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Approximate symmetry and exact solutions of the perturbed nonlinear Klein-Gordon equation

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Abstract In this paper, the Lie approximate symmetry analysis is applied to investigate new exact solutions of the perturbed nonlinear Klein-Gordon equation. The nonlinear Klein-Gordon equation is used to model many nonlinear phenomena. The tanh-coth method, is employed to solve some of the obtained reduced ordinary differential equations. We construct new analytical solutions with small parameter which is effectively obtained by the proposed method.

Keywords. Perturbed Klein-Gordon equation, Exact solutions, Approximate symmetry, Approximate invariant solutions.

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1. Introduction

The classical Lie Symmetry method, originally introduced by Sophus Lie (1895), was popularized in [13] and presented in a modern form using the jet space theory in [11]. This method leads us to one-parameter group of transformations called classical symmetries that leaves the equation unchanged, and hence, they map the set of all solutions to itself. These symmetries are used to reduce the order of ordinary differential equations, or to reduce the number of independent variables of PDEs [4].

The fact that symmetry reductions for many PDEs cannot be determined, via the classical symmetry method, is the source of motivated to create several generalizations of the classical Lie group approach for symmetry reductions. Consequently, several alternative reduction methods have been proposed, such as Lie-Bäcklund symmetry, nonclassical symmetry, potential symmetry, etc. [1, 2]. One of these techniques which is extremely applied particularly for nonlinear problems is perturbation analysis. It is worth mentioning that sometimes differential equations which appear in mathematical modelings are presented with terms involving a parameter called the perturbed term.

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Because of the instability of the Lie point symmetries with respect to perturbation of coefficients of differential equations, a new class of symmetries has been created for such equations, which are known as approximate symmetries. In the last century, in order to have the utmost result from the methods, combination of Lie symmetry method and perturbations are investigated and two different so-called approximate symmetry methods have been developed. The first method was presented by Baikov et al. [3]. The second procedure was proposed by Fushchich and Shtelen [6] and later was followed by Euler et al. [5]. This method is generally based on the perturbation of dependent variables. In [10], a comprehensive comparison of these two methods is presented.

We will investigate the vector fields, approximate symmetry, symmetry reductions and new exact solutions to the perturbed nonlinear Klein-Gordon equation [12]:

$$u_{tt} - b^2 u_{xx} + a^2 u = -\epsilon q u^3 \,, \tag{1.1}$$

where $0 < \epsilon \ll 1$ is a small parameter and a, b and q are arbitrary constants, with the method of Baikov et al. [3, 8].

This work is organized as follows. In section 2, we present approximate symmetry and optimal system of the perturbed nonlinear Klein-Gordon equation. Section 3 is devoted to symmetry reductions of ordinary differential equations. In section 4, the exact analytic solutions to the equation are investigated by means of the tanh-coth method. Finally, the conclusions will be given in section 5.

2. Approximate Symmetry and optimal system

In this paper, the approximate equation $f \approx g$ means that $f(x,\epsilon) = g(x,\epsilon) + o(\epsilon)$ and

$$F(z,\epsilon) \approx F_0(z) + \epsilon F_1(z) = u_{tt} - b^2 u_{xx} + a^2 u + \epsilon q u^3.$$

Recall that the generator of an approximate transformation group admitted by Eq. (1.1) will be written in the form of (see [3, 8]):

$$X = \tau(t,x,u,\epsilon)\frac{\partial}{\partial t} + \xi(t,x,u,\epsilon)\frac{\partial}{\partial x} + \eta(t,x,u,\epsilon)\frac{\partial}{\partial u}, \tag{2.1}$$

where

$$\tau(t,x,u,\epsilon) \approx \tau_0(t,x,u) + \epsilon\tau_1(t,x,u),$$

$$\xi(t,x,u,\epsilon) \approx \xi_0(t,x,u) + \epsilon\xi_1(t,x,u),$$

$$\eta(t,x,u,\epsilon) \approx \eta_0(t,x,u) + \epsilon\eta_1(t,x,u).$$

It is convenient to identify X with its canonical representative: $X = X_0 + \epsilon X_1$. If the vector field (2.1) generates an approximate symmetry of the Eq. (1.1), then X must satisfy the Lie approximate symmetry condition:

$$\Big[X^{(2)} F\left(z,\epsilon\right)\Big]_{F(z,\epsilon)\approx 0} = o(\epsilon), \tag{2.2}$$

or

$$\Big[X_0^{(2)}F_0(z)+\epsilon(X_1^{(2)}F_0(z)+X_0^{(2)}F_1(z))\Big]_{F(z,\epsilon)\approx 0}=o(\epsilon), \tag{2.3}$$

where $X^{(2)}$ denotes the 2-th order prolongation of X. Eq. (2.2) is the determining equation for infinitesimal approximate symmetries.

By solving system (2.3), we obtain

$$\begin{cases} X_0=(c_1x+c_3)\,\partial_t+(c_1t+c_2)\,\partial_x, \\ X_1=(c_4x+c_6)\ \partial_t+(c_4t+c_5)\ \partial_x+c_7u\,\partial_u, \end{cases} \tag{2.4}$$

therefore

$$X=(c_1x+\epsilon c_4x+\epsilon c_6+c_3)\ \partial_t+(c_2+\epsilon tc_4+\epsilon c_5+c_1t)\ \partial_x+\epsilon c_7u\,\partial_u\,, \tag{2.5}$$

where c_i, $i=1,2,\dots,7$ are arbitrary constants. Hence the infinitesimal approximate symmetries of Eq. (1.1) form the seven-dimensional approximate Lie algebra (see [7]) spanned by the following independent operators:

$$\begin{cases} v_1=\ \partial_t, \quad v_2=\ \partial_x, \quad v_3=x\,\partial_t+t\,\partial_x, \quad v_4=\epsilon\,\partial_t, \\ \quad v_5=\epsilon\,\partial_x, \quad v_6=\epsilon x\,\partial_t+\epsilon t\,\partial_x, \quad v_7=\epsilon u\,\partial_u, \end{cases} \tag{2.6}$$

where their approximate commutator (see [7]), evaluated in the first order of precision, is given in Table 1.

TABLE 1. Commutators of approximate symmetry of Eq. (1.1).

$[v_i,v_j]$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	0	v_2	0	0	v_5	0
v_2	0	0	v_1	0	0	v_4	0
v_3	$-v_2$	$-v_1$	0	$-v_5$	$-v_4$	0	0
v_4	0	0	v_5	0	0	0	0
v_5	0	0	v_4	0	0	0	0
v_6	$-v_5$	$-v_4$	0	0	0	0	0
v_7	0	0	0	0	0	0	0

The approximate operator $X=X_0+\epsilon X_1$ generates the one-parameter approximate transformation group given by the following approximate exponential map (see [8]):

$$\bar{x}^i=(1+\epsilon \ll aX_0,aX_1\gg)\exp\left(aX_0\right)(x^i), \quad i=1,2,3,$$

where $x^1=t, x^2=x, x^3=u$ and

$$\exp(aX_0)=1+aX_0+\frac{a^2}{2!}X_0^2+\frac{a^3}{3!}X_0^3+\cdots,$$

and

$$\ll aX_0,aX_1\gg=aX_1+\frac{a^2}{2!}\left[X_0,X_1\right]+\frac{a^3}{3!}\left[X_0,\left[X_0,X_1\right]\right]+\cdots.$$

Furthermore, for Eq. (1.1), the one-parameter approximate transformation group g_i generated by the v_i for $i = 1, 2, \ldots, 7$ are given in the followings:

$$\begin{cases} g_1 & : & (t, x, u) & \mapsto & (t + a, x, u), \\ g_2 & : & (t, x, u) & \mapsto & (t, x + a, u), \\ g_3 & : & (t, x, u) & \mapsto & (\frac{t-x}{2}e^{-a} + \frac{t+x}{2}e^{a}, -\frac{t-x}{2}e^{-a} + \frac{t+x}{2}e^{a}, u), \\ g_4 & : & (t, x, u) & \mapsto & (t + \epsilon a, x, u), \\ g_5 & : & (t, x, u) & \mapsto & (t, x + \epsilon a, u), \\ g_6 & : & (t, x, u) & \mapsto & (t + a\epsilon x, x + a\epsilon t, u), \\ g_7 & : & (t, x, u) & \mapsto & (t, x, (1 + \epsilon a)u). \end{cases}$$

Consequently, if $u = f(t, x)$ is a solution of the Eq. (1.1), so are the functions

$$\begin{cases} g_1(a) \cdot f(t, x) & = & f(t - a, x), \\ g_2(a) \cdot f(t, x) & = & f(t, x - a), \\ g_3(a) \cdot f(t, x) & = & f(t \cosh a - x \sinh a, x \cosh a - t \sinh a), \\ g_4(a) \cdot f(t, x) & = & f(t - \epsilon a, x), \\ g_5(a) \cdot f(t, x) & = & f(t, x - \epsilon a), \\ g_6(a) \cdot f(t, x) & = & f(t - \epsilon a x, x - a\epsilon t), \\ g_7(a) \cdot f(t, x) & = & f(t, x)(1 + a\epsilon). \end{cases} \tag{2.7}$$

Since a solution can be used to generate new solutions using different groups, it would be convenient to identify the minimum collection of subgroups that will generate all possible group invariant solutions. Such a collection is called an optimal system and it is constructed by examining the ways in which group invariant solutions transform among themselves through the adjoint operation [4, 11].

An optimal system of one-dimensional approximate Lie algebras of the perturbed nonlinear Klein-Gordon equation is provided by:

$$\begin{cases} v_1, \quad \alpha v_1 + v_2, \quad v_3, \quad \gamma v_3 + v_4, \quad \gamma v_3 + v_5, \\ \quad \alpha v_1 + \beta v_2 + \gamma v_3 + v_6, \quad \alpha v_1 + \beta v_2 + \gamma v_3 + \zeta v_6 + v_7, \end{cases} \tag{2.8}$$

where α, β, ζ and $\gamma \neq 0$ are arbitrary constants.

3. Symmetry Reductions

In the previous section, we obtained the infinitesimal approximate symmetry, one-parameter approximate transformation group and the optimal systems of Eq. (1.1). Now, we deal with the symmetry reductions, exact solutions and approximate solutions of the equation. We will consider the following similarity reductions and approximate group-invariant solutions based on the optimal system method. From an optimal system of approximate group-invariant solutions to an equation, every other such solution to the equation can be derived.

Reduction 1. Similarity variables related to the generator v_1, are $u(t,x) = f(x)$, Substituting into Eq. (1.1), we reduce it to the following ODE:

$$b^2 f'' - q\epsilon f^3 - a^2 f = 0, \tag{3.1}$$

where $f' = df/dx$.

Reduction 2. In general, the traveling wave solutions to a PDE arise as special group-invariant solutions in which the group under consideration is a translational group on the space of independent variables. In the present case, we consider the generator $v_1 + cv_2$ ($c \neq 0$), in which c is a fixed constant which will determine the speed of the waves. Global invariants of this group are as follows

$$\eta = x - ct, \qquad u(t,x) = f(\eta). \tag{3.2}$$

In view of (3.2), we have $u(t,x) = f(x - ct) = f(\eta)$. Substituting it into (1.1), we find the reduced ordinary differential equation for the traveling wave solutions to be

$$(c^2 - b^2)f'' + \epsilon q f^3 + a^2 f = 0, \tag{3.3}$$

where $f' = df/d\eta$.

Reduction 3. For the generator v_3 the similarity variables and similarity solutions $\eta = x^2 - t^2$, $u(t,x) = f(\eta)$, and for $b = 1$ the reduced Eq. (1.1) is the following ODE:

$$4\eta f'' + 4f' - a^2 f = \epsilon q f^3, \tag{3.4}$$

where $f' = df/d\eta$.

Reduction 4. For the generator

$$X = v_1 + v_5 = \partial_t + \epsilon\, \partial_x = X_0 + \epsilon X_1\,,$$

the approximate invariants are written in the form of $J(t,x,u,\epsilon) = J^0(t,x,u,) + \epsilon J^1(t,x,u) + o(\epsilon)$ and they are determined by the equation, $X(J) = o(\epsilon)$, or

$$\begin{cases} X_0(J^0) = \alpha \frac{\partial J^0}{\partial t} + \beta \frac{\partial J^0}{\partial x} = 0\,, \\ X_0(J^1) + X_1(J^0) = \alpha \frac{\partial J^1}{\partial t} + \beta \frac{\partial J^1}{\partial x} + \epsilon \frac{\partial J^0}{\partial t} = 0\,. \end{cases} \tag{3.5}$$

By solving Eqs. (3.5), we find two functionally independent approximate invariants: $\eta = x - \epsilon t$, $\mu = u$. Similarity variables and approximately invariant solutions are $\eta = x - \epsilon t$ and $u(t,x) = f(\eta)$, so, we have

$$(\epsilon^2 - b^2)f'' + \epsilon q f^3 + a^2 f = 0, \tag{3.6}$$

where $f' = df/d\eta$.

Reduction 5. For the generator $v_2 + v_4$, we have $u(t,x) = f(\eta)$, where $\eta = t - \epsilon x$. Substituting it into Eq. (1.1), we reduce it to the following ODE:

$$(1 - \epsilon^2 b^2)f'' + \epsilon q f^3 + a^2 f = 0, \tag{3.7}$$

where $f' = df/d\eta$.

Reduction 6. Using the generator $v_1 + v_7$, we obtain the similarity variables $\eta = x$, $\mu = u - ut\epsilon$ and approximate invariant solutions $u(t,x) = f(x)/(1-\epsilon t)$, and the reduced Eq. (1.1) is the following ODE:

$$\frac{b^2}{1-\epsilon t} f'' - \epsilon q f^3 - \big(a^2 + \frac{2\epsilon^2}{(1-\epsilon t)^2}\big) f = 0, \tag{3.8}$$

where $f' = df/dx$.

4. Exact solutions and Approximate solutions

Here we consider some reduced equations of previous section.

4.1. **Exact analytical solutions of Eq. (3.1).** We apply the tanh-coth method [9, 14], to solve the Eq. (3.1). Then using solutions of the Eq. (3.1), we can obtain some solutions of Eq. (1.1).

We assume that the solution of Eq. (3.1) can be expressed in the form,

$$f = \sum_{j=1}^{m} b_j T^{-j} + \sum_{i=1}^{m} b_i T^i, \tag{4.1}$$

where $T = \tanh(k\eta)$, $\eta = x$, b_i $(i = -m, \dots, m)$ and k are constants, m is positive integer to be determined later (if possible), so that

$$\begin{cases} \frac{d}{d\eta} = k\left(1-T^2\right)\frac{d}{dT}, \\ \frac{d^2}{d\eta^2} = -kT\left(1-T^2\right)\frac{d}{dT} + k^2\left(1-T^2\right)^2\frac{d^2}{dT^2}. \end{cases} \tag{4.2}$$

In order to determine the value of m, we balance the highest order linear term with the highest order nonlinear term in Eq. (3.1), to obtain $m+2 = 3m$, so that $m = 1$. In this case, the trial equation (4.1) reduces to:

$$f = b_{-1}T^{-1} + b_0 + b_1 T. \tag{4.3}$$

In order to determine the values of b_{-1}, b_0, b_1, k and c, substituting (4.3), (4.2) into (3.1) and collecting all the terms of powers of T, and setting each coefficient to zero, we get the following system of algebraic equations:

$$\begin{cases} T^{-3} & : & -\epsilon q b_{-1}^3 + 2k^2 b_{-1} b^2 = 0, \\ T^{-2} & : & -3\epsilon q b_{-1}^2 b_0 = 0, \\ T^{-1} & : & -3\epsilon q b_{-1}^2 b_1 - 3\epsilon q b_{-1} b_0^2 - 2k^2 b_{-1} b^2 - b_{-1} a^2 = 0, \\ T^0 & : & -6\epsilon q b_{-1} b_0 b_1 - \epsilon q b_0^3 - b_0 a^2 = 0\,, \\ T^1 & : & -3\epsilon q b_{-1} b_1^2 - 3\epsilon q b_0^2 b_1 - 2k^2 b_1 b^2 - b_1 a^2 = 0, \\ T^2 & : & -3\epsilon q b_0 b_1^2 = 0, \\ T^3 & : & -\epsilon q b_1^3 + 2k^2 b_1 b^2 = 0. \end{cases}$$

By solving the above system, we obtain the following different sets of solutions:

$$\left\{ k = \frac{\pm a}{2b}, b_{-1} = -b_1 = \frac{\pm a\sqrt{2}}{2\sqrt{\epsilon q}}, b_0 = 0 \right\}, \tag{4.4}$$

$$\left\{k = \frac{\pm\imath a\sqrt{2}}{4b}, b_{-1} = \frac{\mp a}{2\sqrt{-\epsilon q}}, b_0 = 0, b_1 = \frac{\mp a\sqrt{-\epsilon q}}{2\epsilon q}\right\}, \tag{4.5}$$

$$\left\{k = \frac{\pm\imath a\sqrt{2}}{2b}, b_{-1} = \frac{\mp a}{\sqrt{-\epsilon q}}, b_0 = b_1 = 0\right\}, \tag{4.6}$$

$$\left\{k = \frac{\pm\imath a\sqrt{2}}{2b}, b_{-1} = b_0 = 0, b_1 = \frac{\mp a}{\sqrt{-\epsilon q}}\right\}. \tag{4.7}$$

Now, substituting (4.4), (4.5), (4.6) and (4.7) into (4.3), we obtain the solutions of Eq. (3.1), and consequently solutions for the Eq. (1.1) as follows:

$$u(t,x,\epsilon) = \frac{\zeta a\sqrt{2}}{2\sqrt{\epsilon q}} \tanh^{-1}\left(\frac{ax}{2b}\right) - \frac{\zeta a\sqrt{2}}{2\sqrt{\epsilon q}} \tanh\left(\frac{ax}{2b}\right), \tag{4.8}$$

$$u(t,x,\epsilon) = \frac{\zeta a}{2\sqrt{-\epsilon q}} \tanh^{-1}\left(\frac{\imath ax\sqrt{2}}{4b}\right) + \frac{\zeta a\sqrt{-\epsilon q}}{2\epsilon q} \tanh\left(\frac{\imath ax\sqrt{2}}{4b}\right), \tag{4.9}$$

$$u(t,x,\epsilon) = \frac{\zeta a}{\sqrt{-\epsilon q}} \tanh^{-1}\left(\frac{\imath ax\sqrt{2}}{2b}\right), \tag{4.10}$$

$$u(t,x,\epsilon) = \frac{\zeta a}{\sqrt{-\epsilon q}} \tanh\left(\frac{\imath ax\sqrt{2}}{2b}\right), \tag{4.11}$$

where $\zeta \in \{-1,1\}$ and $\imath^2 = -1$. In Figure 1 we plot the solution (4.9) with $\zeta = 1$, $\epsilon = 0.01$, $q = 2$, $a = 1$ and $b = 0.03$.

4.2. **Exact analytical solutions of Eq. (3.3) and Eq. (3.7).** Using the tanh-coth method, similar to the solving of the equation (3.1), we obtain solution for Eqs. (3.3) and (3.7), therefore the exact solutions for Eq. (1.1):

$$u(t,x,\epsilon) = \frac{\zeta a\sqrt{2}}{2\sqrt{\epsilon q}}\Big[\tanh^{-1}\left(\frac{ax-act}{2\sqrt{b^2-c^2}}\right) - \tanh\left(\frac{ax-act}{2\sqrt{b^2-c^2}}\right)\Big], \tag{4.12}$$

$$u(t,x,\epsilon) = \frac{\zeta a}{2\sqrt{-\epsilon q}}\Big[\tanh^{-1}\left(\frac{ax-act}{2\sqrt{2c^2-2b^2}}\right) + \tanh\left(\frac{ax-act}{2\sqrt{2c^2-2b^2}}\right)\Big], \tag{4.13}$$

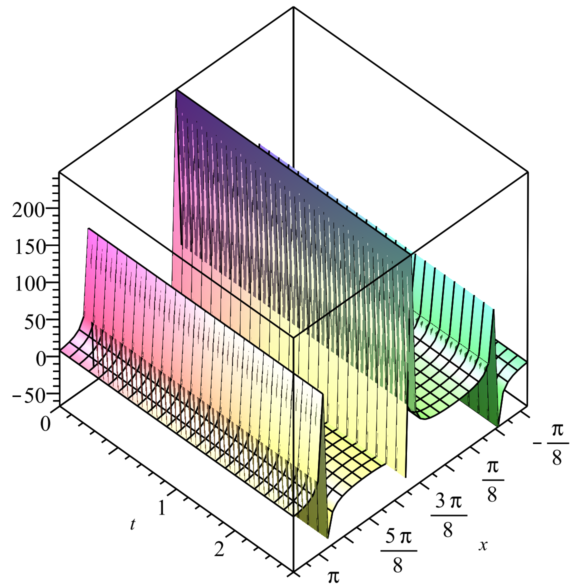
$$u(t,x,\epsilon) = \frac{\zeta a}{\sqrt{-\epsilon q}} \tanh^{-1}\left(\frac{ax-act}{\sqrt{2c^2-2b^2}}\right), \tag{4.14}$$

$$u(t,x,\epsilon) = \frac{\zeta a}{\sqrt{-\epsilon q}} \tanh\left(\frac{ax-act}{\sqrt{2c^2-2b^2}}\right), \tag{4.15}$$

$$u(t,x,\epsilon) = \frac{\zeta a\sqrt{2}}{2\sqrt{\epsilon q}}\Big[\tanh^{-1}\left(\frac{at-a\epsilon x}{2\sqrt{b^2\epsilon^2-1}}\right) - \tanh\left(\frac{at-a\epsilon x}{2\sqrt{b^2\epsilon^2-1}}\right)\Big], \tag{4.16}$$

$$u(t,x,\epsilon) = \frac{\zeta a}{2\sqrt{-\epsilon q}}\Big[\tanh^{-1}\left(\frac{at-a\epsilon x}{2\sqrt{2-2\epsilon^2 b^2}}\right) - \tanh\left(\frac{at-a\epsilon x}{2\sqrt{2-2\epsilon^2 b^2}}\right)\Big], \tag{4.17}$$

FIGURE 1. The solution (4.9).



$$u(t,x,\epsilon) = \frac{\zeta a}{\sqrt{-\epsilon q}} \tanh^{-1}\Big(\frac{at - a\epsilon x}{\sqrt{2 - 2\epsilon^2 b^2}}\Big), \tag{4.18}$$

$$u(t,x,\epsilon) = \frac{\zeta a}{\sqrt{-\epsilon q}} \tanh\Big(\frac{at - a\epsilon x}{\sqrt{2 - 2\epsilon^2 b^2}}\Big), \tag{4.19}$$

where $\zeta \in \{-1, 1\}$. In Figure 2 we plot the solution (4.14) with $\zeta = 1$, $\epsilon = 0.01$, $q = -1$, $a = 4$, $c = 1$ and $b = 0.001$. In Figure 3 we plot the solution (4.17) with $\zeta = 1$, $\epsilon = 0.01$, $q = -2$, $a = 1$, $c = 1$ and $b = 2$.

5. Conclusions

In this paper, Lie approximate symmetry analysis was applied to study the perturbed nonlinear Klein-Gordon equation. We obtained Lie approximate algebra, similarity reductions of this equation. All the group-invariant solutions to the Eq. (1.1) are considered based on the optimal system method. Then, we construct new analytical solutions with a small parameter to the Eq. (1.1) are investigated by means of the tanh-coth method. The basic idea described in this paper is efficient and powerful in solving wide classes of nonlinear differential equations.

Figure 2. The solution (4.14).

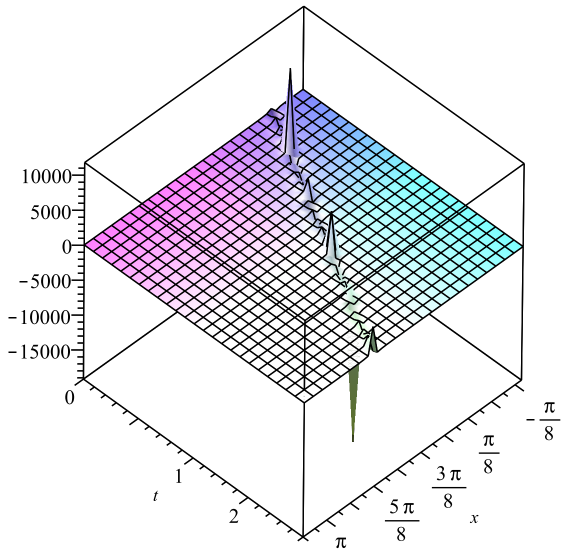
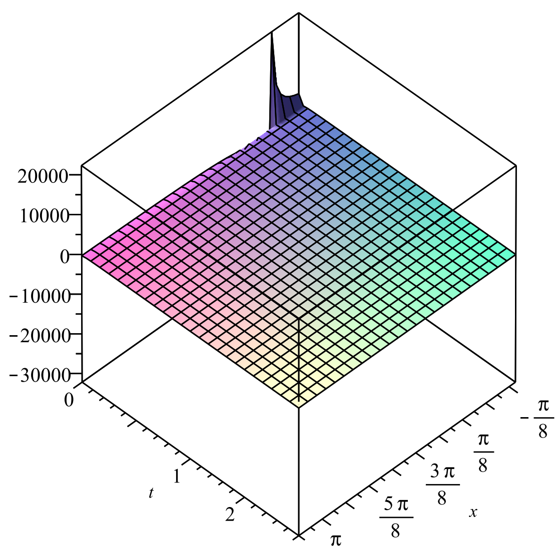


Figure 3. Solution (4.17).



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