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Accurate splitting approach to characterize the solution set of boundary layer problems

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Abstract The boundary layer (BL) is an important concept and refers to the layer of fluid in the immediate vicinity of a bounding surface where the effects of viscosity are significant. This paper studies singularly perturbed fractional differential equations where the fractional derivatives are defined in the Caputo sense. The solution of such equations, with appropriate boundary conditions, displays BL behavior. The solution out of the BL is estimated by the solution of the reduced problem and the layer solution is approximated by means of a modified truncated Chebyshev series. The coefficients of the truncated series are evaluated using a novel operational matrix technique. Moreover, the stability and the error analysis of the proposed method are analyzed. Several examples illustrate the validity and applicability of the method.

Keywords. Singular perturbation, Boundary value problem, Shifted Chebyshev polynomial, Operational matrix, Boundary layer, Caputo fractional derivative.

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1. Introduction

Singularly perturbed equations are ubiquitous in real-life phenomena and pose difficulties for numerical approximations. Examples of application are geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, optimal control and many others [1]. We consider this type of problem where perturbations are operative over narrow regions across which the dependent variables undergo very rapid changes. These regions often adjoin the boundaries of the domain of interest, because a parameter with small numerical value multiplies the derivative of highest order. Such systems are referred, respectively as boundary, edge and skin layers in fluid mechanics, solid mechanics, and electrical applications respectively. We find many physical systems having sharp changes inside the domain of interest. The narrow regions where these changes occur are referred to as shock layers, in fluid and solid

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mechanics, transition points, in quantum mechanics, or Stokes lines and surfaces, in mathematics. The rapid variations cannot be handled by slow scales; however, they can be solved by means of fast, magnified, or stretched scales. For detailed discussion on the analytical and numerical treatment of such problems readers may refer to the works of O'Malley [15], Doolan et al. [5], Roos et al. [18], Nayfeh [12, 13] and Miller et al. [10].

Fractional calculus (FC) has been in the mind of mathematicians during 300 years. In the last decades, FC became the object of increasing interest, due to its applications in different areas of science and engineering. Recently, it was verified that differential equations involving derivatives of non-integer order can model adequately various physical phenomena with long range memory effects [6, 16]. The book by Oldham and Spanier [14] played a key role in the development of the subject. Some fundamental results related to fractional differential equations may be found in the seminal books of Miller and Ross [11], Kilbas et al. [9] and Baleanu et al. [2].

Despite the large number of studies dedicated to the solution of the singularly perturbed boundary value problems (e.g., Attili [1] and the survey by Kadalbajoo and Gupta [8]), we find only a few papers addressing fractional singularly perturbed boundary value problems (e.g., Bijura [4], Roop [17] and the references therein). Therefore, the topic remains an open area that deserves further research.

This paper focuses on the singularly perturbed boundary value problems of fractional order

$$\epsilon D_{*x}^{m\alpha} y(x) + \sum_{i=1}^{m} f_i(x,\epsilon) D_{*x}^{(m-i)\alpha} y(x) = g(x,\epsilon), \quad x \in [0,1], \ \ 0 < \alpha \leq 1, \tag{1.1}$$

with the boundary conditions

$$y^{(i_k)}(0) = a_k, \ \ 0 \leq k \leq i-1, \quad y^{(j_k)}(1) = b_k, \ \ 0 \leq k \leq m-i-1, \tag{1.2}$$

where $i-1 < i\alpha \leq i$, $i = 1, \ldots, m$, and ϵ is a small parameter. The boundary conditions (1.2) constitute integer order derivatives of y with respect to x, $i_k, \ j_k \in \{0, 1, \ldots, m-1\}$, with $i_{k1} \neq i_{k2}$ and $j_{k1} \neq j_{k2}$, if $k1 \neq k2$. Furthermore, $f_i(x,\epsilon), i = 1, \ldots, m$, and $g(x,\epsilon)$ are continuous functions on $[0,1] \times [0,\epsilon]$ and $D_{*x}^{i\alpha}$, $i = 0, 1, \ldots, m$, denotes the Caputo fractional derivative that is defined in the follow-up.

It is known that the solution of this kind of problem has BL at the endpoints [5, 10, 12, 13, 15, 18]. There is a wide class of asymptotic expansion methods available for solving such type of problems. However, often we find some in applying the asymptotic expansion methods, such as finding the appropriate asymptotic expansions in the inner and outer regions, since they are not routine exercises. In this paper, we construct an approximate solution for (1.1)-(1.2) using spectral methods and considering the division of the interval $[0,1]$. The truncated Chebyshev series is adopted for approximating the exact solution inside the layers. The solution out of BL is estimated by the solution of the reduced problem. This procedure gives very accurate results for small values of the perturbation parameter, even when low degree orthogonal polynomials are used. In this perspective, the paper is organized as follows. In section 2, the main idea of the method is discussed. In section 3, the stability analysis is performed considering the properties of the Mittag-Leffler function. In section 4, the error of the method is analyzed. In section 5, some numerical examples are presented. Finally, section 6 outlines the concluding remarks.

2. Main idea of the method

2.1. The shifted Chebyshev polynomial of the first kind. The main tool to construct the spectral approximation of the layer solutions consists of the shifted Chebyshev polynomials. The shifted Chebyshev polynomial [3] of the first kind $T_n^*(x)$, with degree n in $x \in [0, \delta]$, is defined by the relation

$$T_n^*(x) = \cos\left(n\cos^{-1}(\frac{2x}{\delta} - 1)\right), \ n \in \mathbb{N}, \tag{2.1}$$

with following properties:
(i). $T_n^*(x)$ has exactly n real zeroes on the interval $[0, \delta]$. The i-th zero of $T_n^*(x)$ is located at

$$x_{i-1} = \frac{\delta}{2}\left(1 + \cos\left(\frac{(n-i+\frac{1}{2})\pi}{n}\right)\right), \quad i = 1, 2, \ldots, n. \tag{2.2}$$

(ii). The relation between the powers x^n and the shifted Chebyshev polynomials $T_n^*(x)$ is given by:

$$x^n = \frac{\delta^n}{2^{2n-1}} \sum_{k=0}^{n}{}' \begin{pmatrix} 2n \\ k \end{pmatrix} T_{n-k}^*(x), \tag{2.3}$$

where the prime indicates that the term for $k = 0$ to be halved.

2.2. Singularly perturbed boundary value problems. We consider the general case of singularly perturbed boundary value problems with two BL ($f_1(x, \epsilon) = 0$, for more details see [15]). It is straightforward to verify that this class of problems also contains problems includes also the case of a single BL. Let us first examine the reduced problem for (1.1)-(1.2)

$$\sum\nolimits_{i=1}^{m} f_i(x, 0) D_{*x}^{(m-i)\alpha} y_0(x) = g(x, 0), \quad x \in [0, 1], \ \ i - 1 < i\alpha \leq i, \tag{2.4}$$

with the appropriate boundary conditions

$$y^{(i_k)}(0) = a_k, \ \ 0 \leq k \leq i - 2, \quad y^{(j_k)}(1) = b_k, \ \ 0 \leq k \leq m - i - 2. \tag{2.5}$$

The operator $D_{*x}^{\alpha} y(x)$ represents the Caputo derivative which is defined by [7, 19, 20, 21]

$$\frac{1}{\Gamma(n-\alpha)} \int_0^x (x - \chi)^{n-\alpha-1} y^{(n)}(\chi) d\chi, \ \ n - 1 < \alpha \leq n, \ \ n \in \mathbb{N}, \tag{2.6}$$

where Γ is the gamma function.

Problem (2.4)-(2.5) has the solution $y_0(x)$ which displays BL at $x = 0$ and $x = 1$. Therefore, we seek a solution of (1.1)-(1.2) in the form

$$y(x) \approx \begin{cases} y_1(x), & x \in [0, \delta], \\ y_0(x), & x \in [\delta, 1-\gamma], \\ y_2(x), & x \in [1-\gamma, 1], \end{cases} \tag{2.7}$$

where y_1 and y_2 are the layer functions satisfying

$$\epsilon D_{*x}^{m\alpha} y_\varsigma(x) + \sum\nolimits_{i=1}^{m} f_i(x,\epsilon) D_{*x}^{(m-i)\alpha} y_\varsigma(x) = g(x,\epsilon), \quad \varsigma = 1, 2, \tag{2.8}$$

with the boundary conditions

$$y_1^{(i_k)}(0) = a_k, \quad 0 \le k \le i-1, \tag{2.9}$$

$$y_1^{(i_k)}(\delta) = y_0^{(i_k)}(\delta), \quad 0 \le k \le i-1, \tag{2.10}$$

and

$$y_2^{(j_k)}(1) = b_k, \quad 0 \le k \le m-i-1, \tag{2.11}$$

$$y_2^{(j_k)}(1-\gamma) = y_0^{(j_k)}(1-\gamma), \quad 0 \le k \le m-i-1. \tag{2.12}$$

The parameters $\delta, \gamma > 0$ represent lengths of the layers are to be determined in the follow-up.

Let us now, construct the spectral approximation for the layer function $y_1(x)$ and $y_2(x)$ in the form of truncated series

$$y_{\varsigma N}(x) = \sum_{k=0}^{N} a_k T_k^*(x), \quad \varsigma = 1, 2, \tag{2.13}$$

where $a_k, k = 0, 1, \ldots, N$, are unknown coefficients, $T_k^*(x), k = 0, 1, \ldots, N$, are the shifted Chebyshev polynomials of the first kind and N is chosen any positive integer.

2.3. **Fundamental matrix relations.** In this section, we find the matrix forms of terms and conditions of the Equations (2.8)-(2.12). We first consider the solution $y_{\varsigma N}(x)$, $\varsigma = 1, 2$, and its derivatives $D_{*x}^{k\alpha} y_{\varsigma N}(x)$ defined by a truncated Chebyshev series. Then, we can write series in matrix form

$$y_{\varsigma N}(x) = \mathsf{T}^*(x)\mathsf{A}, D_{*x}^{k\alpha} y_{\varsigma N}(x) = D_{*x}^{k\alpha} \mathsf{T}^*(x)\mathsf{A}, \ k = 0, 1, \ldots, m, \ \varsigma = 1, 2, \tag{2.14}$$

where

$$\mathsf{T}^*(x) = [T_0^*(x) \ \ T_1^*(x) \ \ldots \ T_N^*(x)], \tag{2.15}$$

$$D_{*x}^{k\alpha} \mathsf{T}^*(x) = [D_{*x}^{k\alpha} T_0^*(x), \ldots, D_{*x}^{k\alpha} T_0^*(x)], \tag{2.16}$$

$$\mathsf{A} = [a_0 \ \ a_1 \ \ldots \ a_N]^T. \tag{2.17}$$

Let us consider Equation (2.8) for $\varsigma = 1$. Consequently, one will set

$$\mathsf{T}^*(x) = \mathsf{X}(x)(\mathsf{D}^T)^{-1}, \tag{2.18}$$

where

$$\mathsf{X}(x) = [1 \;\; x \ldots x^N], \tag{2.19}$$

and

$$\mathsf{D} = \begin{pmatrix} \delta^0 2^0 \binom{0}{0} & 0 & 0 & \cdots & 0 \\ \delta^1 2^{-2} \binom{2}{1} & \delta^1 2^{-1} \binom{2}{2} & 0 & \cdots & 0 \\ \delta^2 2^{-4} \binom{4}{2} & \delta^2 2^{-3} \binom{4}{3} & \delta^2 2^{-3} \binom{4}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta^N 2^{-2N} \binom{2N}{N} & \delta^N 2^{-2N+1} \binom{2N}{2N+1} & \delta^N 2^{-2N+1} \binom{2N}{N+2} & \cdots & \delta^N 2^{-2N+1} \binom{2N}{2N} \end{pmatrix}. \tag{2.20}$$

Moreover, the relation between the matrix $\mathsf{X}(x)$ and its derivative $D^{k\alpha}_{*x}\mathsf{X}(x)$ is given by

$$D^{k\alpha}_{*x}\mathsf{X}(x) = \mathsf{X}^{-k\alpha}(x)\mathsf{B}^k, \tag{2.21}$$

where

$$\mathsf{X}^{-k\alpha}(x) = [0 \cdots 0 \; x^{j-k\alpha} \cdots x^{N-k\alpha}], \quad k = 0, 1, \ldots, m, \tag{2.22}$$

$$j = \begin{cases} \text{the largest integer such that } j \leq [k\alpha], & k\alpha \in \mathbb{N}, \\ \text{the largest integer such that } j - 1 \leq [k\alpha], & k\alpha \notin \mathbb{N}, \end{cases} \tag{2.23}$$

and

$$\mathsf{B}^k = \begin{pmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \frac{\Gamma(j+1)}{\Gamma((j+1)-k\alpha)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \frac{\Gamma(N+1)}{\Gamma(N+1-k\alpha)} \end{pmatrix}, \quad k = 0, 1, \ldots, m. \tag{2.24}$$

Using the relation (2.21), the derivative of the matrix $\mathsf{T}^*(x)$ defined in (2.18), can be expressed as

$$D^{k\alpha}_{*x}\mathsf{T}^*(x) = D^{k\alpha}_{*x}\mathsf{X}(x)(\mathsf{D}^T)^{-1} = \mathsf{X}^{-k\alpha}(x)\mathsf{B}^k(\mathsf{D}^T)^{-1}, \quad k = 0, 1, \ldots, m, \tag{2.25}$$

where $D^0_{*x}\mathsf{T}^*(x) = \mathsf{T}^*(x)$, $\mathsf{X}^0(x) = \mathsf{X}(x)$ and B^0 is an identity matrix I. Substituting (2.25) into (2.14), we obtain

$$D^{k\alpha}_{*x}y_{1N}(x) = \mathsf{X}^{-k\alpha}(x)\mathsf{B}^k(\mathsf{D}^T)^{-1}\mathsf{A}, \quad k = 0, 1, \ldots, m, \tag{2.26}$$

where $D^0_{*x}y_{1N}(x) = y_{1N}(x)$.

Using (2.26), the matrix form of the conditions given by (2.9)-(2.10) can be written as

$$y_1^{(i_k)}(0) = a_k \Longrightarrow \mathsf{X}^{-i_k}(0)\mathsf{B}^{i_k}(\mathsf{D}^T)^{-1}\mathsf{A} = [a_k], \tag{2.27}$$

$$y_1^{(i_k)}(\delta) = y_0^{(i_k)}(\delta) \Longrightarrow \mathsf{X}^{-i_k}(\delta)\mathsf{B}^{i_k}(\mathsf{D}^T)^{-1}\mathsf{A} = [y_0^{(i_k)}(\delta)], \tag{2.28}$$

where $0 \leq k \leq I-1$, and

$$\mathsf{X}(0) = [1 \;\; 0 \;\; \cdots \;\; 0], \tag{2.29}$$

$$\mathsf{X}(\delta) = [1 \;\; \delta \;\; \cdots \;\; \delta^N], \tag{2.30}$$

and X^{-i_k} is defined by (2.22).

Let us consider Equation (2.8) for $\varsigma = 2$. Then we can obtain the corresponding matrix relation as

$$\mathsf{T}^*(x) = \mathsf{Z}(x)(\mathsf{C}^T)^{-1}, \tag{2.31}$$

such that

$$\mathsf{Z}(x) = [1 \;\; (x-1) \cdots (x-1)^N], \;\; \mathsf{C} = \begin{pmatrix} \mathsf{C}_1 & \mathsf{C}_2 \end{pmatrix}, \tag{2.32}$$

where

$$\mathsf{C}_1 = \begin{pmatrix} \gamma^0 2^0 \binom{0}{0} & 0 \\ -\gamma^1 2^{-2}\binom{2}{1} & \gamma^1 2^{-1}\binom{2}{2} \\ \gamma^2 2^{-4}\binom{4}{2} & -\gamma^2 2^{-3}\binom{4}{3} \\ \vdots & \vdots \\ (-1)^N\gamma^N 2^{-2N}\binom{2N}{N} & (-1)^{N+1}\gamma^N 2^{-2N+1}\binom{2N}{2N+1} \end{pmatrix}, \tag{2.33}$$

$$\mathsf{C}_2 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \gamma^2 2^{-3}\binom{4}{4} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ (-1)^{N+2}\gamma^N 2^{-2N+1}\binom{2N}{N+2} & \cdots & (-1)^{2N}\gamma^N 2^{-2N+1}\binom{2N}{2N} \end{pmatrix}. \tag{2.34}$$

Moreover, the relation between the matrix $\mathsf{Z}(x)$ and its derivative $D_{*x}^{k\alpha}\mathsf{Z}(x)$ is

$$D_{*x}^{k\alpha}\mathsf{Z}(x) = \mathsf{Z}^{-k\alpha}(x)B^k, \tag{2.35}$$

where

$$\mathsf{Z}^{-k\alpha}(x) = [0 \cdots 0 \; (x-1)^{j-k\alpha} \cdots (x-1)^{N-k\alpha}], \quad k = 0, 1, \ldots, m. \tag{2.36}$$

Adopting (2.35), the derivative of the matrix $\mathsf{T}^*(x)$, defined in (2.31), can be expressed as

$$D_{*x}^{k\alpha}\mathsf{T}^*(x) = D_{*x}^{k\alpha}\mathsf{Z}(x)(\mathsf{C}^T)^{-1} = \mathsf{Z}^{-k\alpha}(x)\mathsf{B}^k(\mathsf{C}^T)^{-1}, \quad k = 0, 1, \ldots, m, \tag{2.37}$$

where $D_{*x}^0\mathsf{T}^*(x) = \mathsf{T}^*(x)$, $\mathsf{Z}^0(x) = \mathsf{Z}(x)$, $\mathsf{B}^0 = I$. Substituting (2.37) into (2.14) leads to

$$D_{*x}^{k\alpha}y_{2N}(x) = \mathsf{Z}^{-k\alpha}(x)\mathsf{B}^k(\mathsf{C}^T)^{-1}\mathsf{A}, \quad k = 0, 1, \ldots, m, \tag{2.38}$$

where $D^{0}_{*x}y_{2N}(x) = y_{2N}(x)$.

Using (2.38), the matrix form of the conditions given in (2.11)-(2.12) can be written as

$$y_2^{(j_k)}(1) = b_k \Longrightarrow \mathsf{Z}^{-j_k}(1)\mathsf{B}^{j_k}(\mathsf{C}^T)^{-1}\mathsf{A} = [b_k], \tag{2.39}$$

$$y_2^{(j_k)}(1-\gamma) = y_0^{(j_k)}(1-\gamma) \Longrightarrow \mathsf{Z}^{-j_k}(1-\gamma)\mathsf{B}^{j_k}(\mathsf{C}^T)^{-1}\mathsf{A} = [y_0^{(j_k)}(1-\gamma)], \tag{2.40}$$

where $0 \le k \le m - I - 1$, and

$$\mathsf{Z}(1) = [1 \;\; 0 \;\; \cdots \;\; 0], \tag{2.41}$$

$$\mathsf{Z}(1-\gamma) = [1 \;\; (-\gamma) \;\; \cdots \;\; (-\gamma)^N], \tag{2.42}$$

and Z^{-j_k} is defined by (2.36).

2.4. **Method of solution.** We can now construct the fundamental matrix equation corresponding to Equation (2.8). For simplifying things, and without loss of generality, we assume that the boundary conditions are at most of first order derivative. Substituting the matrix relation (2.26) into (2.8) for $\varsigma = 1$, we obtain

$$\Big(\epsilon \mathsf{X}^{-m\alpha}(x)\mathsf{B}^m(\mathsf{D}^T)^{-1} + \sum_{i=1}^{m} f_i(x,\epsilon)\mathsf{X}^{-(m-i)\alpha}(x)\mathsf{B}^{(m-i)}(\mathsf{D}^T)^{-1}\Big)\mathsf{A} = g(x,\epsilon). \tag{2.43}$$

For computing the Chebyshev coefficient matrix A, the zeroes of the shifted Chebyshev polynomials of the first kind are substituted in Equation (2.43). Consequently, it yields

$$\Big(\epsilon \mathsf{X}^{-m\alpha}(x_i)\mathsf{B}^m(\mathsf{D}^T)^{-1} + \sum_{i=1}^{m} f_i(x_i,\epsilon)\mathsf{X}^{-(m-i)\alpha}(x_i)\mathsf{B}^{(m-i)}(\mathsf{D}^T)^{-1}\Big)\mathsf{A} = g(x_i,\epsilon). \tag{2.44}$$

The fundamental matrix equation is given by:

$$\Big(\mathsf{E}\mathsf{X}^{-m\alpha}\mathsf{B}^m(\mathsf{D}^T)^{-1} + \sum_{i=1}^{m} \mathsf{F}_i\mathsf{X}^{-(m-i)\alpha}\mathsf{B}^{(m-i)}(\mathsf{D}^T)^{-1}\Big)\mathsf{A} = \mathsf{G}, \tag{2.45}$$

where

$$\mathsf{F}_i = \begin{pmatrix} f_i(x_0,\epsilon) & 0 & 0 & \dots & 0 \\ 0 & f_i(x_1,\epsilon) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_i(x_N,\epsilon) \end{pmatrix}, \quad i = 1, \dots, m, \tag{2.46}$$

$$\mathsf{X} = \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ 1 & x_1 & x_1^2 & \dots & x_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{pmatrix}, \tag{2.47}$$

$$\mathsf{E}=\begin{pmatrix}\epsilon & 0 & 0 & \dots & 0\\ 0 & \epsilon & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & \epsilon\end{pmatrix},\quad \mathsf{G}=\begin{pmatrix} g(x_0,\epsilon)\\ g(x_1,\epsilon)\\ \vdots\\ g(x_N,\epsilon)\end{pmatrix}. \tag{2.48}$$

The fundamental matrix Equation (2.45) corresponds to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown coefficients $\{a_0, a_1, \dots, a_N\}$. Therefore, we can write Equation (2.43) as

$$\mathsf{WA}=\mathsf{G},\quad \text{or } [\mathsf{W};\mathsf{G}], \tag{2.49}$$

so that, for $p, q = 0, 1, \dots, N$, it yields

$$\mathsf{W}=[w_{p,q}]=\mathsf{E}\mathsf{X}^{-m\alpha}\mathsf{B}^{m}(\mathsf{D}^{T})^{-1}+\sum_{i=1}^{m}\mathsf{F}_i\mathsf{X}^{-(m-i)\alpha}\mathsf{B}^{(m-i)}(\mathsf{D}^{T})^{-1}. \tag{2.50}$$

The matrix form for conditions (2.9)-(2.10) are

$$\mathfrak{A}_0\mathsf{A}=[a_0],\quad \text{or } [\mathfrak{A}_0;a_0],\quad \mathfrak{B}_0\mathsf{A}=[a_1],\quad \text{or } [\mathfrak{B}_0;a_1], \tag{2.51}$$

$$\mathfrak{A}_\delta\mathsf{A}=[y_0(\delta)],\quad \text{or } [\mathfrak{A}_\delta;y_0(\delta)],\quad \mathfrak{B}_\delta\mathsf{A}=[y_0'(\delta)],\quad \text{or } [\mathfrak{B}_0;y_0'(\delta)], \tag{2.52}$$

where

$$\mathfrak{A}_i=\mathsf{X}(i)(\mathsf{D}^{T})^{-1}\equiv[a_{i0}\ a_{i1}\ \cdots\ a_{iN}],\quad i=0,\delta, \tag{2.53}$$

$$\mathfrak{B}_i=\mathsf{X}^{-1}(i)\mathsf{B}(\mathsf{D}^{T})^{-1}\equiv[b_{i0}\ b_{i1}\ \cdots\ b_{iN}],\quad i=0,\delta. \tag{2.54}$$

For obtaining the solution of Equation (2.8) under the conditions (2.9)-(2.10), we replace the rows matrices (2.51)-(2.52) by the first and last two rows of the matrix (2.49), respectively. Therefore, we have the augmented matrix

$$[\mathsf{W}^*;\mathsf{Y}^*]=\begin{pmatrix} w_{2,0} & w_{2,0} & \dots & w_{2,N} & ; & g(x_2)\\ w_{3,0} & w_{3,1} & \dots & w_{3,N} & ; & g(x_3)\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ w_{(N-2),0} & w_{(N-2),1} & \dots & w_{(N-2),N} & ; & g(x_{N-2})\\ a_{0,0} & a_{0,1} & \dots & a_{0,N} & ; & a_0\\ a_{\delta,0} & a_{\delta,1} & \dots & a_{\delta,N} & ; & y_0(\delta)\\ b_{0,0} & b_{0,1} & \dots & b_{0,N} & ; & a_1\\ b_{\delta,0} & b_{\delta,1} & \dots & b_{\delta,N} & ; & y_0'(\delta)\end{pmatrix}, \tag{2.55}$$

and the corresponding matrix equation

$$\mathsf{W}^*\mathsf{A}=\mathsf{Y}^*. \tag{2.56}$$

If $rank(\mathsf{W}^*)=rank[\mathsf{W}^*;\mathsf{Y}^*]=N+1$, then we can write

$$\mathsf{A}=(\mathsf{W}^*)^{-1}\mathsf{Y}^*. \tag{2.57}$$

Thus, the coefficients $a_i, i = 0, \dots, N$, are uniquely determined by Equation (2.57). Similarly, substituting the matrix relation (2.37) into (2.8) for $\varsigma = 2$ we obtain

$$\Big(\epsilon \mathsf{Z}^{-m\alpha}(x)\mathsf{B}^m(\mathsf{C}^T)^{-1} + \sum_{i=1}^{m} f_i(x,\epsilon)\mathsf{Z}^{-(m-i)\alpha}(x)\mathsf{B}^{(m-i)}(\mathsf{C}^T)^{-1}\Big)\mathsf{A} = g(x,\epsilon). \quad (2.58)$$

The fundamental matrix equation is given by

$$\Big(\mathsf{E}\mathsf{Z}^{-m\alpha}\mathsf{B}^m(\mathsf{C}^T)^{-1} + \sum_{i=1}^{m} \mathsf{F}_i\mathsf{Z}^{-(m-i)\alpha}\mathsf{B}^{(m-i)}(\mathsf{C}^T)^{-1}\Big)\mathsf{A} = \mathsf{G}, \quad (2.59)$$

where

$$\mathsf{Z} = \begin{pmatrix} 1 & (x_0-1) & (x_0-1)^2 & \dots & (x_0-1)^N \\ 1 & (x_1-1) & (x_1-1)^2 & \dots & (x_1-1)^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_N-1) & (x_N-1)^2 & \dots & (x_N-1)^N \end{pmatrix}. \quad (2.60)$$

We can write Equation (2.58) as

$$\mathsf{W}\mathsf{A} = \mathsf{G}, \quad \text{or } [\mathsf{W}; \mathsf{G}], \quad (2.61)$$

so that, for $p, q = 0, 1, \dots, N$, we have

$$\mathsf{W} = [w_{p,q}] = \mathsf{E}\mathsf{Z}^{-m\alpha}\mathsf{B}^m(\mathsf{C}^T)^{-1} + \sum_{i=1}^{m} \mathsf{F}_i\mathsf{Z}^{-(m-i)\alpha}\mathsf{B}^{(m-i)}(\mathsf{C}^T)^{-1}. \quad (2.62)$$

Furthermore, the matrix form for conditions (2.11)-(2.12) are

$$\mathfrak{C}_1\mathsf{A} = [a_0], \quad \text{or } [\mathfrak{C}_1; b_0], \quad \mathfrak{D}_1\mathsf{A} = [b_1], \quad \text{or } [\mathfrak{D}_1; b_1], \quad (2.63)$$

$$\mathfrak{C}_{1-\gamma}\mathsf{A} = [y_0(1-\gamma)], \quad \text{or } [\mathfrak{C}_{1-\gamma}; y_0(1-\gamma)], \quad (2.64)$$

$$\mathfrak{D}_{1-\gamma}\mathsf{A} = [y_0'(1-\gamma)], \quad \text{or } [\mathfrak{D}_{1-\gamma}; y_0'(1-\gamma)], \quad (2.65)$$

where

$$\mathfrak{C}_i = \mathsf{Z}(i)(\mathsf{C}^T)^{-1} \equiv [c_{i0}\ c_{i1} \cdots c_{iN}], \quad i = 1, 1-\gamma, \quad (2.66)$$

$$\mathfrak{D}_i = \mathsf{Z}^{-1}(i)\mathsf{B}(\mathsf{C}^T)^{-1} \equiv [d_{i0}\ d_{i1} \cdots d_{iN}], \quad i = 1, 1-\gamma. \quad (2.67)$$

To obtain the solution of Equation (2.8) for $\varsigma = 2$ under conditions (2.11)-(2.12), we replace the rows matrices (2.63)-(2.64) by the first and the last two rows of the matrix

(2.61) respectively, and we obtain the augmented matrix

$$[\mathsf{W}^*;\mathsf{Y}^*]=\begin{pmatrix} w_{2,0} & w_{2,0} & \dots & w_{2,N} & ; & g(x_2)\\ w_{3,0} & w_{3,1} & \dots & w_{3,N} & ; & g(x_3)\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ w_{(N-2),0} & w_{(N-2),1} & \dots & w_{(N-2),N} & ; & g(x_{N-2})\\ c_{1,0} & c_{1,1} & \dots & c_{1,N} & ; & b_0\\ c_{1-\gamma,0} & c_{1-\gamma,1} & \dots & c_{1-\gamma,N} & ; & y_0(1-\gamma)\\ d_{1,0} & d_{1,1} & \dots & d_{1,N} & ; & b_1\\ d_{1-\gamma,0} & d_{1-\gamma,1} & \dots & d_{1-\gamma,N} & ; & y_0'(1-\gamma) \end{pmatrix}. \quad (2.68)$$

The corresponding matrix equation is:

$$\mathsf{W}^*\mathsf{A}=\mathsf{Y}^*. \quad (2.69)$$

Thus, the coefficients $a_i, i=0,\dots,N$, are uniquely determined by Equation (2.69).

3. Stability and error analysis

In this section, the stability and error of the proposed method are analyzed based on properties of the Mittag-Leffler function and shifted Chebyshev polynomials.

3.1. On the asymptotic Mittag-Leffler stability.

Definition 1. *The Mittag-Leffler function $E_{\alpha,\beta}(z)$ with $\alpha>0$, $\beta>0$ is defined by the following series representation, valid in the whole complex plane*

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)},\quad z\in\mathbb{C}. \quad (3.1)$$

For $\beta=1$, we obtain the Mittag-Leffler function with one parameter:

$$E_{\alpha,1}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}\equiv E_\alpha(z). \quad (3.2)$$

It is to be noted that the fractional differential equation

$$\frac{d^\alpha y}{dx^\alpha}=g(x,y), \quad (3.3)$$

is asymptotic Mittag-Leffler stable if there exist positive constants k and λ such that

$$|y(x)|\le kE_\alpha(-\lambda(x-s)^\alpha),\ 0\le s\le t<\ \infty. \quad (3.4)$$

We consider the changes in the solution that are caused by small variations in the order α and boundary conditions and functions $f_i(x,\epsilon)$ and $g(x,\epsilon)$. Let us introduce small changes in problem (1.1)-(1.2) as follows:

$$\epsilon D_{*x}^{m\tilde{\alpha}}\tilde{y}(x)+\sum_{i=1}^{m}\tilde{f}_i(x,\epsilon)D_{*x}^{(m-i)\tilde{\alpha}}\tilde{y}(x)=\tilde{g}(x,\epsilon),\quad x\in[0,1],\ \ 0<\alpha\le 1, \quad (3.5)$$

with the boundary conditions

$$y^{(i_k)}(0)=\tilde{a}_k,\ \ 0\le k\le i-1,\quad y^{(j_k)}(1)=\tilde{b}_k,\ \ 0\le k\le m-i-1. \quad (3.6)$$

Theorem 3.1. *If $y(x)$ defined as (2.7) is a solution of the Equation (1.1) satisfying the boundary conditions (1.2), and $\tilde{y}(x)$ is a solution of Equation (3.5) satisfying the conditions (3.6), then there exists a constant l such that the following relation holds:*

$$y(x) \approx \tilde{y}(x) + E_{m\alpha}\left(\frac{l}{\epsilon}x^{m\alpha}\right). \tag{3.7}$$

Proof. Substituting $y(x)$ defined as (2.7) in Equation (1.1) and applying the operator I_x^α, the inverse of the operator D_{*x}^α, to both sides of this equation yields

$$y(x) = \sum_{i=0}^{m-1}\frac{c_i x^i}{i!} + \frac{1}{\epsilon\Gamma(m\alpha)}\int_0^x (x-\chi)^{m\alpha-1}\Big(g(\chi,\epsilon) - \sum\nolimits_{i=1}^{m} f_i(\chi,\epsilon) D_{*\chi}^{(m-i)\alpha} y(\chi)\Big)d\chi, \tag{3.8}$$

where the constants c_i can be determined by the boundary conditions (1.2). The same relation holds for $\tilde{y}$. Therefore, we can write

$$\begin{aligned}
|y(x)-\tilde{y}(x)| \le & \Big|\sum_{i=0}^{m-1}\frac{(c_i-\tilde{c}_i)x^i}{i!}\Big| \\
&+\Big|\frac{1}{\epsilon\Gamma(m\alpha)}\int_0^x (x-\chi)^{m\alpha-1}\Big(g(\chi,\epsilon) - \sum_{i=1}^{m} f_i(\chi,\epsilon) D_{*\chi}^{(m-i)\alpha} y(\chi) \\
&-\tilde{g}(\chi,\epsilon) + \sum_{i=1}^{m} \tilde{f}_i(\chi,\epsilon) D_{*\chi}^{(m-i)\tilde{\alpha}} \tilde{y}(\chi)\Big)d\chi\Big| \\
&+\Big|\frac{1}{\epsilon\Gamma(m\alpha)}\int_0^x (x-\chi)^{m\alpha-1}\Big(\tilde{g}(\chi,\epsilon) - \sum_{i=1}^{m} \tilde{f}_i(\chi,\epsilon) D_{*\chi}^{(m-i)\tilde{\alpha}} \tilde{y}(\chi)\Big)d\chi \\
&-\frac{1}{\epsilon\Gamma(m\tilde{\alpha})}\int_0^x (x-\chi)^{m\tilde{\alpha}-1}\Big(\tilde{g}(\chi,\epsilon) - \sum_{i=1}^{m} \tilde{f}_i(\chi,\epsilon) D_{*\chi}^{(m-i)\tilde{\alpha}} \tilde{y}(\chi)\Big)d\chi\Big|.
\end{aligned} \tag{3.9}$$

We now consider the following relation:

$$\begin{aligned}
&\Big|\frac{1}{\epsilon\Gamma(m\alpha)}\int_0^x (x-\chi)^{m\alpha-1}\Big(g(\chi,\epsilon) - \sum_{i=1}^{m} f_i(\chi,\epsilon) D_{*\chi}^{(m-i)\alpha} y(\chi) \\
&-\tilde{g}(\chi,\epsilon) + \sum_{i=1}^{m} \tilde{f}_i(\chi,\epsilon) D_{*\chi}^{(m-i)\tilde{\alpha}} \tilde{y}(\chi)\Big)d\chi\Big| \\
&= \Big|\frac{1}{\epsilon\Gamma(m\alpha)}\int_0^x (x-\chi)^{m\alpha-1}\Big[\Big(g(\chi,\epsilon) - \sum_{i=1}^{m} f_i(\chi,\epsilon) D_{*\chi}^{(m-i)\alpha} y(\chi) \\
&-g(\chi,\epsilon) + \sum_{i=1}^{m} f_i(\chi,\epsilon) D_{*\chi}^{(m-i)\tilde{\alpha}} \tilde{y}(\chi)\Big) \\
&+\Big(g(\chi,\epsilon) - \sum_{i=1}^{m} f_i(\chi,\epsilon) D_{*\chi}^{(m-i)\alpha} y(\chi) - \tilde{g}(\chi,\epsilon) + \sum_{i=1}^{m} \tilde{f}_i(\chi,\epsilon) D_{*\chi}^{(m-i)\tilde{\alpha}} \tilde{y}(\chi)\Big)\Big]d\chi\Big|.
\end{aligned} \tag{3.10}$$

Hence, we obtain

$$|y(x)-\tilde{y}(x)| \leq \sum_{i=0}^{m-1} \frac{|c_i-\tilde{c}_i|x^i}{i!}$$

$$+\frac{1}{\epsilon\Gamma(m\alpha)}\int_0^x (x-\chi)^{m\alpha-1}|y(\chi)-\tilde{y}(\chi)|d\chi + \frac{1}{\epsilon\Gamma(m\alpha)}\int_0^x (x-\chi)^{m\alpha-1}d\chi$$

$$+\sup\Big|\tilde{g}(\chi,\epsilon)-\sum_{i=1}^{m}\tilde{f}_i(\chi,\epsilon)D_{*\chi}^{(m-i)\tilde{\alpha}}\tilde{y}(\chi)\Big|\times\Big|\frac{1}{\epsilon\Gamma(m\alpha)}\int_0^x (x-\chi)^{m\alpha-1}d\chi$$

$$-\frac{1}{\epsilon\Gamma(m\tilde{\alpha})}\int_0^x (x-\chi)^{m\tilde{\alpha}-1}d\chi\Big|. \tag{3.11}$$

Therefore, we conclude that

$$y(x) \approx \tilde{y}(x) + \frac{l}{\epsilon\Gamma(m\alpha)}\int_0^x (x-\chi)^{m\alpha-1}|y(\chi)-\tilde{y}(\chi)|d\chi. \tag{3.12}$$

In summary, we have

$$y(x) \approx \tilde{y}(x) + E_{m\alpha}\left(\frac{l}{\epsilon}x^{m\alpha}\right). \tag{3.13}$$

3.2. **Error analysis for BL problems.** Let $y(x)$ and $y_1(x)$ be the solutions of (1.1)-(1.2) and the boundary value problem (2.8)-(2.12), respectively. Then, we have:

Theorem 3.2. *Consider problems with left BL. If $y_1 \in C^{\alpha+1}[0,1]$, then the error on the BL $[0,\delta]$ can be estimated as follows*

$$\Xi(x) \leq \Xi_1 + \Xi_2, \tag{3.14}$$

where

$$|y(x)-y_1(x)| \leq C|y(\delta)-y_0(\delta)| = \Xi_1, \tag{3.15}$$

and

$$\frac{\delta^{\alpha+1}}{2^{2\alpha+1}(\alpha+1)!}\|y_1^{(\alpha+1)}\|_\infty = \Xi_2,\ n-1<\alpha\leq n,\ n\in\mathbb{N}. \tag{3.16}$$

Proof. First, consider the following definition:
"A real function $y(x)$, $x>0$, is said to be in the space C_α, $\alpha\in\mathbb{R}$, if there exists a real number $p(>\alpha)$, such that $y(x)=x^p y_c(x)$, where $y_c(x)\in C[0,\infty)$, and it said to be in the space C_α^n, $n\in\mathbb{N}\cup\{0\}$, if and only if $y^{(n)}(x)\in C_\alpha$."

Now, inside the BL, for $x\in[0,\delta]$ and based on (2.13), we have

$$\Xi(x) \leq |y(x)-y_1(x)| + |y_1(x)-y_{1N}(x)|. \tag{3.17}$$

Since $\|T_{k+1}^*\|_\infty = 1$, we conclude that if we choose the grid nodes $(x_i)_{0\leq i\leq N}$, it comes

$$\sup\|\prod_{i=0}^{k}(x-x_i)\|_\infty = \frac{\delta^{\alpha+1}}{2^{2\alpha+1}}. \tag{3.18}$$

In particular, for any $y_1 \in C^{\alpha+1}[0,1]$ we have

$$\|y_1 - y_{1N}\|_\infty \leq \frac{\delta^{\alpha+1}}{2^{2\alpha+1}(\alpha+1)!}\|y_1^{(\alpha+1)}\|_\infty. \tag{3.19}$$

By use of (3.19), for $x \in [0,\delta]$ it yields

$$\Xi(x) \leq \Xi_1 + \Xi_2, \tag{3.20}$$

and this proves Theorem 3.2.

Proposition 1. *The error estimate for $x \in [1-\gamma, 1]$ can be provided in the same manner.*

4. Illustrative problems

To demonstrate the applicability of the method, and to assess its performance we consider in the follow-up three problems. The examples are discussed in the literature for $\alpha = 1$. In the last two cases, their exact solutions are available for comparison when $\alpha = 1$. The approximate solutions are calculated by means of the *Maple* software.

Example 1. *Consider the following fractional singularly perturbed differential equation*

$$\begin{aligned} -\epsilon^2 D_{*x}^{2\alpha} y \quad &+ \quad \frac{2+4\epsilon^2-2\epsilon^2 x}{(2-x)^2} D_{*x}^{\alpha} y = \frac{\epsilon^2\pi^2}{(2-x)^4}\cos\left(\frac{\pi(1-x)}{2-x}\right) \\ &+ \quad \frac{2\pi}{(2-x)^4}\sin\left(\frac{\pi(1-x)}{2-x}\right), \end{aligned} \tag{4.1}$$

$$i-1 < i\alpha \leq i, \quad i = 1,2, \quad y(0) = y(1) = 0. \tag{4.2}$$

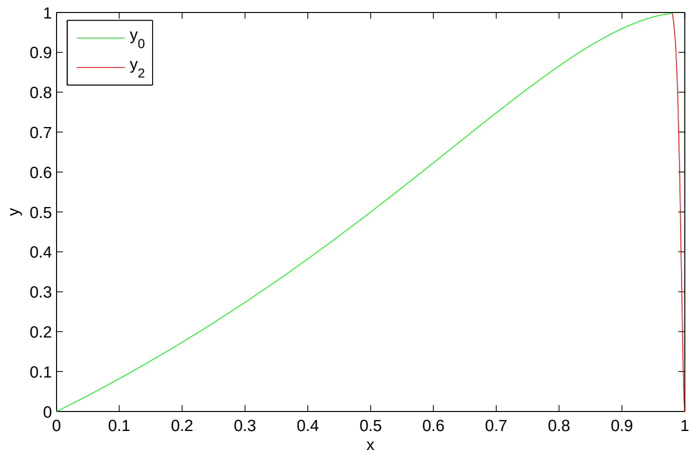
When $\epsilon \to 0$, the order of the differential equation reduces to one and one of the boundary conditions must be dropped. It is clear that, the BL is at the right end, and the reduced equation satisfies the boundary condition $y(0) = 0$. Hence, for sufficiently small ϵ, we have a thin BL on the right hand side, making it an ideal problem for application of the present method.

For $\alpha = 1$ (see[5]), $\epsilon = 0.1$ and $N = 5$, the approximate solutions have the following form:

$$\begin{cases} 1-\gamma = 0.9802, \\ y_0 = cos(\frac{\pi(1-x)}{2-x}), \\ y_2 = 0.5815T^*_{0,[1-\gamma,1]} - 0.5390T^*_{1,[1-\gamma,1]} - 0.1100T^*_{2,[1-\gamma,1]} + 0.0342T^*_{3,[1-\gamma,1]} \\ +0.0276T^*_{4,[1-\gamma,1]} + 0.0057T^*_{5,[1-\gamma,1]}. \end{cases}$$

In this example, the asymptotic expansion method for $\alpha = 1$ does not converge toward the solution in the BL (see [5]).

FIGURE 1. Numerical results for Example 1, when $\epsilon = 0.1$, $\alpha = 1$ and $N = 5$.



Example 2. *Consider the following fractional singularly perturbed differential equation*

$$-\epsilon^2 D_{*x}^{2\alpha} y + y = -\cos^2(\pi x) - 2(\epsilon\pi)^2 \cos(2\pi x),\ 1 < 2\alpha \leq 2, \tag{4.3}$$

$$y(0) = y(1) = 0. \tag{4.4}$$

The order of the highest derivative is greater than the order of the second term. In this case there are two BL, one at each end.

For $\alpha = 1$ (see [5]), $\epsilon = 0.01$ and $N = 5$, the approximate solutions have the following form:

$$\begin{cases} \delta = \gamma = 0.0630, \\ y_0 = -cos^2(\pi x), \\ y_1 = -0.7476T^*_{0,[0,\delta]} - 0.3708T^*_{1,[0,\delta]} + 0.2322T^*_{2,[0,\delta]} - 0.1005T^*_{3,[0,\delta]} \\ +0.0347T^*_{4,[0,\delta]} - 0.0093T^*_{5,[0,\delta]}, \\ y_2 = -0.7476T^*_{0,[1-\gamma,1]} - 0.3708T^*_{1,[1-\gamma,1]} + 0.2322T^*_{2,[1-\gamma,1]} - 0.1005T^*_{3,[1-\gamma,1]} \\ +0.0347T^*_{4,[1-\gamma,1]} - 0.0093T^*_{5,[1-\gamma,1]}. \end{cases}$$

Example 3. *Consider the following fractional singularly perturbed differential equation*

$$-\epsilon D_{*x}^{8\alpha} y + D_{*x}^{6\alpha} y - y = -x - e^{-\frac{x}{\sqrt{\epsilon}}},\ i-1 < i\alpha \leq i,\ \ i = 6, 8, \tag{4.5}$$

$$y(0) = 1, \quad y''(0) = \frac{1}{\epsilon}, \quad y^{(4)}(0) = \frac{1}{\epsilon^2}, \quad y^{(6)}(0) = \frac{1}{\epsilon^3}, \tag{4.6}$$

FIGURE 2. Numerical results for Example 2, when $\epsilon = 0.01, \alpha = 1$ and $N = 5$.

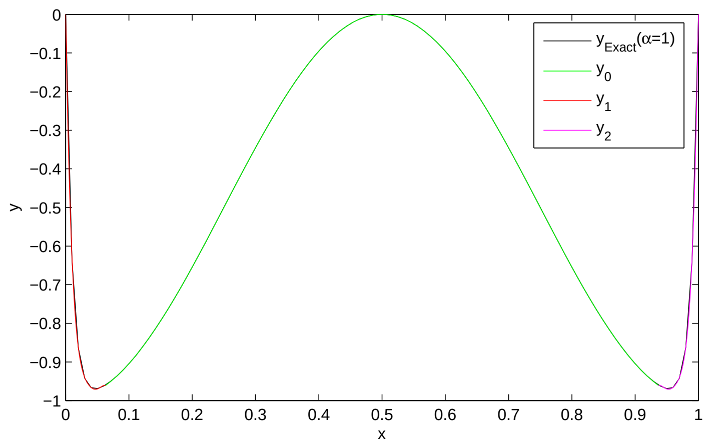
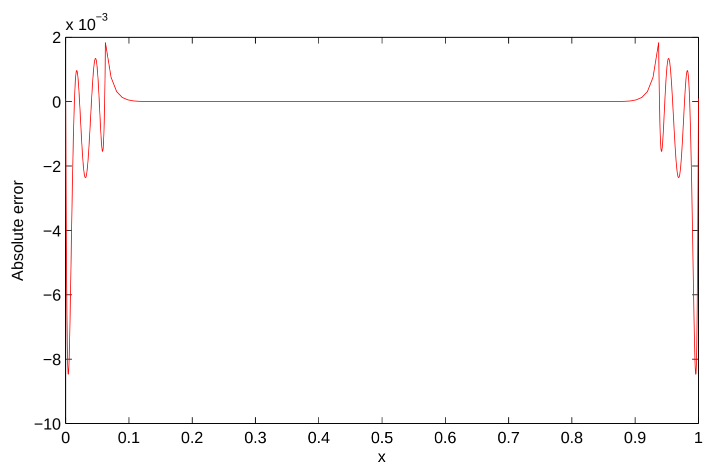


FIGURE 3. Absolute error for Example 2, when $\epsilon = 0.01, \alpha = 1$ and $N = 5$.



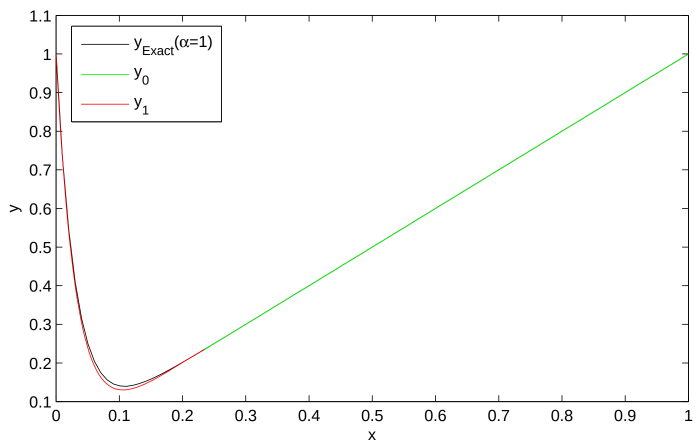
$$y(1) = 1 + e^{\frac{1}{\sqrt{\epsilon}}}, \quad y''(1) = \frac{e^{-\frac{1}{\sqrt{\epsilon}}}}{\epsilon}, \quad y^{(4)}(1) = \frac{e^{-\frac{1}{\sqrt{\epsilon}}}}{\epsilon^2}, \quad y^{(6)}(1) = \frac{e^{-\frac{1}{\sqrt{\epsilon}}}}{\epsilon^3}. \tag{4.7}$$

In this case, the right-hand side of Equation (4.5) is discontinuous. Consequently, the estimation of location of the BL is more complex and, in fact, we verify that there is only a BL at the left end.

TABLE 1. Numerical results for Example 2, when $\epsilon = 0.01$, $\alpha = 1$ and $N = 5$.

x	y_{Exact}	$y_{Our\ method}(N=5)$	*Absolute error*
0.03	-0.941356556996480	-0.939137344081353	2.2192E-3
0.05	-0.968790311148491	-0.969804240304260	1.0139E-3
δ =0.0630	-0.959500065158029	-0.961336369935057	1.8363E-3
0.1	-0.904463097257711	-0.904508497187474	4.5399E-5
0.5	3.857499695890342E-22	-3.749399456654644E-33	3.8574E-22
0.9	-0.904463097257711	-0.904508497187474	4.5399E-5
$1-\gamma = 0.9370$	-0.959500065158029	-0.961336369935057	1.8363E-3
0.95	-0.968790311148491	-0.969804240304260	1.0139E-3
0.97	-0.941356556996480	-0.939137344081354	2.2192E-3

FIGURE 4. Numerical results for Example 3, when $\epsilon = \frac{1}{2^{10}}, \alpha = 1$ and $N = 8$.



For $\alpha = 1$ (see [22]), $\epsilon = \frac{1}{2^{10}}$ and $N = 8$, the approximate solutions have the following form:

$$\begin{cases} \delta = 0.2339, \\ y_0 = x, \\ y_1 = 0.3268T^*_{0,[0,\delta]} - 0.2460T^*_{1,[0,\delta]} + 0.2378T^*_{2,[0,\delta]} - 0.1197T^*_{3,[0,\delta]} \\ +0.0476T^*_{4,[0,\delta]} - 0.0163T^*_{5,[0,\delta]} + 0.0046T^*_{6,[0,\delta]} - 0.0011T^*_{7,[0,\delta]} + 0.0002T^*_{8,[0,\delta]}. \end{cases}$$

FIGURE 5. Absolute error for Example 3, when $\epsilon = \frac{1}{2^{10}}$, $\alpha = 1$ and $N = 8$.

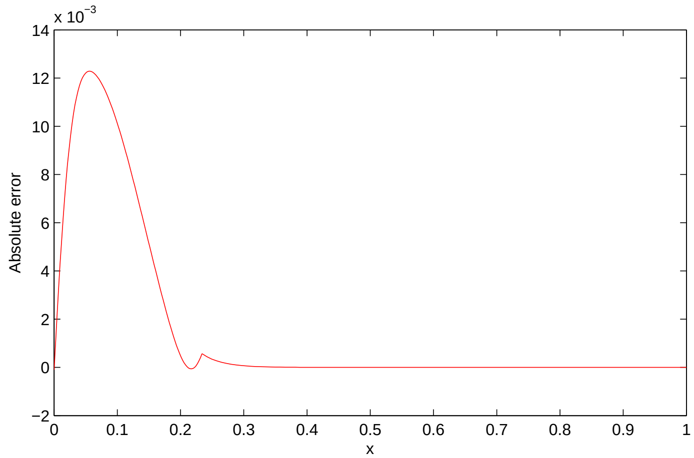


TABLE 2. Numerical results for Example 3, when $\epsilon = \frac{1}{2^{10}}$, $\alpha = 1$ and $N = 8$.

x	y_{Exact}	$y_{Our\ method}(N = 8)$	*Absolute error*
0.15	0.158229747049020	0.153156164497625	5.0735E-3
0.2	0.201661557273174	0.201196415335477	4.6514E-4
δ =0.2339	0.234461555469835	0.233900000000000	5.6155E-4
0.3	0.300067728736491	0.300000000000000	6.7728E-5
0.5	0.500000112535175	0.500000000000000	1.1253E-7
0.9	0.900000000000310	0.900000000000000	3.1068E-13

5. Concluding remarks

A wide class of singularly perturbed boundary value problems of fractional order was considered. The solutions both inside and outside of the BL are approximated by means of truncated Chebyshev series and reduced problem, respectively. For evaluating the coefficients of the truncated series, operational matrices were adopted. The examples confirm that the computational of the method descend to determine the numerical layer length. The new method compares well with previous ones that pose difficulties such as finding the appropriate asymptotic expansions in the inner and outer regions, or determining a composite expansion valid in all domain. The results show that the present method is highly accurate even when adopting low degree orthogonal polynomials.

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