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On using topological degree theory to investigate a coupled system of non linear hybrid differential equations

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Abstract In this work, we discuss the existence of solutions of nonlinear fractional differential equations. By using the topological degree theory, some results on the existence of solutions are obtained. Our analysis relies on the reduction of the problem considered to the equivalent system of Fredholm integral equations. As applications, an examples is also provided to illustrate our main results.

Keywords. Hybrid initial value problem; K-condensing, Existence of fixed point without compactness theorem; Caputo fractional derivative.

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1. Introduction

Nonlinear differential equation are crucial tools in the modeling of nonlinear real phenomena corresponding to a great variety of events, in relation with several fields of the physical sciences and technology. For instance, they appear in the study of the air motion or the fluids dynamics, electricity, electromagnetism, or the control of nonlinear processes, among others (see [2]). Moreover, most of the authors also considered the fractional differential equations as an object of mathematical investigations, we refer the readers to [1, 5, 6, 7, 8, 17] and the references therein for recent development of the theory. Perturbation techniques are useful in the nonlinear analysis for studying the dynamical system represented by nonlinear differential and integral equations.

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Evidently, some differential equation representing a certain dynamical system have no analytical solution, so the perturbation of such problem can be helpful. The perturbed differential equations are categorized into various types. An important type of these such perturbation is called a hybrid differential equation (i.e quadratic perturbation of a nonlinear differential equation), see [3] and the reference therein.
Existence theory for real world problems which can be modeled by of fractional differential equations with multi-point boundary conditions have attracted the attention of many researchers and is a rapidly growing area of investigation, [2, 9, 15].
Recently, the hybrid differential equations have been much more attractive [1, 5, 16, 17] and then there have been many works on the theory of hybrid differential equation. Addition-ally, hybrid fixed point theory can be used to developed the existence theory for the hybrid equation. The topological methods proved to be a powerful tool in the study of various problems which appears in nonlinear analysis. We refer the reader to [2, 9, 11, 12, 20, 21] for some results on existence and uniqueness of solution. In [21], the author has applied the topological degree theory in order to obtain the necessary and sufficient conditions for following nonlocal cauchy problem of the form

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t)); \quad t \in I = [0, T], \\ u(0) + g(u) = u_0, \end{cases}$$

where D^{α} is the Caputo's fractional derivative of order $\alpha \in (0, 1]$, $u_0 \in \mathbb{R}$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous. The result was extended to the case of boundary value problem by Khan and Shah [11], who studied sufficient conditions for existence results for the boundary value problem

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t)); \quad t \in I = [0, T], \\ u(0) = g(u), \quad u(1) = h(u) + \displaystyle\sum_{k=1}^{m-2} \lambda_k u(\eta_k), \end{cases}$$

where D^{α} is Caputo's fractional derivatives, $0 < \lambda_k,\ \eta_k < 1$. Shah *et al*[19], studied the Existence of solution to multi point boundary value problem of degree theory in the form of

$$\begin{cases} D^{\alpha}x(t) = \phi(t, x(t), y(t)),\ t \in I = [0, 1], \\ D^{\beta}y(t) = \psi(t, x(t), y(t)),\ t \in I = [0, 1], \\ x(0) = g(x),\ x(1) = \delta x(\eta),\ 0 < \eta < 1, \\ y(0) = h(y),\ y(1) = \gamma y(\xi),\ 0 < \xi < 1, \end{cases}$$

In (2016) Shah and Khan [12], also studied the coupled system of nonlinear boundary value problem for the existence and uniqueness solution given as

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t)),\ D^{\beta}v(t) = g(t, u(t), v(t))\ t \in I = [0, 1], \\ \lambda_1 u(0) - \gamma_1 u(\eta) - \mu_1 u(1) = \phi(u),\ \lambda_2 v(0) - \gamma_2 v(\eta) - \mu_2 v(1) = \psi(v). \end{cases}$$

. In order to enlarge the class of boundary value problems and to impose less restricted conditions, one need to search for other sophisticated tools of functional analysis. Isaia

[10], applied the technique to obtain the existence result for the integral equation without compactness in the form of

$$u(t) = \varphi(t, u(t)) + \int_a^b \psi(t, s, u(s))ds,$$

where $\varphi : [a, b] \times \mathbb{R} \to \mathbb{R}$ and $\psi : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous function with some special growth conditions.
Our purpose in this paper is to prove the existence of solution to the following system of hybrid differential equation of order "$1 < \alpha < 2$":

$$\begin{cases} D^\alpha[x(t) - f(t, x(t))] = g(t, y(t), I^\alpha y(t)),\ t \in \jmath, \\ D^\alpha[y(t) - f(t, y(t))] = g(t, x(t), I^\alpha x(t)),\ t \in \jmath, \\ D^p x(0) = 0, \quad D^p x(1) = \delta x(\eta),\ 0 < \eta < 1, \\ D^p y(0) = 0, \quad D^p y(1) = \delta y(\eta),\ for\ \alpha > 0,\ where\ \ 0 < p < 1. \end{cases} \tag{1.1}$$

The proof is rooted on a nonlinear integral equation without compactness under appropriate assumptions on operators F and G. The hypothesis imposed on operators F and G are stronger and the result is stronger as well.

2. Background Materials and Lemmas

In this section, we recall some basic definitions, lemmas and notations.

Definition 2.1. The fractional integral of order $\alpha \in \mathbb{R}_+$ of the function $h \in L^1([a, b], \mathbb{R})$ is defined as

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} h(s)\, ds.$$

provided that right hand side is point wise defined on $(0, \infty)$.

Definition 2.2. The Caputo's fractional order derivative of a function h on the interval $[a, b]$ is defined by

$${}^c D^\alpha h(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n - \alpha - 1} h^{(n)}(s)\, ds,$$

provided that right hand side is point wise defined on $(0, \infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ represents an integer part of α.

Lemma 2.3. *The fractional order differential equation of order $\alpha > 0$ of the form*

$${}^c D^\alpha h(t) = 0,\ n - 1 < \alpha \leq n,$$

has a unique solution of the form $h(t) = C_0 + C_1 t + C_2 t^2 + ... + C_{n-1} t^{n-1}$, where $C_i \in R$, $i = 0, 1, 2, ..., n - 1$.

In view of Lemma (2.3), we can easily obtain the following result

Lemma 2.4. *For $x,\ y \in C[0,1]$, $0 < \alpha,\ \beta \leq 1$, $\lambda_i \neq \mu_i + \nu_i (i = 1, 2)$, and $\lambda_i,\ \mu_i,\ \nu_i \in \mathbb{R}$ and the function $\phi(x),\ \psi(y) : C[0,1],\ \mathbb{R} \to \mathbb{R}$, the coupled system of boundary value problem (1.1) has a solution of the form*

$$\begin{cases} x(t) = f(t, x(t)) + \frac{1}{\delta}(D^{\alpha} f(1, x(1))) - f(\eta, x(\eta)) + I^{\alpha} h(t) + \frac{1}{\delta}(I^{\alpha-p} h(1)) - I^{\alpha} h(\eta), \\ y(t) = f(t, y(t)) + \frac{1}{\delta}(D^{\alpha} f(1, y(1))) - f(\eta, y(\eta)) + I^{\alpha} h(t) + \frac{1}{\delta}(I^{\alpha-p} h(1)) - I^{\alpha} h(\eta). \end{cases} \tag{2.1}$$

In the following, X will be a Banach space and $\mathbb{S} \subset P(X)$ will be the family of all its bounded sets.

Definition 2.5. The Kuratowski measure of non-compactness $A : \mathbb{S} \to \mathbb{R}_+$ is defined as

$$A(S) = \{\inf d > 0 \text{ where } s \in \mathbb{S} \text{ admits a finite cover by sets of diameter } \leq d\}$$

Definition 2.6. A topological space $X$is said to be compact if it is both complete and totally bounded.

Proposition 2.7. *The Kuratowski measure A satisfy the following properties:*
(i) $A(S) = 0$ if and only if S is relatively compact;
(ii) A is semi norm, $A(\lambda S) = |\lambda| A(S),\ \lambda \in \mathbb{R}$ and $A(E_1 + E_2) \leq A(E_1) + A(E_2)$;
(iii) $E_1 \subset E_2$ implies $A(E_1) \leq A(E_2)$; $A(E_1 U E_2) = \max\{A(E_1),\ A(E_2)\}$;
(iv) $A(Conv\ S) = A(S)$;
(v) $A(\bar{S}) = A(S)$.

Definition 2.8. Let $\mathbb{F} : \Omega \longrightarrow X$ be a continuous bounded map, where $\Omega \subset X$. Then $\mathbb{F}$ is k-Lipschitz if there exists $\lambda > 0$ such that $k(\mathbb{F}(A)) \leq \lambda k(A)$for all $A \subset \Omega$ is bounded. Further, $\mathbb{F}$ will be strict k-contraction if $\lambda < 1$.

Definition 2.9. The function $\mathbb{F}$ is k-condensing if
$k(\mathbb{F}(A)) \leq k(A)$ for all $A \subset \Omega$ bounded with $k(A) > 0$.
In other words, $k(\mathbb{F}(A)) > k(A)$ implies $k(A)$= 0.

The class of all strict k-contractions $\mathbb{F} : \Omega \longrightarrow X$ is denoted by $\Re C_k(\Omega)$ and the class of all k-condensing maps $\mathbb{F} : \Omega \longrightarrow X$ by $C_k(\Omega)$.

Moreover, recall that $\mathbb{F} : \Omega \longrightarrow X$ is Lipschitz if there exists $\lambda > 0$ such that $\|F(u) - F(v)\| \leq \lambda \|u - v\|$, for all $u, v \in \Omega$, and that if $\lambda < 1$, then $\mathbb{F}$ is a strict contraction.
For the following results, we refer to [10].

Proposition 2.10. *If $\mathbb{F} : \Omega \longrightarrow X$ are k-Lipschitz maps with constants k_1 and k_2 respectively, then $F + G : \Omega \longrightarrow X$ are k-Lipschitz with constants $k_1 + k_2$.*

Proposition 2.11. *If $\mathbb{F} : \Omega \longrightarrow X$ is compact then $\mathbb{F}$ is k-Lipschitz with constants λ.*

Proposition 2.12. *If $\mathbb{F} : \Omega \longrightarrow X$ is Lipschitz with constants λ, then $\mathbb{F}$ is k-Lipschitz with same constants λ.*

The following theorem due to Isaia [10], plays important rule for our main result.

Theorem 2.13. *If $\mathbb{F} : X \longrightarrow X$ be k-condensing and*

$$S = \{x \in X : \text{there exist } \mu \in [0,1] \text{such that } x = \mu\mathbb{F}x\}.$$

If S is a bounded set in X, so there exists $r > 0$ such that $S \subset B_r(0)$, then the degree

$$D(I - \mu\mathbb{F}, B_r(0), 0) = 1, \text{ for all } \mu \in [0,1].$$

Consequently, $\mathbb{F}$ has at least one fixed point and the set of fixed points of $\mathbb{F}$ lies in $B_r(0)$.

Now denoting by $X = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $H = C[0,1] \longrightarrow \mathbb{R}$ with the topological norm $\|x\|= \max\{|x(t)|: t \in [0,1]\}$. Then the product space $X \times Y$ defined by $X \times Y = \{(x,y) : x \in X, y \in Y\}$, is a Banach space under the norm $\|(x,y)\|= \max\{\|x\|, \|y\|\}$. We list the following assumptions:

(A_1) There exist constants $K', K'' \in [0,1)$ such that for $u, v, x, \bar{x} \in C(H,\mathbb{R})$,
$|f(u,x) - f(u,\bar{x})|\leq K'\|x - \bar{x}\|$, $D^p|[f(1,x) - f(1,\bar{x})]|\leq \frac{1}{\delta}K''\|x - \bar{x}\|$.
$|g(v,x) - g(v,\bar{x})|\leq K'\|x - \bar{x}\|$, $D^p|[g(1,x) - g(1,\bar{x})]|\leq \frac{1}{\delta}K''\|x - \bar{x}\|$.

(A_2) There exist constants $C_\phi,\ C_\psi,\ M_\phi$ and $M_\psi > 0$ such that for $(x,y) \in C(H,\mathbb{R})$,
$|\phi(u)|\leq C_\phi\|x\|^{q1}+M_\phi$, $|\psi(v)|\leq C_\psi\|y\|^{q1}+M_\psi$.

(A_3) There exist constants C_f, C_g and M_f, M_g such that for $t \in [0,1]$, and $(x,y) \in C(H,\mathbb{R})$,
$|f(t,x(s),y(s))|\leq C_{f1}\|x\|^{q2}+C_{f2}\|y\|^{q2}+M_f$,
$|g(t,x(s),y(s))|\leq C_{g1}\|x\|^{q2}+C_{g2}\|y\|^{q2}+M_g$.

3. Existence and uniqueness result of the system

For the existence of solution of the coupled system (1.1), it is enough to show that the integral equation (2.1) of the system (1.1) has at least one solution $(x,y) \in X \times Y$. Define the following operators $F, G, T : X \times Y \longrightarrow X \times Y$ by

$$F(x,y)(t) = \big(F_1x(t), F_2y(t)\big),\ \ G(x,y)(t) = (G_1x(t), G_2y(t)).$$

and

$$T(x,y) = F(x,y) + G(x,y).$$

Where

$$F_1x(t) = f(t,x(t)) + \tfrac{1}{\delta}(D^\alpha f(1,x(1)) - f(\eta,x(\eta)),$$

$$F_2y(t) = f(t,y(t)) + \tfrac{1}{\delta}(D^\alpha f(1,y(1)) - f(\eta,y(\eta)),$$

$$G_1x(t) = \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),I^\alpha x(s))ds$$

$$+\tfrac{1}{\delta}\tfrac{1}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1} f(s,x(s),I^\alpha x(s))ds$$

$$-\tfrac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1} f(s,x(s),I^\alpha x(s))ds,$$

$$G_2y(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, y(s), I^\beta y(s))ds$$
$$+\frac{1}{\delta}\frac{1}{\Gamma(\beta-p)}\int_0^1 (1-s)^{\beta-p-1} g(s, y(s), I^\beta y(s))ds$$
$$-\frac{1}{\Gamma(\beta)}\int_0^\eta (\eta-s)^{\beta-1} g(s, y(s), I^\beta y(s))ds.$$

The continuity of f, g shows that the operator T is well define. The integral equation (2.1) can be written as an operator equation

$$(x, y) = T(x, y) = F(x, y) + G(x, y), \tag{3.1}$$

and has a fixed point of the operator equation (3.1) are solution of the integral equation (2.1).

Lemma 3.1. *Under the Assumption A_1 and A_2, the operator $F : X \times Y \longrightarrow X \times Y$ is Lipschitz with constant K and satisfied the growth condition*

$$\|F(x, y)\| \leq C\|(x, y)\|^{q1} + M. \tag{3.2}$$

Proof. For $(x, y), (\bar{x}, \bar{y}) \in X \times Y$, such that

$$|F_1x - F_1\bar{x}| \leq |f(t, x(t) - f(t, \bar{x}(t))|$$
$$+\frac{1}{\delta}|(D^\alpha f(1, x(1)) - D^\alpha f(1, \bar{x}(1)))| - |f(\eta, x(\eta) - \eta, \bar{x}(\eta))|.$$

Using A_1,

$$|F_1x - F_1\bar{x}| \leq k'|x - \bar{x}| + \frac{1}{\delta}\delta k''|x - \bar{x}| - k'''|x - \bar{x}|.$$

$$|F_1x - F_1\bar{x}| \leq k_1|x - \bar{x}|. \tag{3.3}$$

Where $k_1 = max\{k', k'', k'''\}$. Hence F_1 is Lipschitz with constant k_1. Similarly

$$|F_2y - F_2\bar{y}| \leq k_2|y - \bar{y}|. \tag{3.4}$$

Which implies F_2 is Lipschitz with constant k_2, so we have

$$\|F(x, y) - F(\bar{x}, \bar{y})\| \leq \max\Big(k_1, k_2\Big)\|(x, y) - (\bar{x}, \bar{y})\|.$$

$\max(k_1, k_2) = K$,

$$\|F(x, y) - F(\bar{x}, \bar{y})\| \leq K\|(x, y) - (\bar{x}, \bar{y})\|,$$

hence by proposition (2.10) F is Lipschitz with constant K. So F is $\alpha - Lipschitz$ with constant K. For the growth condition, using the assumption (A_2), we obtain

$$|F_1(x)| = |\phi(u)| \leq C_\phi\|x\|^{q1} + M_\phi,$$

$$|F_2(y)| = |\psi(v)| \leq C_\psi\|y\|^{q1} + M_\psi.$$

Hence we get that

$$\|F(x, y)\| \leq C\|(x, y)\|^{q1} + M,$$

where $C = \max(C_\phi, C_\psi)$ and $M = \max(M_\phi, M_\psi)$. □

Lemma 3.2. *The operator $G : X \times Y \longrightarrow X \times Y$ is compact. Consequently G is Lipschitz with constant zero.*

Proof. First, we prove the continuity of G. Chose a bounded subset

$$E_s = \{(x, y) \in X \times Y : \|(x, y)\| \leq E\} \subset X \times Y$$

and consider a sequence $\{k_n = (x_n, y_n)\} \in E_s$, such that $k_n \longrightarrow k = (x, y)$ as $n \longrightarrow \infty$ in E_s we need to show that $\|Gk_n - Gk\| \longrightarrow 0$ as $n \longrightarrow \infty$. From the continuity of $f(t, x, y)$, it follows that $f(s, x_n, y_n) \longrightarrow f(s, x, y)$, as $n \longrightarrow \infty$. In view of (A_3), we obtained the following relations:

$$(t-s)^{\alpha-1}\|f(s, x_n(s), y_n(s)) - f(s, x(s), y(s))\| \leq (t-s)^{\alpha-1}[C_f^1 R + C_f^2 + M_f],$$

$$(1-s)^{\alpha-p-1}\|f(s, x_n(s), y_n(s)) - f(s, x(s), y(s))\| \leq (1-s)^{\alpha-p-1}[C_f^1 R + C_f^2 + M_f],$$

$$(\eta-s)^{\alpha-1}\|f(s, x_n(s), y_n(s)) - f(s, x(s), y(s))\| \leq (\eta-s)^{\alpha-1}[C_f^1 R + C_f^2 + M_f].$$

Which implies that each term on the left is integrable, so by Lebesgue dominated convergent theorem, we have

$$\int_0^t (t-s)^{\alpha-1}|f(s, x_n(s), Y_n(s)) - f(s, x(s), y(s))|ds \longrightarrow 0, \text{as } n \longrightarrow \infty$$

and

$$\int_0^1 (1-s)^{\alpha-p-1}|f(s, x_n(s), Y_n(s)) - f(s, x(s), y(s))|ds \longrightarrow 0, \text{as } n \longrightarrow \infty,$$

$$\int_0^\eta (\eta-s)^{\alpha-1}|f(s, x_n(s), Y_n(s)) - f(s, x(s), y(s))|ds \longrightarrow 0, \text{as } n \longrightarrow \infty.$$

Hence, $\|G_1(x_n, y_n) - G_1(x, y)\| \longrightarrow 0$ as $n \longrightarrow \infty$. Similarly, we obtain $\|G_2(x_n, y_n) - G_2(x, y)\| \longrightarrow 0$ as $n \longrightarrow \infty$. It follow that $\|G(x_n, y_n) - G(x, y)\| \longrightarrow 0$ as $n \longrightarrow \infty$. Which implies the continuity of the operator G. Moreover, G satisfies the following growth conditions

$$\|G(x, y)\| \leq \triangle(\|(x, y)\|^{q2} + M^*). \tag{3.5}$$

For the growth condition, we note that

$$|G_1(x,y)(t)| \leq \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|f(s, x(s), y(s))|ds$$

$$+\tfrac{1}{\delta}\tfrac{1}{\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1}|f(s, x(s), y(s))|ds$$

$$-\tfrac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1}|f(s, x(s), y(s))|ds,$$

$$\|G_1(x,y)(t)\| \leq \frac{|t_2^\alpha - t_1^\alpha|}{\Gamma(\alpha+1)} C_f\|x\|^{q2} + C_f\|y\|^{q2} + M_f.$$

Similarly, we obtain

$$\|G_2(x,y)(t)\| \leq \frac{|t_2^\alpha - t_1^\alpha|}{\Gamma(\alpha+1)} C_g\|x\|^{q2} + C_g\|y\|^{q2} + M_g.$$

So we get the growth condition (3.5) as

$$\|G(x,y)(t)\| \leq \triangle(\|(x,y)\|^{q2}+M^*),$$

where $\triangle = \max(C_f, C_g)\frac{|t_2^\alpha - t_1^\alpha|}{\Gamma(\alpha+1)}$ and $M^* = \max(M_f, M_g)$.
In order to prove the compactness of G, we consider a bounded set $M \subset E_s \subset X \times Y$ and we will show that $G(M)$ is relatively compact in $X \times Y$. For any $k_n = (x_n, y_n) \in M \subset E_s$, the growth condition (3.5) implies that

$$\|G(x,y)(t)\| \leq \triangle(\|(x,y)\|^{q2}+M^*)$$

That is, $G(M)$is uniformly bounded. For equi-continuity of G, choose $0 \leq t \leq \tau \leq 1$. Then we have

$$\begin{aligned}
&\|G_1(x_n,y_n)(t) - G_1(x_n,y_n)(\tau)\| = \\
&\qquad \Big|\frac{1}{\Gamma(\alpha)}\int_0^t [(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}] f(s, x_n(s), y_n(s)) ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_t^\tau (\tau-s)^{\alpha-1} f(s, x_n(s), y_n(s)) ds\Big| \\
&\qquad \leq \frac{1}{\Gamma(\alpha)}\int_0^t [(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}] |f(s, x_n(s), y_n(s))| ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_t^\tau (\tau-s)^{\alpha-1} |f(s, x_n(s), y_n(s))| ds \\
&\qquad \leq \frac{1}{\Gamma(\alpha+1)}\Bigg[t^\alpha - \tau^\alpha + (\tau-t)^\alpha + (\tau-t)^\alpha\Bigg]\Bigg(C_{f1}\|x\|^{q2} + C_{f2}\|y\|^{q2} + M_f\Bigg),
\end{aligned}$$

$$\begin{aligned}
&\|G_1(x_n,y_n)(t) - G_1(x_n,y_n)(\tau)\| \leq \\
&\qquad \Bigg(\frac{C_{f1}\|x\|^{q2} + C_{f2}\|y\|^{q2} + M_f}{\Gamma(\alpha+1)}\Bigg)\Bigg[t^\alpha - \tau^\alpha + (\tau-t)^\alpha + (\tau-t)^\alpha\Bigg].
\end{aligned}$$

Thus

$$\begin{aligned}
&\|G_1(x_n,y_n)(t) - G_1(x_n,y_n)(\tau)\| \leq \\
&\qquad \Bigg(\frac{(C_{f1} + C_{f2})\|E\|^{q2} + M_f}{\Gamma(\alpha+1)}\Bigg)\Bigg[t^\alpha - \tau^\alpha + (\tau-t)^\alpha + (\tau-t)^\alpha\Bigg].
\end{aligned} \tag{3.6}$$

Similarly, we obtain

$$\begin{aligned}
&\|G_2(x_n,y_n)(t) - G_2(x_n,y_n)(\tau)\| \leq \\
&\qquad \Bigg(\frac{(C_{g1} + C_{g2})\|E\|^{q2} + M_g}{\Gamma(\beta+1)}\Bigg)\Bigg[t^\beta - \tau^\beta + (\tau-t)^\beta + (\tau-t)^\beta\Bigg].
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we follow that

$$\|G_1(x_n,y_n)(t) - G_1(x_n,y_n)(\tau)\| \longrightarrow 0, \quad \|G_2(x_n,y_n)(t) - G_2(x_n,y_n)(\tau)\| \longrightarrow 0.$$

as $t \longrightarrow \tau$, which implies that $G(x, y)$ is equi-continues.
For every $(x, y) \in M$, the set $G(M) \subset X \times Y$ satisfies the hypothesis of Arzela-Ascoli theorem, so $G(M)$ is relatively compact in $X \times Y$. Hence G is k-Lipschitz with constant 0. □

Theorem 3.3. *Assume the assumption $(A_1) - (A_3)$ are satisfied. Then the BVP (1) has at least one solution $(x, y) \in X \times Y$ and the set of solutions is bounded in $X \times Y$.*

Proof. As we proved in Lemma (3.1), F is k-Lipschitz with constant K. and by Lemma (3.2), G is k-Lipschitz with constant 0. Consequently T is k-Lipschitz with constant K. Hence T is strict k-contraction with constant K. Since $K \in [0, 1)$, so T is k-condensing.

Now consider the following set

$$S = \{(x, y) \in X \times Y : \text{there exist } \lambda \in [0, 1], \text{ such that } (x, y) = \lambda T(x, y).$$

We need to prove S is bounded. For $(x, y) \in S$, we have

$$(x, y) = \lambda T(x, y) = \lambda(F(x, y) + G(x, y)),$$

which implies that

$$\|x\| = \lambda[\|F_1(x)\| + \|G_1(x)\|],$$

$$\|x\| \leq \lambda[C_\phi \|x\|^{q1} + M_\phi + C_{f1}\|x\|^{q2} + C_{f2}\|y\|^{q2} + M_f]. \tag{3.8}$$

Similarly we can prove that

$$\|y\| \leq \lambda[C_\psi \|y\|^{q1} + M_\psi + C_{g1}\|x\|^{q2} + C_{g2}\|y\|^{q2} + M_g]. \tag{3.9}$$

The inequalities (3.8) and (3.9) combine with $0 \leq q_1, q_2 < 1$ show that S is bounded in $X \times Y$, on other words if we dividing (3.8) by $\|x\|$ and letting $\|x\| \longrightarrow \infty$ we write as

$$1 \leq \lim_{\|x\| \to \infty} \lambda \left(\frac{C_\phi}{\|x\|^{1-q1}} + \frac{C_{f1}}{\|x\|^{1-q2}} + \frac{\|y\|^{q2} + M_\phi + M_f}{\|x\|^{1-q1}} \right) = 0. \tag{3.10}$$

Which is a contradiction. A similar contradiction aries. when we divide (3.9) by $\|y\|$ and let $\lim_{\|y\| \to \infty}$. Thus T has at least one fixed point, which corresponds to a solution of (1.1). The set of the solution is bounded. □

Example 3.4.

$$\begin{cases} D^{1.5}[x(t) - f(t, x(t))] = g(t, y(t), I^{1.5}y(t)), \ t \in \jmath, \\ D^{1.5}[y(t) - f(t, y(t))] = g(t, x(t), I^{1.5}x(t)), \ t \in \jmath, \\ D^{0.5}x(0) = 0, \quad D^{0.5}x(1) = \dfrac{1}{2}x(\dfrac{1}{2}), \\ D^{0.5}y(0) = 0, \quad D^{0.5}y(1) = \dfrac{1}{2}y(\dfrac{1}{2}), \end{cases} \tag{3.11}$$

where $f(t, x(t)) = \frac{t^2 \sin x(t)}{50+t^2}$, $g(t, x(t), I^{1.5}x(t)) = \frac{\exp(-t)}{50} \Bigg[\cos x(t) + I^{1.5} \cos x(t) \Bigg]$. Obviously the above functions satisfy the assumptions from $(A_1) - (A_5)$. Also the auschwitz conditions of Lemma 3.1 are satisfied. Further $q_1 = q_2 = 1$ and $C_{f_1} = C_{f_2} = \frac{1}{50}$, $C_{g_1} = C_{g_2} = \frac{1}{50}, M_f = 0, M_g = 0$. Now by simple calculation one can verify that all the conditions of Theorem 3.3 are satisfied so the given problem (3.11) has at least one solution. Also one can easily obtain that the solution set is bounded.

4. Conclusion

By the use of topological degree theory, we have developed sufficient conditions for the existence of at least one solution to a coupled system of nonlinear hybrid differential equations with fractional order. The mentioned conditions guarantee the existence of solutions the considered problem with boundary conditions. Finally by suitable examples we have verified the established results.

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