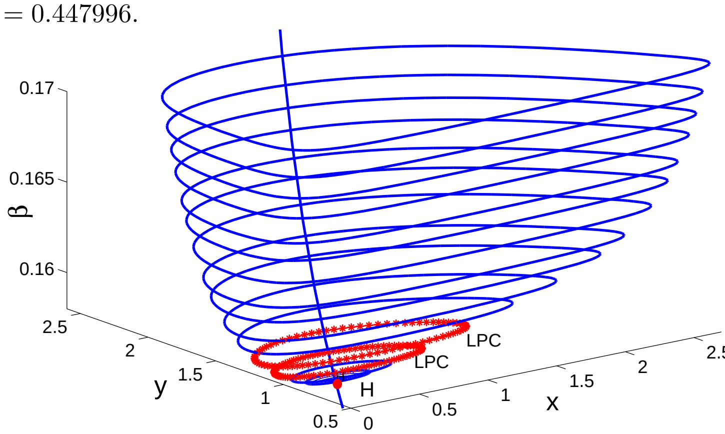


FIGURE 12. The family of limit cycles starting from the Hopf point with the variation of the parameter $c' = 0.190741$.

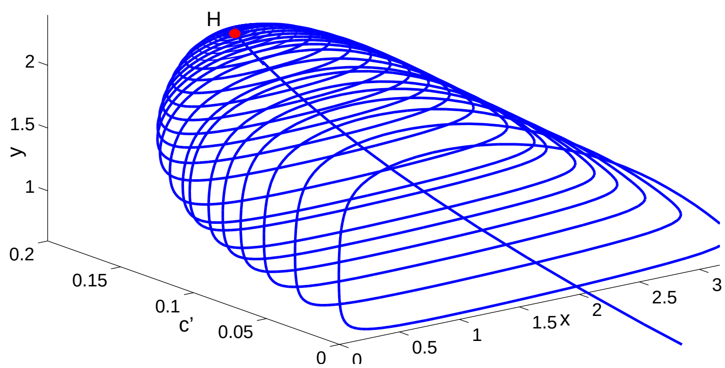


FIGURE 13. Hopf bifurcation occurs at E_* and bifurcating periodic solution for system (1.3) with the variation of β.

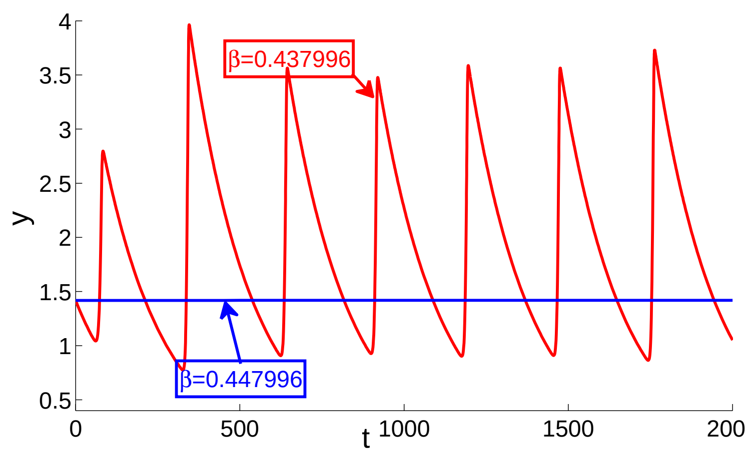


Figure 14. Periodic solution for system (1.3) with c'

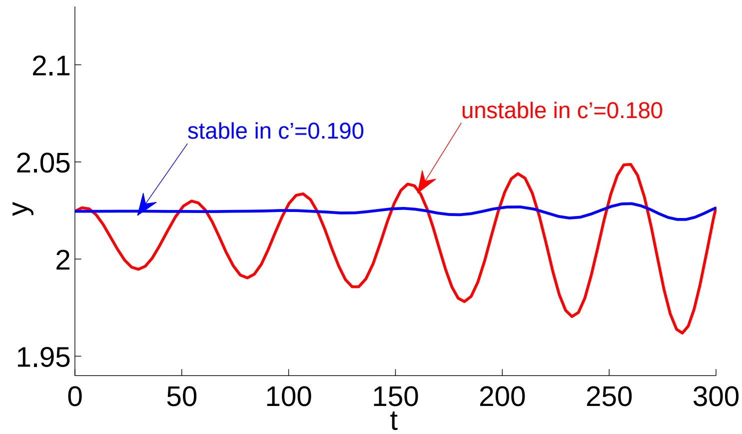


Figure 15. Hopf curve in model (1.3), two-parameter bifurcation diagram when we take ξ, α as the free parameter with fixed $\beta, \delta, \gamma, c'$.

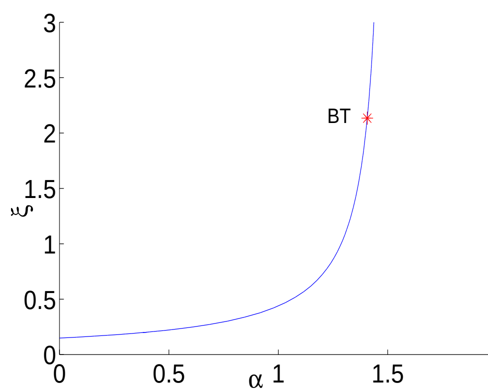
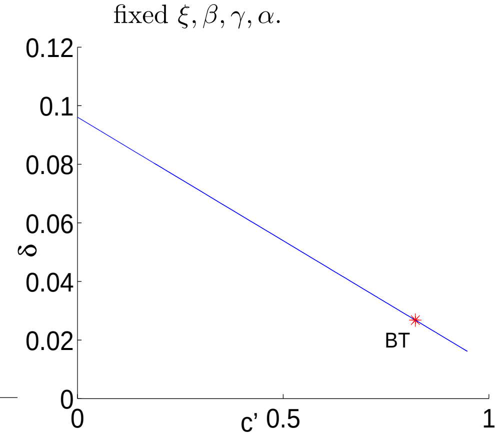


Figure 16. Hopf curve in model (1.3), two-parameter bifurcation diagram when we take δ, c' as the free parameter with fixed $\xi, \beta, \gamma, \alpha$.



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