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Spectral solution of fractional fourth order partial integro-differential equations

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Abstract In this paper, mixed spectral method is applied to solve the fractional fourth order partial integro-differential equations together with weak singularity. Eigenfunctions of the fourth order self-adjoint positive-definite differential operator are used for the discretization of spatial variable and its derivatives. Also, shifted Legendre polynomials are applied to the discretization of time variable. Numerical results are presented for some problems to demonstrate the usefulness and accuracy of this approach. The method is easy to apply and produces very accurate numerical results.

Keywords. Integro-differential equation, Weakly singular kernel, Collocation method, Operational matrix.
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1. Introduction

New spectral solution is suggested for the fractional fourth order partial integro-differential equations:

$$ {}^{c}D_t^{\alpha}U(x,t) + \int_0^t (t-s)^{-\beta}\frac{\partial^4}{\partial x^4}U(x,s)ds = f(x,t), \quad x\in(0,a),\ t\geqslant 0, \tag{1.1}$$

subject to the boundary condition

$$\begin{cases} U(0,t) = U(a,t) = 0, \\ U_{xx}(0,t) = U_{xx}(a,t) = 0, \end{cases} \tag{1.2}$$

and the initial condition

$$ U(x,0) = \psi(x), \tag{1.3}$$

where $0<\alpha$ and $\beta<1$ and ${}^{c}D_t^{\alpha}$ is the Caputo-type fractional derivative of order α with respect to t [5]. It can be seen that, in Eq. (1.1), the kernel function has weak singularity [27] that it induces sharp transitions in the solutions [32]. Mathematical modelling and simulation of real-life problems usually result in functional equations,

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including partial differential equations (PDEs), integral and integro-differential equations and others. However, many mathematical formulations of physical phenomena contain integro-differential equations. The Eq. (1.1), can be found in the modeling of heat flow in materials with memory [16], linear viscoelastic mechanics [7], and modeling of thin beams and plates, strain gradient elasticity, and phase separation in binary mixtures [9]. There are few studies in the literature about the numerical solution of fractional fourth order integro-differential equations. But Xu [29, 30] used finite element method to solve the parabolic partial integro-differential equations in 1993. Recently, Zhang et al. [31] developed Quintic B-spline collocation method for fourth order partial integro-differential equations with a weakly singular kernel and Sweilam [25] used variational iteration method for this problem.

There is rapidly increasing interest in the study of fractional differential equations, because recent investigations in science and engineering have indicated that the dynamics of many systems can be described more accurately using differential equations of non-integer order [14]. Compared with the classical integer order one, fractional derivatives are more excellent instruments for the description of memory and hereditary properties of various materials and processes [2], because, in the integer order derivatives, such effects are in fact neglected. Also, the advantage of fractional derivatives becomes apparent in modeling mechanical and electrical properties of real materials as well as in describing the rheological properties of rocks, and in many other fields(for details,see [5, 11, 20]). So for the foregoing reasons, we consider the equation with fractional derivative on time variable. Numerous methods have been devoted to the solution of various fractional problems [2] such as Adomian's decomposition method [17], He's variational iteration method [12], homotopy perturbation method [26], collocation method [21], Galerkin method [10], and others [8, 22], among which spectral methods have a highlighted efficiency for solving fractional problems.

Spectral methods play an important role in recent studies for the numerical solution of differential equations [4, 6]. These methods have shown their efficiency and convergence in solving numerous problems [3, 4, 23]. They convert a differential equation into an algebraic one, which not only simplifies the problem, but also accelerates the computation [4, 18]. Many authors have presented the operational matrix of fractional differential or integration operators based on various orthogonal functions and have applied them to solve various fractional differential equations. In [22], Saadatmandi and Dehghan proposed an operational matrix of derivatives with fractional order for Legendre polynomials and used it for solving fractional differential equation. In [8], Doha et al. applied Chebyshev spectral method for solving fractional differential equations. Also, enthusiastic readers can refer to Wu [28] and Kilicman [15].

In this article, first, eigenfunctions of the fourth order self-adjoint positive-definite differential operator [24] are used for the discretization of spatial variable and reduction of the problem to a system of integro-differential equation; then, Legendre collocation method is applied to solve this system. It is notable that we use roots of Legendre polynomial for collocation points. The swell trait of this approach is in its flexibility and ability for the problems with high dimensions.

This paper is organized as follows. In sections 2, some basic concepts are reviewed. In section 3, the method is implemented shown. In section 4, some numerical results are reported to show how this technique works efficiently. The final section presents the conclusion.

2. Preliminaries

In this section, some notations, definitions, and preliminary facts are presented that will be used further in this work.

2.1. Caputo fractional derivative.

Definition 2.1. Caputo fractional order derivative [5] is defined as:

$$ {}^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}, \tag{2.1} $$

where $\alpha > 0$ is the order of derivative and n is the smallest integer of greater than or equal to α.

For the Caputo derivative, we have [5]:

$$ {}^cD_t^\alpha C = 0, \quad \text{where } C \text{ is constant} \tag{2.2} $$

and

$$ {}^cD_t^\alpha t^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil\alpha\rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil\alpha\rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor\alpha\rfloor. \end{cases} \tag{2.3} $$

The ceiling function $\lceil\alpha\rceil$ is used to denote the smallest integer greater than or equal to α and the floor function $\lfloor\alpha\rfloor$ to denote the largest integer less than or equal to α. Also, $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Recall that, for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of an integer order. The Caputo fractional differentiation is a linear operation

$$ {}^cD_t^\alpha(\lambda f(t) + \mu g(t)) = \lambda {}^cD_t^\alpha f(t) + \mu {}^cD_t^\alpha g(t), \tag{2.4} $$

where λ and μ are constant.

2.2. Shifted Legendre polynomials and its operational matrix.

Let $I = [-1, 1]$, and $L_k(x)$ be the Legendre polynomial [1] with degree k,

$$ L_k(x) = \frac{1}{2^n} \sum_{i=0}^{k} \binom{k}{i}^2 (x-1)^{k-i}(x+1)^i, \qquad k = 0, 1, \ldots. \tag{2.5} $$

They can be also determined by the aid of the following recurrence formula:

$$ L_{k+1}(x) = \frac{(2k+1)}{k+1} x L_k(x) - \frac{k}{k+1} L_{k-1}(x), \; k = 1, 2, \ldots, \tag{2.6} $$

where $L_0(x) = 1$, and $L_1(x) = x$. We use these polynomials on the interval $x \in [0, t_f]$; so, we define the so-called shifted Legendre polynomials (SL polynomials) by introducing the change of variable $x = \frac{2t}{t_f} - 1$. Let SL polynomials $L_i(\frac{2t}{t_f} - 1)$ be denoted by $L_{t_f,i}(t)$ and satisfy the orthogonality relation

$$\int_0^{t_f} L_{t_f,i}(t) L_{t_f,j}(t) dt = \begin{cases} 0, & \text{for } i \neq j, \\ \frac{t_f}{2j+1}, & \text{for } i = j. \end{cases} \tag{2.7}$$

They may be obtained as follows:

$$L_{t_f,k}(t) = \sum_{i=0}^{k} c_i^{(k)} t^i, \qquad k = 0, 1, 2, \dots. \tag{2.8}$$

where $c_i^{(k)} = (-1)^{k-i} \binom{k+i}{i} \binom{k}{k-i} t_f^{-i}$.

In the single domain, a function $f(t)$, square integrable in $[0, t_f]$, can be expressed in terms of SL polynomials as

$$f(t) = \sum_{j=0}^{\infty} c_j L_{t_f,j}(t), \tag{2.9}$$

where the coefficients c_j are obtained from

$$c_j = \frac{2j+1}{t_f} \int_0^{t_f} f(t) L_{t_f,j}(t) dt, \quad j = 0,\ 1,\ 2,\ \dots. \tag{2.10}$$

Actually, in numerical methods, only the first (N+1)-terms of SL polynomials are considered. So we have:

$$f_N(t) \simeq \sum_{j=0}^{N} c_j L_{t_f,j}(t). \tag{2.11}$$

Now, the following lemma can be presented an upper bound for estimating the error.

Lemma 2.2. *Let the function $f : [t_0, t_f] \to R$ is $(N+1)$ times continuously differentiable and $\boldsymbol{P}^{N+1} = Span\{L_{t_f,0}(t), L_{t_f,1}(t), \dots, L_{t_f,N}(t)\}$, than we can calculate following upper bound for approximation $f_N(t)$,*

$$\|f(t) - f_N(t)\|_{2,t_f} \leq \frac{\sqrt{t_f} M C^{N+1}}{(N+1)!}, \tag{2.12}$$

where $M = \max_{t \in [t_0, t_f]} f^{(N+1)}(t)$ and $C = \max\{t_f - t_0, t_0\}$.

Proof. Similar to proof of [11, Theorem 2.1.8]. □

We set $\phi(t) = [L_{t_f,0}(t), L_{t_f,1}(t), \dots, L_{t_f,N}(t)]^T$. then, the derivative of vector $\phi(t)$ can be expressed by:

$$\frac{d}{dt}\phi(t) = D^{(1)}\phi(t), \tag{2.13}$$

where $D^{(1)}$ is the $(N+1)\times(N+1)$ operational matrix of the derivative and given by:

$$\{D^{(1)}\}_{i,j} = \begin{cases} \frac{2}{t_f}(2j+1), & \text{for } j = i-s, \begin{cases} s=1,3,...,n, & \text{if } N \text{ is odd,} \\ s=1,3,...,N-1, & \text{if } N \text{ is even,} \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

For example, if $N = 4$ then we have

$$D^{(1)} = \frac{2}{t_f}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 \\ 0 & 3 & 0 & 7 & 0 \end{pmatrix}.$$

Caputo fractional differential operator of order $\alpha > 0$ of the vector $\phi(t)$ can be expressed by

$$^cD_t^{\alpha}\phi(t) \simeq {}^c\mathbf{D}^{(\alpha)}\phi(t), \tag{2.14}$$

where $^c\mathbf{D}^{(\alpha)}$ is the $(N+1)\times(N+1)$ SL operational matrix of Caputo fractional differential operator of order α.

Theorem 2.3. *Let $^c\boldsymbol{D}^{(\alpha)}$ is the $(N+1)\times(N+1)$ SL operational matrix of Caputo fractional differential operator of order α then the elements of this matrix are obtained as:*

$$\{{}^c\boldsymbol{D}^{(\alpha)}\}_{i,j=0}^{N} =$$

$$\begin{cases} 0, & \text{if } i = 0,1,\dots,\lceil\alpha\rceil-1, \\ \sum_{k=\lceil\alpha\rceil}^{i}\sum_{l=0}^{j} c_k^{(i)}c_l^{(j)}\frac{(2j+1)\Gamma(k+1)t_f^{l+k-\alpha}}{\Gamma(k+1-\alpha)(l+k-\alpha+1)}, & \text{if } \lceil\alpha\rceil \le i \le N. \end{cases} \tag{2.15}$$

Proof. The proof is similar to [22, Theorem (1)]. □

3. Application of the scheme

In this section, the scheme is illustrated. First, by means of eigenfuntions technique [24], equations (1.1)-(1.3) are transformed into a system of integro-differential equations with initial condition (1.3).

Let

$$L = \frac{\partial^4}{\partial x^4}, \tag{3.1}$$

be defined on the domain $D(A) = \{w \in H^4 | w \text{ satisfies } (1.2)\}$ where $H = L^2([0,a])$ and $w \in H^4$ means that w and its derivatives up to order 4 are elements of H. The operator L is self-adjoint compact and positive on D(A), which leads to a countable infinite set of positive real eigenvalues $\{\lambda_m = (\frac{m\pi}{a})^4\}$ and corresponds to the set of orthonormal eigenfunctions $\{\phi_m(x) = \sqrt{\frac{2}{a}}\sin(\frac{m\pi}{a})x\}$.

Equations (1.1) can be transformed by expanding the function $U(x,t)$ and $f(x,t)$ in terms of the finite eigenfunctions $\{\phi_m(x)\}_{m=0}^{M}$ of the operator L {Eq. (3.1)},

$$U(x,t) = \sum_{m=1}^{M} u_m(t)\phi_m(x) = \vec{U}^{\mathrm{T}}(t)\Phi(x), \tag{3.2}$$

$$f(x,t) = \sum_{m=1}^{M} < f, \phi_m > \phi_m(x) = \vec{F}^{\mathrm{T}}(t)\Phi(x), \tag{3.3}$$

where

$$\vec{U}^{\mathrm{T}}(t) = [u_1(t), u_2(t), \dots, u_M(t)], \tag{3.4}$$

$$\Phi^{\mathrm{T}}(x) = [\phi_1(x), \phi_2(x), \dots, \phi_m(x)], \tag{3.5}$$

$$\vec{F}^{\mathrm{T}}(t) = [< f, \phi_1 >, < f, \phi_2 >, \dots, < f, \phi_M >], \tag{3.6}$$

which $< f(x,t), \phi_m(x) >= \int_0^a f(x,t)\phi_m(x)dx$ and T stands for a vector transpose. Substituting (3.2) and (3.3) in Eq. (1.1) gives:

$$ {}^cD_t^{\alpha}\vec{U}^{\mathrm{T}}(t)\Phi(x) + \int_0^t (t-s)^{-\beta}\frac{\partial^4}{\partial x^4}\vec{U}^{\mathrm{T}}(s)\Phi(x)ds = \vec{F}^{\mathrm{T}}(t)\Phi(x), \tag{3.7}$$

$$\Rightarrow \ {}^cD_t^{\alpha}\vec{U}^{\mathrm{T}}(t)\Phi(x) + \int_0^t (t-s)^{-\beta}\vec{U}^{\mathrm{T}}(s)\Lambda_M\Phi(x)ds = \vec{F}^{\mathrm{T}}(t)\Phi(x), \tag{3.8}$$

which Λ_M is a $M \times M$ diagonal matrix that is obtained from eigenfunction definition with the corresponding eigenvalues of $\frac{\partial^4}{\partial x^4}$ on its diagonal; i.e.

$$\Lambda_M = \text{ diag } [\lambda_m = (\frac{m\pi}{a})^4], \quad m = 1, 2, \dots, M.$$

Taking the dot product of resulting expression (3.8) by $\Phi(x)$, and integrating with respect to x over $[0,a]$, result in:

$$D_c^{\alpha}\vec{U}^{\mathrm{T}}(t) + \int_0^t (t-s)^{-\beta}\vec{U}^{\mathrm{T}}(s)ds\Lambda_M = \vec{F}^{\mathrm{T}}(t). \tag{3.9}$$

For appropriate initial condition, the initial condition (1.3) can be transformed in a similar process. To this end, substituting (3.2) in (1.3) and expanding $\psi(x)$ by the same eigenfunctions lead to:

$$\sum_{m=0}^{M} u_m(0)\phi_m(x) = \sum_{m=0}^{M} < \psi(x), \phi_m(x) > \phi_m(x), \tag{3.10}$$

or in the vector form:

$$\vec{U}^{\mathrm{T}}(0)\Phi(x) = \vec{\Psi}^{\mathrm{T}}.\Phi(x), \tag{3.11}$$

which $\vec{\Psi}^{\mathrm{T}} = [< \psi(x), \phi_1(x) >, < \psi(x), \phi_2(x) >, \dots, < \psi(x), \phi_M(x) >]$. Taking the dot product of resulting expression (3.11) by $\Phi(x)$ and integrating with respect to x over $[0,a]$ provide

$$\vec{U}(0) = \vec{\Psi}. \tag{3.12}$$

As the second stage of our scheme, we solve the new fractional integro-differential equations (3.9), (3.12) via SL polynomials collocation methods. Suppose that the solution is needed on interval $[0, t_f]$ and

$$\vec{U}(t) = A.\vec{L}(t), \tag{3.13}$$

where $A = [A_{ij}]_{M\times(N+1)}$ is the unknown coefficient matrix and

$$\vec{L}(t) = [L_{t_f,0}(t), L_{t_f,1}(t), \ldots, L_{tf,N}(t)]^{\mathrm{T}},$$

is vector of SL polynomials. By transposing expression (3.9) and substituting (3.13) in it, we get:

$${}^{c}D_t^{\alpha} A.\vec{L}(t) + \Lambda_M. \int_0^t (t-s)^{-\beta} A.\vec{L}(s)ds = \vec{F}(t) \tag{3.14}$$

$$\Rightarrow A\, {}^{c}\mathbf{D}^{(\alpha)}\vec{L}(t) + \lambda A \int_0^t (t-s)^{-\beta}\vec{L}(s)ds = \vec{F}(t), \tag{3.15}$$

which ${}^{c}\mathbf{D}^{(\alpha)}$ is the operational matrix of fractional derivative of SL polynomials obtained in Theorem 2.3. Also, by substituting (3.13) in Eq. (3.12), we get:

$$A.\vec{L}(0) = \vec{\Psi}, \tag{3.16}$$

which makes M equations. Paying attention to the point that $M(N+1)$ unknowns exist and M equations are obtained from initial conditions (3.16), so we need MN equation. To find the required equations, we collocate Eqs. (3.15) at N points. For suitable collocation point, we use the roots of SL polynomial $L_{tf,N}$. These equations together with Eqs. (3.16) generate $M(N+1)$ linear equations which can be solved using various known methods. Consequently, $U(x,t)$ given in Eq. (1.1) can be approximated.

Note: We use Gaussian numerical integration in Eq. 3.15 to solve singularity.

4. Illustrative examples

In order to illustrate the method and demonstrate its simplicity, efficiency, and accuracy, some numerical examples are presented. The numerical algorithms are programmed in Maple.

4.1. **Example.** As the first example, consider Eqs. (1.1)-(1.3) for $\beta = 1/2$, $x \in [0, 2]$, $t_f = 1$ and the exact solution

$$U(x,t) = x^3(x-2)^3t^2.$$

In Table 1–3 and 4, absolute errors of $|U(x,t) - U_{M,N}(x,t)|$ are shown for $N = 4, M = 20$, and $\alpha = \frac{1}{2}, \frac{3}{4}, \frac{8}{9}$ and 1, respectively, from which interesting information could be extracted. It can be observed that, if α approach to 1, then absolute errors become much smaller and the greatest errors are usually relevant to $x = 1$, namely the middle point of interval $[0, 2]$ and also absolute errors are symmetric with respect to $x = 1$. In Figure 1, the absolute error of the solution at $t = 1$ for $M = 20, N = 4$ and $\alpha = \frac{1}{2}$ on interval $[0, 2]$ is plotted, which shows the accuracy of the method. In Figure 2,

Table 1. Absolute errors at $\alpha = \frac{1}{2}$ and $M = 20, N = 4$ for Example 4.1.

points	x=0.25	x=0.5	x=0.75	x=1	x=1.25	x=1.5	x=1.75
t=0.5	0.00009	0.00016	0.00021	0.00023	0.00021	0.00016	0.00009
t=0.75	0.0001	0.00021	0.00027	0.00029	0.00027	0.00021	0.0001
t=1	0.0002	0.0004	0.00064	0.00072	0.00064	0.0004	0.0002

Table 2. Absolute errors at $\alpha = \frac{3}{4}$ and $M = 20, N = 4$ for Example 4.1.

points	x=0.25	x=0.5	x=0.75	x=1	x=1.25	x=1.5	x=1.75
t=0.5	0.0001	0.00018	0.00024	0.00026	0.00024	0.00018	0.0001
t=0.75	0.00006	0.00013	0.00017	0.00018	0.00017	0.00013	0.00006
t=1	0.00008	0.00014	0.00023	0.00027	0.00023	0.00014	0.00008

Table 3. Absolute errors at $\alpha = \frac{8}{9}$ and $M = 20, N = 4$ for Example 4.1.

points	x=0.25	x=0.5	x=0.75	x=1	x=1.25	x=1.5	x=1.75
t=0.5	0.00006	0.0001	0.00014	0.00015	0.00014	0.0001	0.00006
t=0.75	0.000012	0.000043	0.000052	0.000053	0.000052	0.000043	0.000012
t=1	0.000009	0.000001	0.000033	0.000053	0.000033	0.000001	0.000009

Table 4. Absolute errors at $\alpha = 1$ and $M = 20, N = 4$ for Example 4.1.

points	x=0.25	x=0.5	x=1	x=1.5	x=1.75
t=0.5	2.3×10^{-6}	3×10^{-6}	2.2×10^{-6}	3×10^{-6}	2.3×10^{-6}
t=0.75	5.3×10^{-6}	6.7×10^{-6}	5×10^{-6}	6.7×10^{-6}	5.3×10^{-6}
t=1	9×10^{-6}	1.2×10^{-5}	9×10^{-6}	1.2×10^{-5}	9×10^{-6}

both exact and approximated solutions are plotted for $\alpha = 1$ at $t = 1$, in which the efficiency and accuracy of our scheme could be easily observed.

4.2. **Example.** Consider the problem Eqs. (1.1)-(1.3) for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}$, $x \in [0, 1]$, $t_f = 1$ and the exact solution

$$U(x, t) = t^2 \sin(\pi x).$$

The absolute errors of $|U(x, 1) - U_{M,N}(x, 1)|$ for $M = 20, N = 9$ are illustrated in Table 5 and plotted in Figure 3, as the evidence for simplicity and accuracy of the method. Also, the comparison plot of exact solution and various approximated solutions for different M, N at $t = 1$, namely $U(x, 1), U_{M,N}(x, 1)$, is indicated in Figure 4 and represents the convergence of the scheme.

4.3. **Example.** For the last example, the problem Eqs. (1.1)-(1.3) for $\alpha = \frac{2}{3}$, $\beta = \frac{1}{3}$, $x \in [0, 2]$, $t_f = 1$ and the exact solution

$$U(x, t) = t \cos(\frac{\pi}{2}(x - 1)),$$

are solved. In Table 6, absolute errors of $|U(x, 1) - U_{M,N}(x, 1)|$ for $M = 50, N = 7$ are presented and plotted in Figure 5, which shows the existence of good approximation

FIGURE 1. Absolute errors of the solution at $t = 1$ for $M = 20, N = 4$ and $\alpha = \frac{1}{2}$ on interval $[0, 2]$ for Example 4.1.

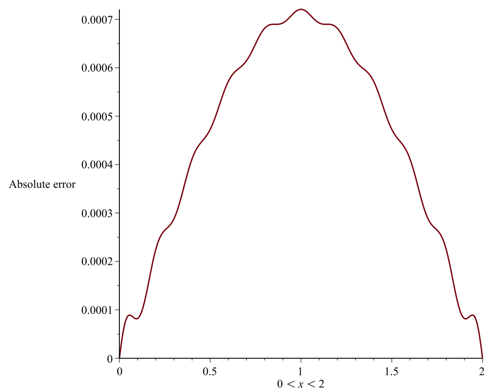

FIGURE 2. Comparison of exact solution and approximated solution for $\alpha = 1$ at $t = 1$ for Example 4.1.

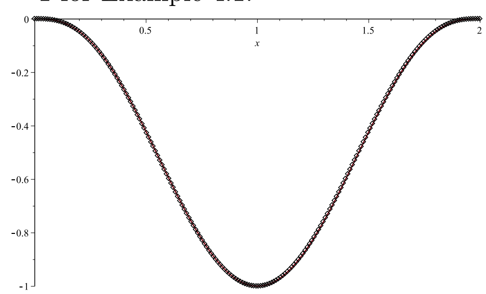

FIGURE 3. Absolute errors of the solution at $t = 1$ for $M = 20, N = 4$ and $\alpha = \frac{1}{3}$ on interval $[0, 1]$ for Example 4.2.

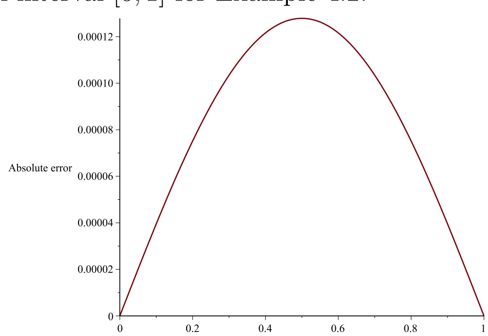

FIGURE 4. Comparison plot of u(x,1) and some obtained solutions from the scheme at $t = 1$, namely $U_{M,N}(x, 1)$, for $\alpha = \frac{1}{3}$ on interval $[0.45, 0.55]$ for Example 4.2.

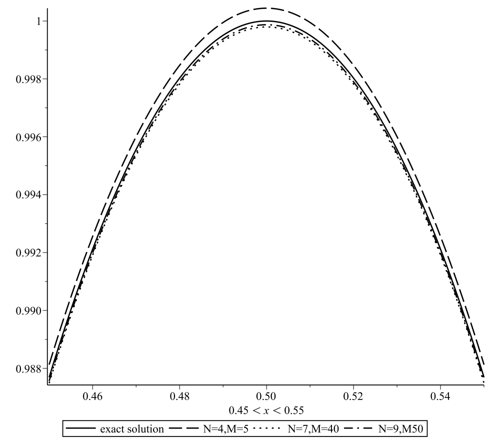

FIGURE 5. Absolute errors of the solution at $t = 1$ for $M = 50, N = 4$ and $\alpha = \frac{2}{3}$ on interval $[0, 2]$ for Example 4.3.

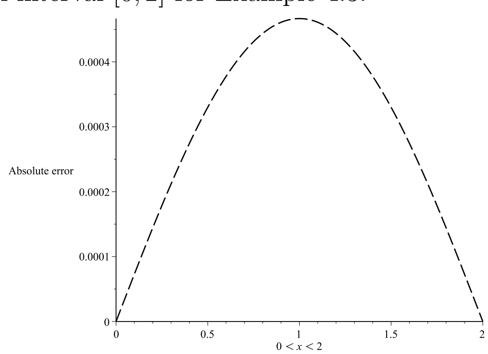

TABLE 5. Absolute errors of the solution at $\alpha = \frac{1}{3}$ and $M = 20, N = 9$ for Example 4.2.

points	x=0.1	x=0.3	x=0.5	x=0.7	x=0.9
t=0.25	1×10^{-6}	4×10^{-6}	5×10^{-6}	4×10^{-6}	1×10^{-6}
t=0.5	2×10^{-6}	6×10^{-6}	8×10^{-6}	6×10^{-6}	2×10^{-6}
t=0.75	1×10^{-6}	4×10^{-6}	6×10^{-6}	4×10^{-6}	1×10^{-6}
t=1	3×10^{-5}	1×10^{-4}	1.2×10^{-4}	1×10^{-4}	3×10^{-5}

for the problem. Comparison of exact solution and obtained solution is portrayed in Figure 6.

FIGURE 6. Comparison plot of u(x,1) and $U_{50,4}(x,1)$ at $\alpha = \frac{2}{3}$ on interval $[0.75, 1.25]$ for Example 4.3.

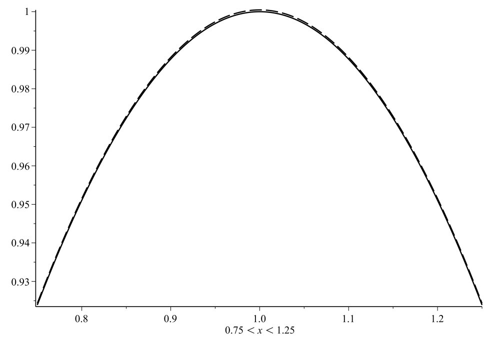

TABLE 6. Absolute errors of solution at $\alpha = \frac{2}{3}$ and $M = 50, N = 7$ at $t_f = 1$ for Example 4.3.

points	x=0.25	x=0.5	x=0.75	x=1	x=1.25	1.5	1.75
t=0.25	0.00036	0.00067	0.00087	0.00094	0.00087	0.00067	0.00036
t=0.5	0.00001	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002
t=0.75	0.00006	0.00012	0.00015	0.00017	0.00015	0.00012	0.00006
t=1	0.00017	0.00033	0.00043	0.00046	0.00043	0.00033	0.00017

5. Conclusions

A combined spectral method has been presented here to solve Eq. (1.1) numerically. The method uses a combination of a fourth order self-adjoint operator of a fractional order of eigenfunctions and shifted Legendre polynomials. This method showed to be convergent and also the solutions obtained from this method are well approximating the exact solutions and also giving the exact ones for some problems. Moreover, It is easy to apply this method in practice.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, Dover Publications, New York, 1992.
[2] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional calculus, models and numerical methods*, World Scientific, Singapure, 2012.
[3] C. Bernardi and Y. Maday, *Spectral method*, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, vol. 5, North-Holland, Amesterdam, 1997, 209–485.
[4] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral methods*, Fundamentals in Single Domains, Springer, Berlin, 2006.
[5] M. Caputo, *Elasticita e dissipazione*, Bologna, Italy, Zanichelli, 1969.
[6] Y. Chen, X. Li, and T. Tang, *A note on Jacobi spectral-collocation methods for weakly singular Volterra integral equations with smooth solutions*, J. Comput. Math., *31* (2013), 47–56.
[7] R. M. Christensen, *Theory of viscoelasticity*, Academic Press, New York, 1971.

[8] E. H. Doha, A. H. Bharwy, and S. S. Ezz-Eldien. *A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order*, Cumput. Math. Appl., *62* (2011), 2364–2373

[9] G. Engel, K. Garikipati, T. J. R. Hughes, M. G. Larson, L. Mazzei, and R. L. Taylor, *Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity*, Comput. Methods Appl. Mech. Eng., *191* (2002), 3669–3750.

[10] V. J. Ervin and J. P. Roop, *Variational formulation for the stationary fractional advection dispersion equation*, Numer. Methods Partial Differential Equations, *22* (2005), 558–576.

[11] R. Hiflfr, *Applications of fractional calculus in Physics*, Publishing company singapore, world scientific, 2000.

[12] M. Inc, *The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method*, J. Math. Anal. Appl., *345* (2008), 476–484.

[13] Y. J. Jiang, *On spectral methods for Volterra-type integro-differential equations*, J. Comput. Appl. Math., *230* (2009), 333–340.

[14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.

[15] A. Kilicman and Z. A. A. Al Zhour, *Kronecker operational matrices for fractional calculus and some applications*, Appl. Math. Comp, *187* (2007), 250–265.

[16] R. K. Miller, *An integro-differential equation for rigid heat conductors with memory*, J. Math. Anal. Appl., *66* (1978), 313–332.

[17] S. Momani and M. A. Noor, *Numerical methods for fourth-order fractional integro-differential equations*, Appl. Math. Comput., *182* (2006), 754–760.

[18] A. Pedas and E. Tamme, *Piecewise polynomial collocation for linear boundary value problems on fractional differential equations*, J. Comput. Appl. Math., *236* (2012), 3349–3359.

[19] G. M. Phillips, *Interpolation and approximation by polynomials*, Springer, New York, 2003.

[20] I. Pudlumbny, *Fractional differential equations*, Academic press, New York, 1990.

[21] E. A. Rawashdeh, *Numerical solution of fractional integro-differential equations by collocation method*, Appl. Math. Comput., *176* (2006), 1–6.

[22] A. Saadatmandi and M. Dehghan, *A new operational matrix for solving fractional-order differential equations*, Comput. Math. Appl., *59* (2010), 1326–1336.

[23] J. Shen and T. Tang, *Spectral and high-order methods with applications*, SciencePress, Beijing, 2006.

[24] I. sadek, T. Abualarub, and M. Abukhaled, *A computational method for solving optimal control of a system of parallel beams using Legendre wavelets*, Mathematical and computer modelling, *45* (2007), 1253–1264.

[25] N. H. Sweilam, *Fourth order integro-differential equations using variational iteration method*, Comput. Math. Appl., *54* (2007), 1086–1091.

[26] N. H. Sweilam, M. M. Khader, and R. F. Al-Bar, *Numerical studies for a multi-order fractional differential equation*, Phys. Lett. A, *371* (2007), 26–33.

[27] T. Tang, *A finite difference scheme for a partial integro-differential equation with a weakly singular kernel*, Appl. Numer. Math., *2* (1993), 309–319.

[28] J. L. Wu, *A wavelet operational method for solving fractional partial differential equations numerically*, Appl. Math .Comput., *214* (2009), 31–40.

[29] D. Xu, *On the discretization in time for a parabolic integro-differential equation with a weakly singular kernel I: smooth initial data*, Appl. Math. Comput., *58* (1993), 1–27.

[30] D. Xu, *Finite element methods for the nonlinear integro-differential equations*, Appl. Math. Comput., *58* (1993), 241–273.

[31] H. Zhang, X. Han, and X. yang, *Quintic B-spline collocation method for fourth order partial integro-differential equations with a weakly singular kernel*, Appl. Math. Comput., *219* (2013), 6565–6575.

[32] X. Yang, D. Xu, and H. Zhang, *Quasi-wavelet based numerical method for fourthorder partial integro-differential equations with a weakly singular kernel*, International Journal of Computer Mathematics, *88* (2011), 3236–3254.