A numerical study using finite element method for generalized Rosenau-Kawahara-RLW equation

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Abstract
In this paper, we are going to obtain the soliton solution of the generalized Rosenau-Kawahara-RLW equation that describes the dynamics of shallow water waves in oceans and rivers. We confirm that our new algorithm is energy-preserved and unconditionally stable. In order to determine the performance of our numerical algorithm, we have computed the error norms $L_2$ and $L_\infty$. Convergence of full discrete scheme is firstly studied. Numerical experiments are implemented to validate the energy conservation and effectiveness for longtime simulation. The obtained numerical results have been compared with a study in the literature for similar parameters. This comparison clearly shows that our results are much better than the other results.

Keywords. Generalized Rosenau-Kawahara-RLW equation, Finite element method, Collocation.

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1. INTRODUCTION
The theory of nonlinear partial differential equations (PDEs) has made a significant development during the last decade. Especially we know that analyzing traveling-wave solutions for PDEs, using various methods, have long been an extensive interest for mathematicians, physicists, engineers and other scientists. At the same time, the existence of soliton kind solutions for PDEs is of special concern due to their common implementations in many physical structures for instance nonlinear optics, optical fibers, chemical physics, capillary-gravity, fluid dynamics and mechanics, plasmas, condensed matter, electro magnetics and any more. As known, a soliton is a specific kind of solitary waves which preserve its wave shape after inconformity with other
solitons. Some of the prevalently studied equations with soliton solutions are contain: Korteweg-de Vries (KdV) equation,

\[ U_t + aU_x + bU_{xxx} = 0, \tag{1.1} \]

was suggested propagation of long waves in shallow water waves \[18, 27, 33, 34, 35, 48\]; Regularized Long Wave (RLW) equation,

\[ U_t + U_x + aU_x - bU_{xxt} = 0, \tag{1.2} \]

was first introduced as a sample for small-amplitude long wave of water in a channel by Peregrine \[6, 7, 8, 12, 14, 15, 16, 28, 31, 36, 42\]; Kawahara equation

\[ U_t + U_x + U_{xxx} + U_{xxxx} - U_{xxxxx} = 0, \tag{1.3} \]

came to exist in the theory of shallow water waves with surface tension \[1, 9, 20, 24, 26, 41\]; Rosenau equation

\[ U_t + U_x + U_{xxx} + U_{xxxx} + U_{xxxxx} = 0, \tag{1.4} \]

and many others. In the research of the dynamics of intensive discrete systems, the status of wave-wave and wave-wall interactions can not be described using the well-recognized KdV equation. Moreover, the slope and the action of high amplitude waves are not be well estimated under the assumption of weak anharmonicity \[29\]. To cope with the imperfection of the Eq. \((1.1)\), Eq. \((1.4)\) was proposed by Rosenau \[38, 39\]. If the viscous term \(U_{xxx}\) is added in Eq. \((1.4)\), the newly equation is usually named Rosenau-KdV equation \[2, 3, 4, 17, 25, 37, 38, 39, 40\],

\[ U_t + U_x + U_{xxx} + U_{xxxx} + U_{xxxxx} = 0. \tag{1.5} \]

When the viscous term \(-U_{xxt}\) is involved in the Eq. \((1.4)\), the obtained equation is usually termed the following Rosenau-RLW equation \[5, 29, 30, 46, 47\],

\[ U_t + U_x - U_{xxt} + U_{xxxx} + U_{xxxx} = 0. \tag{1.6} \]

If the another viscous term \(-U_{xxxx}\) is involved in the Eq. \((1.5)\), the newly obtained equation is usually called the following Rosenau-Kawahara equation \[10, 23, 50\],

\[ U_t + U_x + U_{xxx} + U_{xxxx} + U_{xxxxx} + U_{xxxxx} = 0. \tag{1.7} \]

The aim of this work is to search for the soliton solutions of the following generalized Rosenau-Kawahara-RLW equation

\[ U_t + aU_x + bU_{x} + cU_{xxx} - \alpha U_{xxt} + \lambda U_{xxxx} + \zeta U_{xxxxx} = 0 \tag{1.8} \]
with the homogeneous boundary conditions
\begin{align}
U(a, t) &= 0, \quad U(b, t) = 0, \\
U_x(a, t) &= 0, \quad U_x(b, t) = 0, \\
U_{xx}(a, t) &= 0, \quad U_{xx}(b, t) = 0, \quad t > 0
\end{align}
(1.9)
and an initial condition
\begin{equation}
U(x, 0) = U_0(x) \quad a \leq x \leq b,
\end{equation}
(1.10)
where \( p \) is a positive integer, \( \alpha, \lambda \) are positive constants and \( a, b, c, \zeta \) are all real constants. Generalized Rosenau-Kawahara-RLW equation has a limited number of numerical studies in the literature. Zuo [49] have implemented sech ansatz and tanh ansatz method to produce exact bright and dark 1-soliton solutions of the general Rosenau-Kawahara-RLW equation. Hea and Pan [21] have developed a three-level linearly implicit finite difference scheme based on sine-cosine method for solving the equation. Hea [22] generates the exact solitary wave solution for the perturbed Rosenau–Kawahara-RLW equation with power law nonlinearity and then improves a three-level linearly implicit difference algorithm for solving the equation. A three-level conservative fourth-order finite difference scheme for the initial boundary value problem of the generalized Rosenau-Kawahara-RLW equation has been developed by Wang and Dai [45].

In this work, the dynamics of shallow water waves that are modeled by generalized Rosenau–Kawahara-RLW equation along with septic B-spline collocation method will be studied. The rest of the paper can be outlined concisely as follows: In the second section, collocation finite element method has been applied to generalized Rosenau–Kawahara-RLW equation. The resulting system can be efficiently solved with a sort of the Thomas algorithm. Section 3 contains a linear stability analysis of the algorithm followed by section 4 which contains convergence of full discrete scheme. In the fifth section, motion of single soliton has examined. The obtained numerical results are given both in tabular and graphical format and the computed results are also compared with the studies available in the literature. Finally, the main findings of the paper are briefly summarized in a concluding section.

2. Collocation Method with Septic B-splines

For our numerical computations, solution area of the problem is limited over an interval \( a \leq x \leq b \). Let the partition of the space interval \([a, b]\) into equally sized finite elements of length \( h \) at the points \( x_m \) like that \( a = x_0 < x_1 < ... < x_N = b \) and \( h = \frac{b-a}{N} \). The set of septic B-spline functions \( \{\phi_{-3}(x), \phi_{-2}(x), \ldots, \phi_{N+3}(x)\} \) form a basis over the solution domain \([a, b]\). The numerical solution \( U_N(x, t) \) is expressed in terms of the septic B-splines as
\begin{equation}
U_N(x, t) = \sum_{m=-3}^{N+3} \phi_m(x)\delta_m(t).
\end{equation}
(2.1)
where $\delta_m(t)$ are time dependent parameters. Septic B-splines $\phi_m(x)$, $(m = -3, -2, ..., N + 3)$ at the knots $x_m$ are designated over the interval $[a, b]$ by Prenter [32]

$$
\phi_m(x) = \frac{1}{n!} \times
$$

$$
\begin{align*}
(x - x_{m-4})^7, \\
(x - x_{m-4})^7 - 8(x - x_{m-3})^7, \\
(x - x_{m-4})^7 - 8(x - x_{m-3})^7 + 28(x - x_{m-2})^7, \\
(x - x_{m-4})^7 - 8(x - x_{m-3})^7 + 28(x - x_{m-2})^7 - 56(x - x_{m-1})^7, \\
(x_{m+4} - x)^7 - 8(x_{m+3} - x)^7 + 28(x_{m+2} - x)^7, \\
(x_{m+4} - x)^7 - 8(x_{m+3} - x)^7 + 28(x_{m+2} - x)^7 - 56(x_{m+1} - x)^7, \\
(x_{m+4} - x)^7 - 8(x_{m+3} - x)^7, \\
(x_{m+4} - x)^7 - 8(x_{m+3} - x)^7, \\
0,
\end{align*}
$$

(2.2)

A characteristic finite interval $[x_m, x_{m+1}]$ is matched to the interval $[0, 1]$ by a local coordinate transformation defined by $h\xi = x - x_m$, $0 \leq \xi \leq 1$. Therefore septic B-splines (2.2) in terms of $\xi$ over $[0, 1]$ are given as follows:

$$
\begin{align*}
\phi_{m-3} &= 1 - 7\xi + 21\xi^2 - 35\xi^3 + 35\xi^4 - 21\xi^5 + 7\xi^6 - \xi^7, \\
\phi_{m-2} &= 120 - 392\xi + 504\xi^2 - 280\xi^3 + 84\xi^4 - 42\xi^5 + 7\xi^6, \\
\phi_{m-1} &= 1191 - 1715\xi + 315\xi^2 + 665\xi^3 - 315\xi^4 - 105\xi^5 + 105\xi^6 - 21\xi^7, \\
\phi_{m} &= 2416 - 1680\xi + 560\xi^2 - 140\xi^3 + 35\xi^4, \\
\phi_{m+1} &= 1191 + 1715\xi + 315\xi^2 + 665\xi^3 - 315\xi^4 + 105\xi^5 + 105\xi^6 - 35\xi^7, \\
\phi_{m+2} &= 120 + 392\xi + 504\xi^2 - 280\xi^3 + 84\xi^4 - 42\xi^5 + 21\xi^6, \\
\phi_{m+3} &= 1 + 7\xi + 21\xi^2 + 35\xi^3 + 35\xi^4 + 21\xi^5 + 7\xi^6 - \xi^7, \\
\phi_{m+4} &= \xi^7.
\end{align*}
$$

(2.3)

For the problem, the finite elements are described with the interval $[x_m, x_{m+1}]$. Using Eq. (2.2) and Eq. (2.1), the nodal values of $U_m, U_m', U_m'', U_m'''$ and $U_m^{iv}$ are given in concern with the element parameters $\delta_m$ by

$$
U_N(x_m, t) =
$$

$$
\begin{align*}
U_m &= \delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_{m} + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3}, \\
U_m' &= \frac{\xi}{\pi} (-\delta_{m-3} - 56\delta_{m-2} - 245\delta_{m-1} + 245\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3}), \\
U_m'' &= \frac{\xi^2}{\pi^2} (\delta_{m-3} + 24\delta_{m-2} + 15\delta_{m-1} - 80\delta_{m} + 15\delta_{m+1} + 24\delta_{m+2} + \delta_{m+3}), \\
U_m''' &= \frac{\xi^3}{\pi^3} (-\delta_{m-3} - 8\delta_{m-2} + 19\delta_{m-1} - 19\delta_{m+1} + 8\delta_{m+2} + \delta_{m+3}), \\
U_m^{iv} &= \frac{\xi^4}{\pi^4} (\delta_{m-3} - 9\delta_{m-2} + 16\delta_{m-1} - 9\delta_{m+1} + \delta_{m+3})
\end{align*}
$$

(2.4)
and the variation of $U$ over the element $[x_m, x_{m+1}]$ is given by

$$U = \sum_{m=-3}^{N+3} \phi_m \delta_m.$$  

(2.5)

When we define the collocation dots with the knots and use Eqs. (2.4) to evaluate $U_m$, its space derivatives and substitute into Eq. (1.8), this brings to a set of ordinary differential equations of the form

$$\begin{aligned}
\dot{\delta}_{m-3} + 120\dot{\delta}_{m-2} + 1191\dot{\delta}_{m-1} + 2416\dot{\delta}_m + 1191\dot{\delta}_{m+1} + 120\dot{\delta}_{m+2} + \dot{\delta}_{m+3} \\
- \frac{25}{12} \alpha (\dot{\delta}_{m-3} + 24\dot{\delta}_{m-2} + 15\dot{\delta}_{m-1} - 80\delta_m + 15\dot{\delta}_{m+1} + 24\dot{\delta}_{m+2} + \dot{\delta}_{m+3}) \\
+ \frac{840}{M} \Lambda (\dot{\delta}_{m-3} - 9\dot{\delta}_{m-1} + 16\delta_m - 9\dot{\delta}_{m+1} + \dot{\delta}_{m+3}) \\
+ (a + bZ_m) \zeta (\dot{\delta}_{m-3} - 56\dot{\delta}_{m-2} - 245\dot{\delta}_{m-1} + 245\delta_{m+1} + 56\dot{\delta}_{m+2} + \dot{\delta}_{m+3}) \\
+ \frac{210}{M^2} \zeta (\dot{\delta}_{m-3} - 8\dot{\delta}_{m-2} + 19\delta_m + 19\dot{\delta}_{m+1} + 8\dot{\delta}_{m+2} + \dot{\delta}_{m+3}) + \\
+ \frac{2520}{840} \zeta (\dot{\delta}_{m-3} - 4\dot{\delta}_{m-2} - 5\delta_m + 5\delta_{m+1} + 4\dot{\delta}_{m+2} + \dot{\delta}_{m+3}) = 0,
\end{aligned}$$  

(2.6)

where

$$Z_m = U_m$$

$$= (\delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3})p.$$  

If $\delta_i$ and its derivatives $\dot{\delta}_i$ in Eq. (2.6) are decoupled by Crank-Nicolson formula

$$\delta_i = \frac{\delta_{i+1}^n + \delta_i^n}{2},$$  

(2.7)

and usual finite difference approximation

$$\dot{\delta}_i = \frac{\delta_{i+1}^n - \delta_i^n}{\Delta t},$$  

(2.8)

we derive a repetition relationship between two time levels $n$ and $n + 1$ relating two unknown parameters $\delta_i^{n+1}, \delta_i^n$ for $i = m - 3, m - 2, ..., m + 2, m + 3$

$$\begin{aligned}
\gamma_1 \delta_{i+1}^{n+1} + \gamma_2 \delta_{i+1}^{n+1} + \gamma_3 \delta_{i+1}^{n+1} + \gamma_4 \delta_{i+1}^{n+1} + \gamma_5 \delta_{i+1}^{n+1} + \gamma_6 \delta_{i+1}^{n+1} + \gamma_7 \delta_{i+1}^{n+1} \\
= \gamma_8 \delta_{i-3}^{n+1} + \gamma_9 \delta_{i-2}^{n+1} + \gamma_{10} \delta_{i-1}^{n+1} + \gamma_{11} \delta_{i}^{n+1} + \gamma_{12} \delta_{i+1}^{n+1} + \gamma_{13} \delta_{i+2}^{n+1} + \gamma_{14} \delta_{i+3}^{n+1},
\end{aligned}$$  

(2.9)

where

$$\gamma_1 = [1 - E + M - K(a + bZ_m) - L - T],$$  

$$\gamma_2 = [120 - 24E - 56K(a + bZ_m) - 8L + 4T],$$  

$$\gamma_3 = [1191 - 15E - 9M - 245(a + bZ_m) + 19L - 5T],$$  

$$\gamma_4 = [2416 + 80E + 16M],$$  

$$\gamma_5 = [1191 - 15E - 9M + 245(a + bZ_m) - 19L + 5T],$$  

$$\gamma_6 = [120 - 24E + 56K(a + bZ_m) + 8L - 4T],$$  

$$\gamma_7 = [1 - E + M + K(a + bZ_m) + L + T],$$  

$$m = 0, 1, ..., N, E = \frac{12}{K}, M = \frac{360}{\pi^2}, K = \frac{7\Delta t}{2\lambda}, L = \frac{210\Delta t}{2\kappa}, \gamma = \frac{250\Delta t}{2\kappa}. $$  

(2.10)
The system (2.9) involves of \((N+1)\) linear equations containing \((N+7)\) unknown coefficients \((\delta_{-3}, \delta_{-2}, \delta_{-1}, \ldots, \delta_{N+1}, \delta_{N+2}, \delta_{N+3})^T\). We need six additional restraints to obtain a unique solution for this system. These are obtained from the boundary conditions (1.9) and can be used to remove \(\delta_{-3}, \delta_{-2}, \delta_{-1}\) and \(\delta_{N+1}, \delta_{N+2}, \delta_{N+3}\) from the systems (2.9) which occurs a matrix equation for the \(N+1\) unknowns \(d^n = (\delta_0, \delta_1, \ldots, \delta_N)^T\) of the form
\[
Ad^{n+1} = Bd^n. \tag{2.11}
\]
Two or three inner iterations are implemented to the term \(\delta^{n*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})\) at each time step to overcome the non-linearity caused by \(Z_m\). Before the beginning of the solution procedure, initial parameters \(d^0\) are established by using the initial condition and following derivatives at the boundaries;
\[
(U_N)(x, 0) = U(x_m, 0); \quad m = 0, 1, 2, ..., N
\]
\[
(U_N)_x(a, 0) = 0, \quad (U_N)_x(b, 0) = 0,
\]
\[
(U_N)_xx(a, 0) = 0, \quad (U_N)_xx(b, 0) = 0,
\]
\[
(U_N)_xxx(a, 0) = 0, \quad (U_N)_xxx(b, 0) = 0.
\]
So we obtain the following matrix form for the initial vector \(d^0\);
\[
Vd^0 = w,
\]
where
\[
V = \begin{bmatrix}
1536 & 2712 & 768 & 24 & 1 \\
82731 & 210568.5 & 104796 & 10063.5 & 81 \\
82731 & 96597 & 96597 & 96597 & 81 \\
96474 & 96474 & 96474 & 81 & 81 \\
... & ... & ... & ... & ... \\
1 & 120 & 1191 & 2416 & 1191 \\
1 & 120 & 195796 & 10063.5 & 10063.5 \\
1 & 120 & 104796 & 210568.5 & 82731 \\
1 & 120 & 768 & 2712 & 1536
\end{bmatrix}
\]
\[
d^0 = (\delta_0, \delta_1, \delta_2, ..., \delta_{N-2}, \delta_{N-1}, \delta_N)^T,
\]
and
\[
w = (U(x_0, 0), U(x_1, 0), ..., U(x_{N-1}, 0), U(x_N, 0))^T.
\]

3. Stability of the Algorithm

To investigate the stability analysis of the presented algorithm, it is suitable to use Von-Neumann theory. Presuming that the quantity \(U^\sigma\) in the nonlinear term \(U^\sigma U_x\) is locally constant. Substituting the Fourier mode \(\delta_m^n = \xi^m e^{i\sigma m h}, (i = \sqrt{-1})\) into the form of (2.9) we obtain,
\[
\xi^{n+1}(\eta_1 e^{i(m-3)\theta} + \eta_2 e^{i(m-2)\theta} + \eta_3 e^{i(m-1)\theta} + \eta_4 e^{i m\theta} + \eta_5 e^{i(m+1)\theta} + \eta_6 e^{i(m+2)\theta} + \eta_7 e^{i(m+3)\theta})
\]
\[
= \xi^n (\eta_1 e^{i(m-3)\theta} + \eta_2 e^{i(m-2)\theta} + \eta_3 e^{i(m-1)\theta} + \eta_4 e^{i m\theta} + \eta_5 e^{i(m+1)\theta} + \eta_6 e^{i(m+2)\theta} + \eta_7 e^{i(m+3)\theta})
\]
\[
(3.1)
\]
where $\sigma$ is mode number, $h$ is the element size, $\theta = \sigma h$

\begin{align*}
\eta_1 &= 1 - E + M - K(a + bZ_m) - L - T, \\
\eta_2 &= 120 - 24E - 56K(a + bZ_m) - 8L + 4T, \\
\eta_3 &= 1191 - 15E - 9M - 245(a + bZ_m) + 19L - 5T, \\
\eta_4 &= 2416 + 80E + 16M, \\
\eta_5 &= 1191 - 15E - 9M + 245(a + bZ_m) - 19L + 5T, \\
\eta_6 &= 120 - 24E + 56K(a + bZ_m) + 8L - 4T, \\
\eta_7 &= 1 - E + M + K(a + bZ_m) + L + T, \\
&\quad m = 0, 1, \ldots, N.
\end{align*}

If we simplify the Eq. (3.1),

$$\xi = \frac{A + iB}{A - iB}$$

is obtained where

\begin{align*}
A &= (2382 - 30E - 18M) \cos(\theta) + (240 - 48E) \cos(2\theta) + (2 - 2E + 2M) \cos(3\theta) + \\
&\quad (2416 + 80E + 16M), \\
B &= (490K(1 + Z_m) - 38L - 10T) \sin(\theta) + (112K(1 + Z_m) + 16L - 8T) \sin(2\theta) + \\
&\quad (2K(1 + Z_m) + 2L + 2T) \sin(3\theta).
\end{align*}

(3.2)

According to Fourier stability analysis, for the given scheme to be stable, the condition $|\xi| < 1$ must be satisfied. Using a symbolic programming software or using simple calculations, since $a^2 + b^2 = a^2 + (-b)^2$ it becomes evident that the modulus of $|\xi|$ is 1. Hence the linearized algorithm is unconditionally stable.

4. Convergence of the full discrete scheme

In this article, we approximate a non-linear partial differential equation (1.8) using a higher order B-splines collocation approach at space predefined points for the space variable and a simple one step scheme for time. Efficiency of a computational algorithm depends on its implementation issues, high accuracy and numerical flexible stability conditions. Here we study the relevant concepts and key results without proof and cite sources of a more complete treatment. To be specific, we aim for a short discussion about the accuracy of the above mentioned space time scheme without a formal proof. One may consult [2, 8, 11, 43, 44] and the references therein for a detailed and settled theories. It is to note that we use some constants $C_i \geq 0$ here which not necessarily the same in all the cases. Usually global polynomial interpolations are used to integrate the solutions of differential equations for simple computational domain and when unknown curves are considered to be smooth enough. However, most engineering and physical problems are considered when the solutions are not sufficiently smooth to support global polynomial approximation and the computational domain is complicated. For these types of cases, finite element approximations play an important role and work very well to represent the solutions of the modelled problem.

Piecewise polynomial basis functions are smooth on each user defined sub-intervals which is one of the very important properties in approximation theory. It is very much helpful to analyze solutions approximated using such basis functions. On each spatial...
sub-intervals we have \( p + 1 \) data values. Then there exists a polynomial of degree at most \( p \) passing through the data points. The error in such polynomial approximation is proportional to a power of the distance (usually denoted by \( h \)) between the data points \([11, 43, 44]\). As stated in the approximation above the scheme we used is based on septic B-splines along with a spatial collocation approach. So the proposed scheme gives super-convergence at collocation points \([19]\). Also the scheme does not require an extra evaluation of an integral as of the Galerkin approach. So this approach is simpler and efficient to compute solutions.

Here by \( H^k(\Omega) \) we mean the usual space of \( k \) times differentiable functions and by \( \| \cdot \|_p \) we mean the standard \( H^k(\Omega) \) norm. Here \( \| \cdot \|_0 \) stands for \( L_2(\Omega) \) norm.

Let \( v_h \) be an approximation to a function \( v(x) \in H^k(\Omega) \) in \( \Omega \). Let \( h \) be the distance between the grids and \( \Omega = \cup \Omega_i \), where \( \Omega_i = [x_i, x_{i+1}] \), \( x_{i+1} = x_i + h \). It is easy to observe \([7, 13, 43, 44]\) that
\[
\| v(x) - v_h(x) \| \leq Ch^{p+1} \| v \|_{p+1},
\]
where \( 1 \leq p < k \) and \( v_h \) stands for interpolation by piecewise-polynomials of degree \( r \) (considering \( \Omega = \cup \Omega_i \)). This error is preserved by Galerkin finite element approximation as well \([2, 44]\). Thus it is well established that
\[
\| w(x) - w_h(x) \| \leq Ch^{m+1} \| w \|_{m+1},
\]
where \( 1 \leq m < p \) and if \( w_h \) is a suitable \( B \)-splines defined by a polynomial of degree less or equal \( k \) for any \( w \in H_p(\Omega) \) \([2, 11, 13]\). In our present spatial approximation we apply septic B-splines. Thus it is evident that theoretically \( O(h^8) \) accuracy has been achieved from such a spatial approximation in \( L_2(\Omega) \) \([2]\). Here a forward difference scheme is accurate of \( O(\Delta t) \) in \( L_2([0 T]) \) norm for some \( T > 0 \) \([44]\). So for the full discrete scheme the accuracy can be presented by the following inequality
\[
\| u(x, t) - u_h(x, t) \| \leq C_1 h^8 + C_2 \Delta t,
\]
for a suitable \( C_1 \geq 0 \) and \( C_2 \geq 0 \).

5. **Numerical Results and Discussion**

In this section, in order to confirm the correction of our algorithm, some numerical test problems will calculate. For the numerical simulations of the motion of single solitary wave for which exact solutions have known before, two sets of parameters are used and probed. The initial value problem \((1.8) - (1.10)\) has following conservative quantity;
\[
I_E = \frac{1}{2} \int_a^b [U^2(x, t) + \alpha U^2_x(x, t) + \lambda U^2_{xx}(x, t)] dx
\]
which corresponds to the energy of the shallow water waves \([22]\). The accuracy and efficiency of the numerical algorithm is controlled by both the error norm \([2]\)
\[
L_2 = \| U^{exact} - U_N \|_2 \simeq \sqrt{h \sum_{j=0}^N U_{exact}^{j} - (U_N)^j}^2,
\]
and the error norm
\[ L_\infty = \| U^{\text{exact}} - U_N \|_\infty \simeq \max_j \| U_j^{\text{exact}} - (U_N)_j \|. \]

5.1. Case 1. For the first numerical calculation, we choose the parameters \( p = 2 \), \( \Delta t = 0.005 \) with various values of \( h \) over the interval \([-40, 200]\). For this case, since the exact solution of the problem is
\[
U(x, t) = \frac{3}{4} \sqrt{370 - 5\sqrt{10}} \sec h^2 \left[ \frac{\sqrt{37} - 5}{4} \left( x - \frac{33 - 5\sqrt{37}}{5\sqrt{37} - 29} \right) t \right], \tag{5.2}
\]
the initial condition for the problem is taken as
\[
U(x, 0) = \frac{3}{4} \sqrt{370 - 5\sqrt{10}} \sec h^2 \left[ \frac{\sqrt{37} - 5}{4} x \right]. \tag{5.3}
\]

The values of invariant and error norms for single solitary wave over the interval \(-40 \leq x \leq 200\) are tabulated in Table 1. From Table 1, it is seen that the error norms obtained by our method are found much better than the others for \( h = 0.8 \) and the invariant \( I_E \) changes from its initial values by less than \( 7.9 \times 10^{-4} \) for \( h = 1.6 \) whereas the changes of invariant approach to zero for \( h = 1.0, 0.8, \) and \( 0.6 \). Also, we have detected that the quantities of the error norms \( L_2 \) and \( L_\infty \) are sufficiently small during the computer run. Thus, we can say our method is marginally conservative.

![Table 1](image)

The profiles of the single soliton for \( h = 0.8, \Delta t = 0.005 \) at times \( t = 1, 2, 3, \ldots, 10 \) are given in Figure 1. We observed from the Figure 1, single soliton travels to the right at a constant speed and keeps its amplitude and form with increasing time as not surprisingly. The amplitude is 2.1597256 at \( t = 0 \) and located at \( x = 0 \), while it is 2.1580500 at \( t = 10 \) and located at \( x = 18.4 \). The absolute difference in amplitudes at times \( t = 0 \) and \( t = 10 \) is \( 1.67 \times 10^{-3} \) so that there is a little change between amplitudes. To demonstrate the errors between the exact and numerical results over...
Figure 1. Motion of single soliton for $p = 2$, $h = 0.8$, $\Delta t = 0.005$ over the interval $[-40, 200]$ at time $t = 0, 1, 2, ..., 10$.

the solution domain, error distributions at time $t = 1, 2, ..., 10$ are depicted graphically in Figure 2. The maximum errors are between $-3 \times 10^{-5}$ and $2 \times 10^{-5}$ and occur around the central position of the solitary wave.

Figure 2. Errors for $p = 2$, $h = 0.8$, $\Delta t = 0.005$ at $t = 1, 2, ..., 10$.

5.2. Case 2. For the second numerical calculation, we have used the parameters $p = 4$, $\Delta t = 0.005$ with different values of $h$ over the interval $[-40, 240]$. For this case,
we have found the analytical solution of the problem [21]

\[ U(x,t) = \left(\frac{40(\sqrt{127} - 10)}{3(10\sqrt{127} - 109)}\right)^{\frac{3}{4}} \sec h\left[\frac{\sqrt{127} - 10}{3}\left(x - \frac{118 - 10\sqrt{127}}{10\sqrt{127} - 109}\right)\right] t, \]  
(5.4)

so the initial condition for this problem is taken as

\[ U(x,0) = \left(\frac{40(\sqrt{127} - 10)}{3(10\sqrt{127} - 109)}\right)^{\frac{3}{4}} \sec h\left[\frac{\sqrt{127} - 10}{3}\left(x - \frac{118 - 10\sqrt{127}}{10\sqrt{127} - 109}\right)\right]. \]  
(5.5)

The experiment is run from \( t = 0 \) to \( t = 10 \) and values of the invariant quantities and error norms are listed in Table 2. Table 2 shows that values of invariant are almost constant as the time increases. It is noticeably seen from the table that the error norms obtained by our method are found much better than the others for \( h = 0.8 \) and the invariant \( I_E \) changes from its initial value by less than \( 1.10 \times 10^{-3} \) for \( h = 1.6 \) and \( 8 \times 10^{-7} \) for \( h = 1.0 \) whereas the changes of invariant approach to zero for \( h = 0.8 \) and 0.7. Also, we have found out error norms \( L_2 \) and \( L_\infty \) are obtained sufficiently small during the computer run. Therefore our method is sensibly conservative.

For visual representation, the simulations of single soliton for values of \( h = 0.8, \Delta t = 0.005 \) at times \( t = 1, 2, 3, \ldots, 10 \) are illustrated in Figure 3. It is understood from this figure that the numerical scheme performs the motion of propagation of a single solitary wave, which moves to the right at nearly unchanged speed and conserves its amplitude and shape with increasing time. The amplitude is 1.5528307 at \( t = 0 \) and located at \( x = 0.0 \), while it is 1.5528307 at \( t = 10 \) and located at \( x = 14.4 \). The absolute difference in amplitudes at times \( t = 0 \) and \( t = 10 \) is \( 1.552 \times 10^{-4} \) so that there is a little change between amplitudes. Error distributions at time \( t = 1, 2, \ldots, 10 \) are shown graphically in Figure 4. As it is seen, the maximum errors are between \( 1 \times 10^{-4} - 6 \times 10^{-5} \) and occur around the central position of the solitary wave.

Table 2. Invariant and error norms for single soliton for \( p = 4 \) over the interval \(-40 \leq x \leq 240\).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( I_E )</th>
<th>( L_2 \times 10^3 )</th>
<th>( L_\infty \times 10^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t = 1 )</td>
<td>13.5736473</td>
<td>8.49940</td>
<td>4.09076</td>
</tr>
<tr>
<td>( t = 5 )</td>
<td>13.5549680</td>
<td>17.44637</td>
<td>7.88485</td>
</tr>
<tr>
<td>( t = 10 )</td>
<td>13.5725435</td>
<td>31.25806</td>
<td>1185995</td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t = 1 )</td>
<td>13.5658564</td>
<td>0.17182</td>
<td>0.09781</td>
</tr>
<tr>
<td>( t = 5 )</td>
<td>13.5658605</td>
<td>0.25653</td>
<td>0.12195</td>
</tr>
<tr>
<td>( t = 10 )</td>
<td>13.5658562</td>
<td>0.39970</td>
<td>0.18451</td>
</tr>
<tr>
<td>0.8</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>( t = 1 )</td>
<td>13.5658560</td>
<td>0.07310</td>
<td>0.04803</td>
</tr>
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<td>0.12211</td>
<td>0.05817</td>
</tr>
<tr>
<td>( t = 10 )</td>
<td>13.5658560</td>
<td>0.20207</td>
<td>0.09192</td>
</tr>
</tbody>
</table>

[21] 13.5656656 | 154.3 | 58.39 |

[45] 13.5637615 | 1.684 | 0.7812 |

<table>
<thead>
<tr>
<th>( h )</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td></td>
<td></td>
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<tr>
<td>( t = 1 )</td>
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<td>0.04322</td>
<td>0.02816</td>
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<td>0.06912</td>
<td>0.03244</td>
</tr>
<tr>
<td>( t = 10 )</td>
<td>13.5658553</td>
<td>0.11088</td>
<td>0.05002</td>
</tr>
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6. Conclusion

In this paper, we have derived soliton solutions of the generalized Rosenau-Kawahara-RLW equation based on septic B-spline collocation finite element method. Numerical solutions of the 1-soliton are obtained for two cases. The newly suggested scheme produces very highly accurate results and the conserved quantities are almost constant during the simulation both $p = 2$ and $p = 4$. The main energy conservative property is preserved by the current numerical scheme. The method is shown to be
unconditionally stable. Also convergence of full discrete scheme is generated. In summary, the derived method is highly accurate and appropriate for simulated solution of the generalized Rosenau-Kawahara-RLW equation. Therefore, we can say that the suggested method is also very efficient for solving nonlinear partial differential equations.

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REFERENCES


