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Numerical solution of nonlinear mixed Fredholm-Volterra integro-differential equations of fractional order by Bernoulli wavelets

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Abstract In this paper, a numerical method for solving a class of nonlinear mixed Fredholm-Volterra integro-differential equations of fractional order is presented. The method is based upon Bernoulli wavelets approximations. The operational matrix of fractional order integration for Bernoulli wavelets is utilized to reduce the solution of the nonlinear fractional integro-differential equations to system of algebraic equations. Illustrative examples are included to demonstrate the efficiency and accuracy of the method.

Keywords. Bernoulli wavelets, Fractional calculus, Fredholm-Volterra integro-differential equations, Caputo derivative, Operational matrix.

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1. Introduction

The fractional integro-differential equations (FIDEs) have drawn increasing attention and interest due to their important applications in science and engineering.

Many mathematical modelings of various physical phenomena contain FIDEs [6, 11, 36]. Generally speaking, the analytical solutions of most FIDEs are not easy to obtain. Therefore, seeking numerical solutions of these equations becomes more and more important [18]. Recently, there has been considerable interest in developing numerical schemes for the solution of FIDEs. These methods include the variational iteration method [9, 32], homotopy perturbation method [4, 5, 10], Adomian's decomposition method [7], homotopy analysis method [8], collocation method [16, 17, 29], CAS

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wavelets [30, 31], second kind Chebyshev wavelets [35], Legendre wavelets [24], SCW method [33], an operational Jacobi Tau method [25] and hat functions [26]. As is stated in [30, 31], most of the methods have been utilized in linear problems and a few number of works have considered nonlinear problems.

In recent years, wavelets have found their way into many different fields of science and engineering. Many researchers started using various wavelets [1, 2, 12, 15, 21, 28, 34] for analyzing problems of greater computational complexity and proved wavelets to be powerful tools to explore a new direction in solving dynamic systems. In this paper, we focus on approximating the solution of a class of nonlinear fractional mixed Fredholm-Volterra integro-differential equations of the type

$$D^q f(x) - \lambda_1 \int_0^1 k_1(x,t)[f(t)]^{r_1} dt - \lambda_2 \int_0^x k_2(x,t)[f(t)]^{r_2} dt = g(x), \ x \in [0,1], \tag{1.1}$$

with the the initial conditions

$$f^{(i)}(0) = \delta_i, \ i = 0, 1, \ldots, n-1, \ n-1 < q \leq n, \ n \in \mathbb{N}, \tag{1.2}$$

where, $g \in L^2([0,1])$, $k_1, k_2 \in L^2([0,1] \times [0,1])$ are known functions, $f(x)$ is the unknown function and D^q is the Caputo fractional derivative of order q. In addition r_1 and r_2 are positive integers and $f^{(i)}(x)$ stands for the ith-order derivative of $f(x)$. Our method is based upon Bernoulli wavelets approximation. The Bernoulli wavelets are first given. We then present the operational matrix of fractional integration for Bernoulli wavelets by expanding these wavelets into block-pulse functions. This matrix is then utilized to reduce the solution of the Eqs. (1.1) and (1.2) to the solution of algebraic equations. For approximating an arbitrary time function the advantages of Bernoulli polynomials over shifted Legendre polynomials are given in [23].

The outline of this paper is as follows: In section 2, we introduce some necessary definitions and mathematical preliminaries of fractional calculus and the basic formulation of the Bernoulli wavelets. In section 3, we present the Bernoulli wavelets operational matrix of fractional integration. Section 4 is devoted to the numerical method for solving problem (1.1)–(1.2), and in section 5 we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme by considering five numerical examples.

2. Preliminaries and notations

2.1. **The fractional integral and derivative.** In this section, we first state some definitions and basic properties regarding fractional derivatives and integrals.

Definition 2.1. The Riemann-Liouville fractional integral operator of order q is defined as [27]

$$I^q f(x) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^x \frac{f(s)}{(x-s)^{1-q}} ds, & q > 0, \\ f(x), & q = 0. \end{cases}$$

Definition 2.2. Caputo's fractional derivative of order q is defined as [14]

$$D^q f(x) = \begin{cases} \frac{1}{\Gamma(n-q)} \int_0^x \frac{f^{(n)}(s)}{(x-s)^{q+1-n}} ds, & n-1 < q < n, \ n \in \mathbb{N}, \\ f^{(n)}(x), & q = n, \end{cases}$$

where $q > 0$ is the order of the derivative.

For the Caputo derivative and Riemann-Liouville integral we have the following two basic properties [19]

$$D^q I^q f(x) = f(x),$$

and

$$I^q D^q f(x) = f(x) - \sum_{i=0}^{n-1} f^{(i)}(0) \frac{x^i}{i!}. \tag{2.1}$$

2.2. **Bernoulli wavelets.** Bernoulli wavelets $\psi_{n,m}(x) = \psi(k, \hat{n}, m, x)$ have four arguments; $\hat{n} = n-1$, $n = 1, 2, ..., 2^{k-1}$, k can assume any positive integer, m is the order for Bernoulli polynomials and x is the normalized time. These wavelets are defined on the interval $[0, 1)$ as [13]

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{\beta}_m(2^{k-1}x - \hat{n}), & \frac{\hat{n}}{2^{k-1}} \le x < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases}$$

with

$$\tilde{\beta}_m(x) = \begin{cases} 1, & m = 0, \\ \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}} \beta_m(x), & m > 0, \end{cases}$$

where $m = 0, 1, 2, ..., M-1$, $n = 1, 2, ..., 2^{k-1}$. Here, $\beta_m(x)$ are the well-known Bernoulli polynomials of order m which can be defined by [3]

$$\beta_m(x) = \sum_{i=0}^{m} \begin{pmatrix} m \\ i \end{pmatrix} \alpha_{m-i} x^i,$$

where $\alpha_i, \ i = 0, 1, ..., m$ are Bernoulli numbers.

2.3. **Function approximation.** Suppose $f(x) \in L^2[0, 1]$. We can expand $f(x)$ in terms of the Bernoulli wavelets as

$$f(x) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) = C^T \Psi(x),$$

where C and $\Psi(x)$ are $m' \times 1$ ($m' = 2^{k-1}M$) vectors given by

$$C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1},M-1}]^T,$$

$$\Psi(x) =$$

$$[\psi_{10}(x), \psi_{11}(x), \ldots, \psi_{1M-1}(x), \psi_{20}(x), \ldots, \psi_{2M-1}(x), \ldots, \psi_{2^{k-1}0}(x), \ldots, \psi_{2^{k-1},M-1}(x)]^T.$$

The vector C can be obtained from [13]

$$\Gamma^T = C^T D,$$

where

$$\Gamma = [f_{10}, f_{11}, \dots, f_{1M-1}, f_{20}, \dots, f_{2M-1}, \dots, f_{2^{k-1}0}, \dots, f_{2^{k-1}M-1}]^T,$$

with

$$f_{ij} = \int_0^1 f(x)\psi_{ij}(x)dx,$$

and $D = [d_{nm}^{ij}]$ is a matrix of order $m' \times m'$ given by

$$D = \int_0^1 \Psi(x)\Psi(x)^T dx.$$

By similar manner, we can approximate the function $k(x,t) \in L^2([0,1]\times[0,1])$ as [22]

$$k(x,t) \simeq \Psi(x)^T K^T \Psi(t),$$

where K is an $m' \times m'$ matrix given by

$$K = D^{-1}[\int_0^1 K_1(s)\Psi(s)ds]D^{-1},$$

with

$$K_1(s) = \int_0^1 k(s,t)\Psi(t)dt.$$

3. Bernoulli wavelets operational matrix of fractional integration

In this section, we recall Bernoulli wavelets operational matrix of fractional integration by first expanding Bernoulli wavelets to block-pulse functions (BPFs) [12]. The set of BPFs over the interval [0,1) is defined as

$$b_i(x) = \begin{cases} 1, & \frac{i}{m'} \le x \le \frac{i+1}{m'}, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 0, 1, 2, \dots, m'-1$, with a positive integer value for m'. The disjointness property is

$$b_i(x)b_j(x) = \begin{cases} 0, & i \neq j, \\ b_i(x), & i = j, \end{cases} \tag{3.1}$$

and the orthogonality property is

$$\int_0^1 b_i(x)b_j(x)dx = \begin{cases} 0, & i \neq j, \\ \frac{1}{m'}, & i = j. \end{cases} \tag{3.2}$$

From the orthogonality property of BPFs, it is possible to expand functions into their block-pulse series [30]. Similarly to Chebyshev wavelets [35], Bernoulli wavelets may be expanded into an m'-term BPFs as

$$\Psi(x) \simeq \Phi B(x), \tag{3.3}$$

where Φ is an $m^{'} \times m^{'}$ matrix given by

$$\Phi \triangleq \left[\Psi\left(\frac{1}{2m^{'}}\right), \Psi\left(\frac{3}{2m^{'}}\right), \dots, \Psi\left(\frac{2m^{'}-1}{2m^{'}}\right)\right],$$

and $B(x) \triangleq [b_0(x), b_1(x), \ldots, b_{m'-1}(x)]^T$. In [20], authors have given the block-pulse operational matrix of fractional integration $F^{(q)}$ as follows:

$$I^q B(x) \simeq F^{(q)} B(x), \tag{3.4}$$

where

$$F^{(q)} = \frac{1}{m'^q} \frac{1}{\Gamma(q+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \ldots & \xi_{m'-1} \\ 0 & 1 & \xi_1 & \ldots & \xi_{m'-2} \\ 0 & 0 & 1 & \ldots & \xi_{m'-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

with $\xi_k = (k+1)^{q+1} - 2k^{q+1} + (k-1)^{q+1}$.

Now, let

$$I^q \Psi(x) \simeq P^{(q)} \Psi(x), \tag{3.5}$$

where matrix $P^{(q)}$ is the Bernoulli wavelets operational matrix of fractional integration. Using Eqs. (3.3) and (3.4) we get

$$I^q \Psi(x) \simeq I^q \Phi B(x) = \Phi I^q B(x) \simeq \Phi F^{(q)} B(x). \tag{3.6}$$

From Eqs. (3.5) and (3.6) we have

$$P^{(q)} \Psi(x) \simeq P^{(q)} \Phi B(x) \simeq \Phi F^{(q)} B(x).$$

Then, the Bernoulli wavelets operational matrix of fractional integration is given by

$$P^{(q)} = \Phi F^{(q)} \Phi^{-1}.$$

In particular, for $k = 2$, $M = 3$ and $q = 0.5$ we get

$$P^{(0.5)} = \left[\begin{array}{ccc|ccc} 0.528223 & 0.181881 & -0.0297821 & 0.443844 & -0.087099 & 0.0256378 \\ -0.14516 & 0.224295 & 0.132924 & 0.0798823 & -0.0449052 & 0.0198105 \\ -0.0598166 & -0.096441 & 0.168799 & -0.0417244 & -0.000185889 & 0.00286805 \\ \hline 0 & 0 & 0 & 0.528223 & 0.181881 & -0.0297821 \\ 0 & 0 & 0 & -0.14516 & 0.224295 & 0.132924 \\ 0 & 0 & 0 & -0.0598166 & -0.096441 & 0.168799 \end{array}\right].$$

4. Solution of the nonlinear fractional mixed Fredholm-Volterra integro-differential equation

Consider the nonlinear fractional-order integro-differential equation given by Eq. (1.1). The functions $D^q f(x)$, $g(x)$, $k_1(x,t)$ and $k_2(x,t)$ may be approximated by the Bernoulli wavelets as

$$D^q f(x) \simeq C^T \Psi(x), \tag{4.1}$$

$$g(x) \simeq G^T \Psi(x), \tag{4.2}$$

$$k_1(x,t) \simeq \Psi(x)^T K_1^T \Psi(t), \tag{4.3}$$

$$k_2(x,t) \simeq \Psi(x)^T K_2^T \Psi(t). \tag{4.4}$$

By using Eqs. (2.1), (3.5) and (4.1) we get

$$f(x) \simeq C^T P^{(q)} \Psi(x) + \sum_{i=0}^{m-1} f^{(i)}(0) \frac{x^i}{i!}. \tag{4.5}$$

Assume that

$$\sum_{i=0}^{m-1} f^{(i)}(0) \frac{x^i}{i!} \simeq J^T \Psi(x),$$

then we have

$$f(x) \simeq (C^T P^{(q)} + J^T)\Psi(x). \tag{4.6}$$

From Eqs. (3.3) and (4.6) we get

$$f(x) \simeq (C^T P^{(q)} + J^T)\Phi B(x) = AB(x),$$

where

$$A = [a_0, a_1, \ldots, a_{m'-1}] = (C^T P^{(q)} + J^T)\Phi.$$

By using Eq. (3.1) we have

$$\begin{aligned} [f(x)]^{r_i} &\simeq [AB(x)]^{r_i} = [a_0 b_0(x) + a_1 b_1(x) + \ldots + a_{m'-1} b_{m'-1}(x)]^{r_i} \\ &= a_0^{r_i} b_0(x) + a_1^{r_i} b_1(x) + \ldots + a_{m'-1}^{r_i} b_{m'-1}(x) = A_{r_i} B(x), \; i = 1, 2, \end{aligned} \tag{4.7}$$

where

$$A_{r_i} = [a_0^{r_i}, a_1^{r_i}, \ldots, a_{m'-1}^{r_i}].$$

Using Eqs. (3.2), (3.3), (4.3) and (4.7) we get

$$\begin{aligned} \int_0^1 k_1(x,t)[f(t)]^{r_1} dt &\simeq \int_0^1 \Psi(x)^T K_1^T \Psi(t) B(t)^T A_{r_1}^T dt \simeq \int_0^1 \Psi(x)^T K_1^T \Phi B(t) B(t)^T A_{r_1}^T dt \\ &= \Psi(x)^T K_1^T \Phi [\int_0^1 B(t) B(t)^T dt] A_{r_1}^T = \Psi(x)^T K_1^T \Phi diag(\tfrac{1}{m'}, \ldots, \tfrac{1}{m'}) A_{r_1}^T \\ &= \tfrac{1}{m'} \Psi(x)^T K_1^T \Phi A_{r_1}^T. \end{aligned} \tag{4.8}$$

Also, by using Eqs. (3.3), (3.4), (4.4) and (4.7) we have

$$\begin{aligned} \int_0^x k_2(x,t)[f(t)]^{r_2} dt &\simeq \int_0^x \Psi(x)^T K_2^T \Psi(t) B(t)^T A_{r_2}^T dt = \Psi(x)^T K_2^T \Phi [\int_0^x B(t) B(t)^T A_{r_2}^T dt] \\ &= \Psi(x)^T K_2^T \Phi [\int_0^x diag(A_{r_2}) B(t) dt] = \Psi(x)^T K_2^T \Phi diag(A_{r_2}) \int_0^x B(t) dt \\ &\simeq \Psi(x)^T K_2^T \Phi diag(A_{r_2}) F^{(1)} B(x) \simeq B(x)^T \Phi^T K_2^T \Phi diag(A_{r_2}) F^{(1)} B(x) \\ &= B(x)^T Q B(x) = \widetilde{Q}^T B(x), \end{aligned} \tag{4.9}$$

where $\widetilde{Q}$ is an m'-vector whose elements are diagonal entries of the following matrix

$$Q = \Phi^T K_2^T \Phi diag(A_{r_2}) F^{(1)}.$$

By substituting Eqs. (4.1), (4.2), (4.8) and (4.9) into Eq. (1.1), we obtain

$$\Psi(x)^T C - \lambda_1 \frac{1}{m'} \Psi(x)^T K_1^T \Phi A_{r_1}^T - \lambda_2 \widetilde{Q}^T B(x) \simeq \Psi(x)^T G.$$

From Eq. (3.3) we have

$$B(x)^T \Phi^T C - \lambda_1 \frac{1}{m'} B(x)^T \Phi^T K_1^T \Phi A_{r_1}^T - \lambda_2 B(x)^T \widetilde{Q} \simeq B(x)^T \Phi^T G.$$

By multiplying both sides of the above equation in $B(x)$ and integrating in the interval $[0,1]$, we get

$$\begin{array}{l} diag(\frac{1}{m'}, \ldots, \frac{1}{m'}) \Phi^T C - \frac{\lambda_1}{m'} diag(\frac{1}{m'}, \ldots, \frac{1}{m'}) \Phi^T K_1^T \Phi A_{r_1}^T \\ -\lambda_2 diag(\frac{1}{m'}, \ldots, \frac{1}{m'}) \widetilde{Q} \simeq diag(\frac{1}{m'}, \ldots, \frac{1}{m'}) \Phi^T G, \end{array}$$

or

$$\Phi^T C - \frac{\lambda_1}{m'} \Phi^T K_1^T \Phi A_{r_1}^T - \lambda_2 \widetilde{Q} \simeq \Phi^T G.$$

These equations give a nonlinear system of algebraic equations which can be solved for the unknown vector C, using Newton's iterative method. The initial guesses for Newton's iterative method are very important. For this problem, by using Eqs. (1.2) and (4.5), we choose the initial guesses such that $C^T P^{(q)} \Psi(0) = 0$ or $C^T P^{(q)} \Phi B(0) = 0$. Then the approximate solution of Eq. (1.1) can be obtained by Eq. (4.5).

5. Illustrative examples

In this section, five examples are given to demonstrate the applicability and accuracy of our method. Example 1-3 were considered in [31] and [30] by using CAS wavelets. Example 4 was first considered in [31], it was also studied in [35] by using Chebyshev wavelets. Example 5 was considered in [24] by applying Legendre wavelets. For Examples 1-4, we use the root mean -square error (RMSE) to show the accuracy of two methods. RMSE is defined by

$$\|e_{m'}(x)\|_2 = \left(\int_0^1 e_{m'}^2(x)dx\right)^{\frac{1}{2}} \simeq \left(\frac{1}{N}\sum_{i=0}^{N} e_{m'}^2(x_i)\right)^{\frac{1}{2}} = \left(\frac{1}{N}\sum_{i=0}^{N} (f(x_i) - f_{m'}(x_i))^2\right)^{\frac{1}{2}},$$

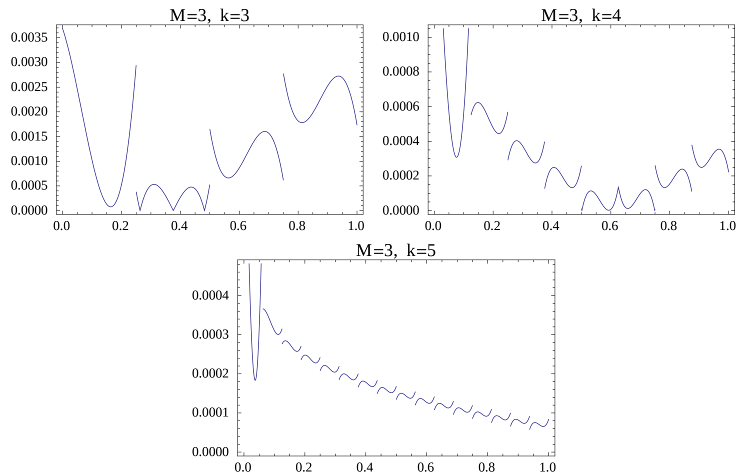
where $f(x)$ and $f_{m'}(x)$ are the exact and approximate solutions, respectively. For all examples the package of Mathematica version (10.0) has been used to solve the test problems considered in this paper.

Example 1. Consider the following fractional Fredholm integro-differential equation [31]

$$D^{\frac{5}{6}} f(x) - \int_0^1 x e^t [f(t)]^2 dt = \frac{3}{\Gamma(\frac{1}{6})} (2\sqrt[6]{x} - \frac{432}{91} \sqrt[6]{x^{13}}) - x(248e - 674),\ x \in [0,1),$$

subject to the initial conditions $f(0) = 0$. The exact solution of this equation is $f(x) = x - x^3$. The RMSE for $M = 3$ with $k = 3$, 4 and $k = 5$ of Bernoulli wavelets and CAS wavelets are shown in Table 1. We can see that as the value of k becomes

FIGURE 1. The variation of absolute error for $M = 3$ with $k = 3$, 4 and $k = 5$, for Example 1.



large, the error obtained by Bernoulli wavelets are less than the error obtained by CAS wavelets. Also, the variation of absolute error for $M = 3$ with $k = 3$, 4 and $k = 5$ are given in Figure 1.

TABLE 1. The RMSE for $M = 3$ and some k of present method and CAS wavelets, for Example 1.

Method	$\|e_{12}(x)\|_2$ $(M = 3, k = 3)$	$\|e_{24}(x)\|_2$ $(M = 3, k = 4)$	$\|e_{48}(x)\|_2$ $(M = 3, k = 5)$
CAS wavelets	2.0862×10^{-3}	6.3440×10^{-4}	2.5659×10^{-4}
Bernoulli wavelets	2.48912×10^{-6}	1.99986×10^{-7}	4.60097×10^{-8}

Example 2. Consider the following fractional Volterra integro-differential equation [30]

$$D^q f(x) - \int_0^x e^{-t}[f(t)]^2 dt = 1, \; x \in [0, 1],$$

subject to the initial conditions $f(0) = f'(0) = f''(0) = f'''(0) = 1$. The numerical results for various q between 3 and 4 are presented in Table 2. In the nonfractional case, $q = 4$, the exact solution is $f(x) = e^x$. Table 3 shows the RMSE when $q = 4$, for $M = 3$ with $k = 3$, 4 and $k = 5$ of Bernoulli wavelets and CAS wavelets.

Example 3. Consider the following fractional Volterra integro-differential equation [30]

$$D^q f(x) + \int_0^x [f(t)]^2 dt = sinh(x) + \frac{1}{2} cosh(x) sinh(x) - \frac{x}{2}, \; x \in [0, 1],$$

TABLE 2. Numerical results for various q between 3 and 4, for Example 2.

x	q=3.75	q=3.8	q=3.9	q=4	Exact
0	1.00003	1.00003	1.00002	1.00002	1
0.1	1.10518	1.10517	1.10517	1.10517	1.10517
0.2	1.22150	1.22148	1.22144	1.22141	1.22140
0.3	1.35022	1.35013	1.34998	1.34987	1.34986
0.4	1.49286	1.49261	1.4922	1.49187	1.49182
0.5	1.65111	1.65058	1.64967	1.64883	1.64872
0.6	1.82624	1.82524	1.82358	1.82219	1.82212
0.7	2.02086	2.01921	2.01633	2.01367	2.01375
0.8	2.23678	2.2342	2.22961	2.22563	2.22554
0.9	2.47661	2.47275	2.46585	2.45971	2.45960

TABLE 3. The RMSE when $q = 4$, for $M = 3$ and some k of present method and CAS wavelets, for Example 2.

Method	$\|e_{12}(x)\|_2$ $(M = 3, k = 3)$	$\|e_{24}(x)\|_2$ $(M = 3, k = 4)$	$\|e_{48}(x)\|_2$ $(M = 3, k = 5)$
CAS wavelets	1.697834×10^{-6}	8.5880067×10^{-7}	2.859761×10^{-8}
Bernoulli wavelets	2.07606×10^{-8}	1.26957×10^{-9}	7.890×10^{-11}

subject to the initial conditions $f(0) = 0$, $f^{'}(0) = 1$. In the nonfractional case $q = 2$, the exact solution is $f(x) = sinh(x)$. The comparison of the numerical results for $q = 1.8, 1.9, 1.99$ and $q = 2$ are shown in Figure 2. Table 4 shows the RMSE when $q = 2$, for $M = 3$ with $k = 3$, 4 and $k = 5$ of Bernoulli wavelets and CAS wavelets.

TABLE 4. The RMSE when $q = 2$, for $M = 3$ and some k of present method and CAS wavelets, for Example 3.

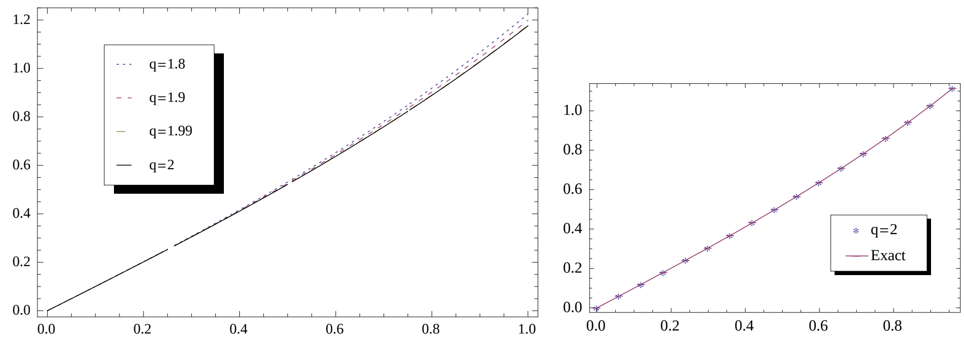
Method	$\|e_{12}(x)\|_2$ $(M = 3, k = 3)$	$\|e_{24}(x)\|_2$ $(M = 3, k = 4)$	$\|e_{48}(x)\|_2$ $(M = 3, k = 5)$
CAS wavelets	1.396927×10^{-6}	1.884591×10^{-7}	1.94042×10^{-8}
Bernoulli wavelets	2.01031×10^{-7}	1.272739×10^{-8}	7.9803×10^{-10}

Example 4. Consider the following fractional Fredholm integro-differential equation [35]

$$D^{\frac{5}{3}} f(x) - \int_0^1 (x+t)^2 [f(t)]^3 dt = \frac{6}{\Gamma(\frac{1}{3})} x^{\frac{1}{3}} - \frac{x^2}{7} - \frac{x}{4} - \frac{1}{9}, \; x \in [0,1],$$

subject to the initial conditions $f(0) = f'(0) = 0$. The exact solution of this equation is $f(x) = x^2$. This example was first considered in [31] by using CAS wavelets, it was also studied in [35] by using Chebyshev wavelets. The results in [35] have been shown to be comparable or superior to [31]. Table 5 shows the RMSE for $M = 2$

FIGURE 2. Comparison of $f(x)$ for $M = 3$, $k = 3$,with $q = 1.8, 1.9, 1.99, 2$ and exact solution, for Example 3.



with $k = 3, 4$ and $k = 5$ of Chebyshev wavelets and Bernoulli wavelets. Also, Figure 3 shows the variation of absolute error for $M = 2$ with $k = 3, 4$ and $k = 5$.

TABLE 5. The RMSE for $M = 2$ and some k of Chebyshev wavelets and present method, for Example 4.

Method	$\|e_{12}(x)\|_2$ $(M = 2, k = 3)$	$\|e_{24}(x)\|_2$ $(M = 2, k = 4)$	$\|e_{48}(x)\|_2$ $(M = 2, k = 5)$
Chebyshev wavelets	3.1863×10^{-5}	6.1566×10^{-6}	2.4897×10^{-7}
Bernoulli wavelets	1.55586×10^{-5}	1.60654×10^{-6}	1.00406×10^{-7}

Example 5. Consider the following fractional mixed Fredholm-Volterra integro-differential equation [24]

$$D^{q+1} f(x) = \int_0^x (e^t + 1)[f(t)]^2 dt + \int_0^1 xt[f(t)]^2 dt + g(x), \ \ x \in [0, 1],$$

where $g(x) = e^x - \frac{(e^x - x - 1)^3}{3} - x(\frac{e^2}{4} - 2e + \frac{11}{3})$ and $f(0) = f'(0) = 0$. The numerical results for $f(x)$ with $M = 3$, $k = 3$ and for $q = 0.75, 0.85, 0.95$, and $q = 1$ are plotted in Figure 4. For $q = 1$, the exact solution is $f(x) = e^x - x - 1$. It is noted that as q approaches 1, the numerical solution converges to the analytical solution $f(x) = e^x - x - 1$. Numerical results for different values of q are shown in Table 6. From Table 6, we see that as q approaches an integer value the error is reduced, as expected. Also, In Table 7, the CPU time with $M = 2$ and for different values of k using Legendre wavelets and Bernoulli wavelets are listed.

6. CONCLUSION

This paper proposed a numerical solution for fractional nonlinear mixed Fredholm-Volterra integro-differential equations by using the operational matrix of the fractional integration of Bernoulli wavelets. This matrix was used to transform our nonlinear integro-differential equations to nonlinear system of algebraic equations that can be

FIGURE 3. The variation of absolute error for $M = 2$ with $k = 3$, 4 and $k = 5$, for Example 4.

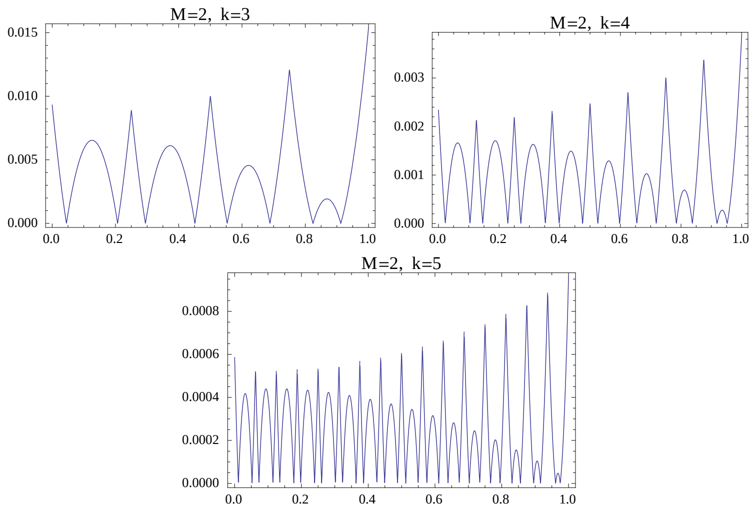


TABLE 6. Numerical results for various q, for Example 5.

x	q=0.75	q=0.85	q=0.95	q=1	Exact solution
0.1	0.012147	0.008890	0.006478	0.005320	0.005170
0.2	0.040886	0.031925	0.024849	0.021697	0.021402
0.3	0.085874	0.069557	0.056168	0.050215	0.049858
0.4	0.147568	0.122702	0.101752	0.092160	0.091824
0.5	0.227254	0.192749	0.163113	0.149211	0.148721
0.6	0.325282	0.280178	0.240852	0.222817	0.222119
0.7	0.444932	0.388064	0.337983	0.314198	0.313753
0.8	0.587842	0.517963	0.455976	0.426592	0.425541
0.9	0.757957	0.673165	0.597786	0.568149	0.559603

FIGURE 4. Comparison of $f(x)$ for $M = 3, k = 3$,with $q = 0.75, 0.85, 0.95, 1$ and exact solution, for Example 5.

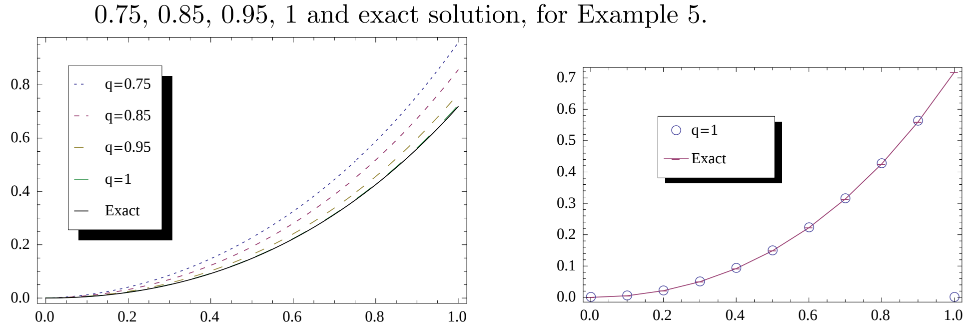


TABLE 7. CPU time for $M = 2$ and with different values of k, for Example 5.

methods	CPU times
Legendre wavelets	
k=4	1.152 s
k=5	2.794 s
k=6	9.358 s
Bernoulli wavelets	
k=4	0.359 s
k=5	1.029 s
k=6	8.720 s

solved by Newton's method. Also, as q approaches an integer value, the scheme provides solutions for the integer-order integro-differential equations. As it is seen from the numerical examples, the method provides accurate solutions.

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