Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 7, No. 1, 2019, pp. 86-95

Multi-soliton of the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation and KdV equation

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Abstract A direct rational exponential scheme is offered to construct exact multi-soliton solutions of nonlinear partial differential equations. We have considered the Calogero-Bogoyavlenskii-Schiff equation and KdV equation as two concrete examples to show efficiency of the method. As a result, one wave, two wave and three wave soliton solutions are obtained. Corresponding potential energy of the soliton are found. Furthermore, three-dimensional plots of the wave solutions and its potential functions are given to visualize the dynamics of the model and its energy. We also provided the corresponding density plot of the solutions to understand the real direction and particles density in the waves which help to realize the elastic situations of the achieved solutions.

Keywords. Direct rational exponential scheme, Calogero-Bogoyavlenskii-Schiff equation, KdV equation, Multi-soliton solutions.

2010 Mathematics Subject Classification. 35K99, 35P05, 35P99.

1. Introduction

The effort in finding exact solutions to non-linear differential equations is an important task for understanding most of the nonlinear physical phenomena in applied mathematics, physics, in fluid dynamics, plasma and optical fibers, biology and issues related to engineering. For instance, the nonlinear wave phenomena observed are often modeled by the bell shaped sech solutions and the kink shaped tanh solutions and are founded by different methods such as, inverse scattering transform [1], analytical methods [4], the exp-function method [5, 14, 17], the Hirota's bilinear method [6], the Jacobi elliptic function expansion method [7], the (G'/G)-expansion method [2, 18, 21], Backlund transformation [13], Darboux transformation [12], the multiple exp-function method [9], the symmetry algebra method [10], the Wronskian technique [11], the $exp(-\Phi(\xi))$-expansion method [19, 20] and few analytical methods [3, 8, 15, 16, 22, 23, 24, 25, 26, 27]. Studies of completely integrable equations and nonlinear phenomena are affluent in relation to solitary wave fields and engineering concepts. In soliton theory, non-elastic phenomena are rear case and there are rear model in the literature in which this phenomena exist. Actually, the interactions between two or more soliton solutions for integrable models are considered

Received: 28 November 2016 ; Accepted: 9 December 2018.

to be completely elastic and their amplitude, velocity, wave shape do not change after the non-linear interaction. Furthermore, some models exist in the literature are completely non-elastic, depending conditions between the wave vectors and velocities. Wazwaz [24, 25, 26] investigated multiple soliton solutions such type of non-elastic phenomena. Burgers equation and Sharma-Tasso-Olver equation are such types of model are studies in previous literature. Wang et al. [27] found non-elastic soliton fission and fusion: Burgers equation and Sharma-Tasso-Olver equations with only two dispersion relations.

In this article, we investigate completely non-elastic multi-soliton (two wave and three wave solutions) the Calogero-Bogoyavlenskii-Schiff equations and KdV equation.

2. One, two and three wave solutions of the Calogero-Bogoyavlenskii-Schiff

In this section, we bring to bear a direct rational exponential approach to find one, two and three wave solutions of the Calogero-Bogoyavlenskii-Schiff equation which is an extremely important nonlinear evolution equation in mathematical physics and have been paid attention by a lot of researchers. We start with the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation in the form

$$u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx} u_z = 0, \tag{2.1}$$

which has both the non-linear radiation and the diffusion effect.
For single soliton solution we first consider trial solution as

$$u(x, z, t) = r\frac{k_1 c_1 exp(k_1 x + l_1 z + w_1 t)}{a_0 + c_1 exp(k_1 x + l_1 z + w_1 t)}. \tag{2.2}$$

Inserting Eq. (2.2) into Eq. (2.1) and then maintenance all the coefficients of $exp(k_1 x + l_1 z + w_1 t)^i, (i = \cdots - 2, -1, 0, 1, 2, \cdots)$ is zero, yields a system of algebraic equations about a_0, c_1, w_1 and k_1 as follows:

$$-11k_1^2 l_1 c_1 a_0^2 + w_1 a_0^2 c_1 + 6rc_1 l_1 a_0^2 k_1^2 = 0,$$

$$11k_1^2 l_1 c_1^2 a_0 - w_1 c_1^2 a_0 - 6ra_0 l_1 c_1^2 k_1^2 = 0,$$

$$-w_1 c_1^3 - k_1^2 c_1^3 l_1 = 0,$$

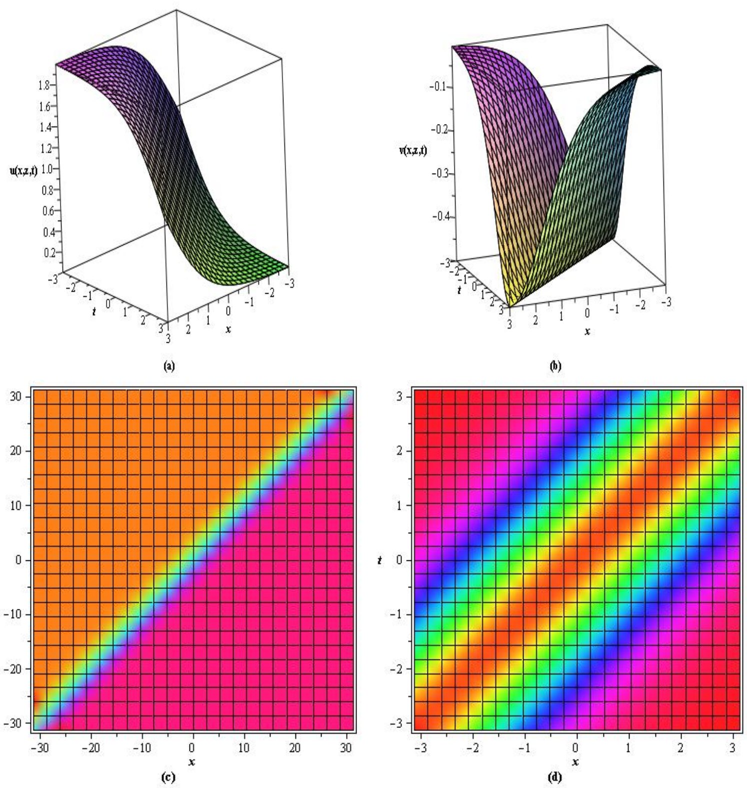
$$w_1 a_0^3 + k_1^2 a_0^3 l_1 = 0.$$

Solving the above over-determined system of algebraic equations for a_0, w_1, l_1, r with the aid of commercial software *Maple 13*, we arrive at the following solutions:
$a_0 = const.$, $l_1 = const.$, $w = -l_1 k_1^2$, $r = 2$ and c_1 is free parameter.
Thus the solution is

$$u(x, z, t) = 2\frac{k_1 c_1 exp(k_1 x + l_1 z - l_1 k_1^2 t)}{a_0 + c_1 exp(k_1 x + l_1 z - l_1 k_1^2 t)}, \tag{2.3}$$

and corresponding potential energy of the soliton Eq. (2.3) is read as $v(x, z, t) = -u_x(x, z, t)$.

FIGURE 1. (a) Profile of the single solitary wave solution Eq. (2.3) of CBS equation, (b) Potential field with $c_1 = k_1 = l_1 = a_0 = 1$. Along $z = 0$, (c) Density plot of (a) and (d) Density plot of (b).

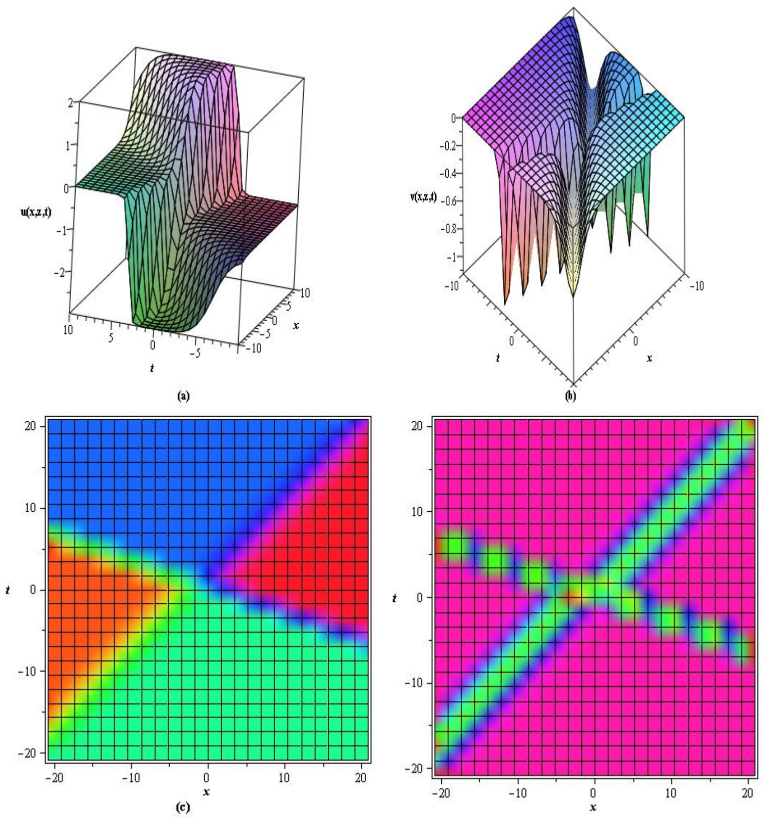


To obtain two wave soliton solutions, we assume

$$u(x,z,t) = r\frac{\Upsilon_1}{\Upsilon_2}, \tag{2.4}$$

where $\Upsilon_1 = k_1c_1exp(\xi_1) + k_2c_2exp(\xi_2) + a_{12}(k_1 + k_2)c_1c_2exp(\xi_1 + \xi_2), \Upsilon_2 = a_o + c_1exp(\xi_1) + c_2exp(\xi_2) + a_{12}c_1c_2exp(\xi_1 + \xi_2), \xi_1 = k_1x + l_1z + w_1t, \xi_2 = k_2x + l_2z + w_2t$; c_1, c_2 are free parameters and the corresponding potential field reads $v = -u_x(x,z,t)$. Directly inserting Eq.(2.4) in the Eq. (2.1) via commercial software *Maple 13*, and solving for r, a_0, k_1, k_2, l_1, l_2, w_1, w_2 and a_{12}, we have $r = 2$, $a_0 = const.$, $a_{12} = \frac{(k_1-k_2)^2}{a_0(k_1+k_2)^2}$, $w_1 = -l_1k_1^2$, $w_2 = -l_2k_2^2$.

FIGURE 2. (a) Profile of two solitary wave elastic solution Eq. (2.5) of CBS equation, (b) Potential field with $c_1 = k_1 = l_1 = c_2 = a_0 = 1$, $k_2 = -1.5$ along $z = 0$, (c) density plot of (a) and (d) density plot of (b).



Thus solution is

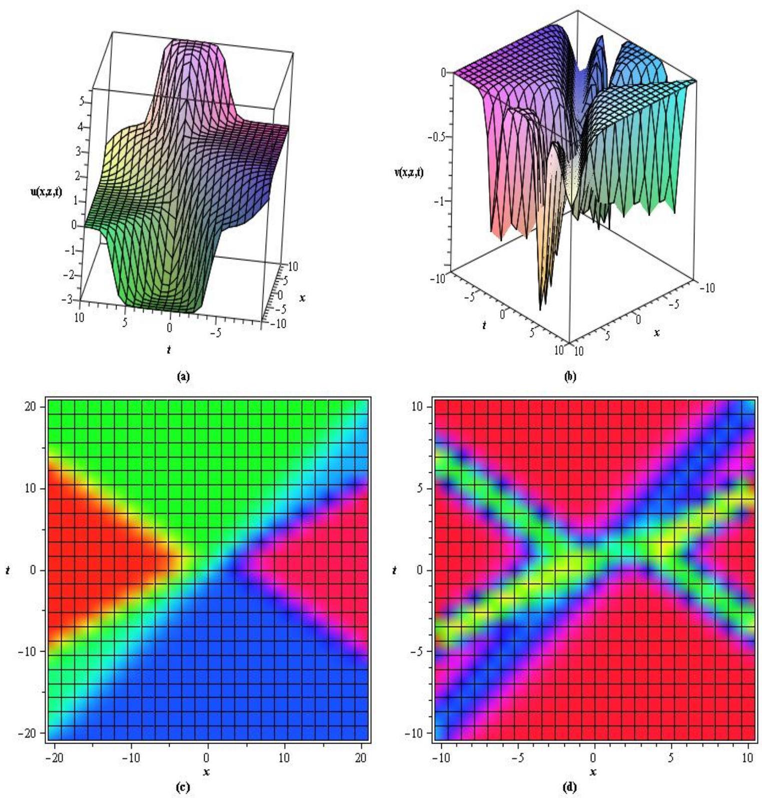
$$u(x, z, t) = 2\frac{\Upsilon_1}{\Upsilon_2}, \tag{2.5}$$

where $\Upsilon_1 = k_1c_1exp(\xi_1)+k_2c_2exp(\xi_2)+\frac{(k_1-k_2)^2}{a_0(k_1+k_2)}c_1c_2exp(\xi_1+\xi_2)$, $\Upsilon_2 = a_o+c_1exp(\xi_1)+c_2exp(\xi_2) + \frac{(k_1-k_2)^2}{a_0(k_1+k_2)^2}c_1c_2exp(\xi_1 + \xi_2)$, $\xi_1 = k_1x + l_1z - l_1k_1^2t$, $\xi_2 = k_2x + l_2z - l_2k_2^2t$, and $a_0, c_1, c_2, k_1, k_2, l_1, l_2$ are arbitrary constants. The corresponding potential field reads $v(x, z, t) = -u_x(x, z, t)$.

To obtain three soliton solutions, we assume

$$u(x, t) = r\frac{\Upsilon_1}{\Upsilon_2}, \tag{2.6}$$

FIGURE 3. Profile of three solitary wave elastic solution Eq. (2.6) of CBS equation, (b) Corresponding potential field with $c_1 = k_1 = l_1 = c_2 = a_0 = 1$, $k_2 = -1.5$ along $z = 0$, (c) density plot of (a) and (d) density plot of (b).



where $\Upsilon_1 = k_1c_1exp(\xi_1) + k_2c_2exp(\xi_2) + k_3c_3exp(\xi_3) + a_{12}(k_1 + k_2)c_1c_2exp(\xi_1 + \xi_2) + a_{23}(k_2 + k_3)c_2c_3exp(\xi_2 + \xi_3) + a_{13}(k_1 + k_3)c_1c_3exp(\xi_1 + \xi_3) + a_{123}(k_1 + k_2 + k_3)c_1c_2c_3exp(\xi_1 + \xi_2 + \xi_3)$,
$\Upsilon_2 = a_0 + c_1exp(\xi_1) + c_2exp(\xi_2) + c_3exp(\xi_3) + a_{12}c_1c_2exp(\xi_1 + \xi_2) + a_{23}c_2c_3exp(\xi_2 + \xi_3) + a_{13}c_1c_3exp(\xi_1 + \xi_3) + a_{123}c_1c_2c_3exp(\xi_1 + \xi_2 + \xi_3), \xi_1 = k_1x + l_1z + w_1t, \xi_2 = k_2x + l_2z + w_2t, \xi_3 = k_3x + l_3z + w_3t$ and and the corresponding potential field reads $v(x, z, t) = -u_x(x, z, t)$.
Similarly, inserting Eq. (2.6) in the Eq. (2.1) via commercial software *Maple 13*, and solving, we gain the unknown parameters $r = 2$, $a_0 = const.$, $a_{12} = \frac{(k_1-k_2)^2}{a_0(k_1+k_2)^2}$, $a_{23} = \frac{(k_2-k_3)^2}{a_0(k_2+k_3)^2}$, $a_{13} = \frac{(k_1-k_3)^2}{a_0(k_1+k_3)^2}$, $a_{123} = a_{12}a_{23}a_{13}$, $w_1 = -l_1k_1^2$, $w_2 = -l_2k_2^2$, $w_3 = -l_3k_3^2$. Now setting these in Eq. (2.6), we attained three wave solution of CBS equation. The corresponding potential field reads $v(x, z, t) = -u_x(x, z, t)$.

Observing figures and density plot of the multi-solutions of the CBS equations, we see that the solutions are completely elastic after collision. That is, before ($t < 0$) and after ($t > 0$) collision of the wave density and direction remain same except at phase shift $t = 0$.

3. Multi-soliton of KdV equation

In this section, we bring to bear a direct rational exponential approach to study the KdV equation [26] as

$$u_t + 6uu_x + u_{xxx} = 0, \tag{3.1}$$

which describes the nonlinear long waves in physics, engineering and fluid mechanics. For single soliton solution we first consider trial solution as

$$u(x,t) = 2\frac{k_1^2 c_1 exp(k_1 x + w_1 t)}{a_0 + c_1 exp(k_1 x + w_1 t)} - 2(\frac{k_1 c_1 exp(k_1 x + w_1 t)}{a_0 + c_1 exp(k_1 x + w_1 t)})^2. \tag{3.2}$$

Inserting Eq. (3.2) into Eq. (3.1) and then maintenance all the coefficients of $exp(k_1 x + w_1 t)^i, (i = \cdots - 2, -1, 0, 1, 2, \cdots)$ is zero, yields a system of algebraic equations about a_0, c_1, w_1 and k_1 and then solving via *Maple 13* for unknown parameters, we arrive $a_0 = const., c_1 = const.$ and $w_1 = -k_1^3$.
Thus the solution is

$$u(x,t) = 2\frac{k_1^2 c_1 exp(k_1 x - k_1^3 t)}{a_0 + c_1 exp(k_1 x - k_1^3 t)} - 2(\frac{k_1 c_1 exp(k_1 x - k_1^3 t)}{a_0 + c_1 exp(k_1 x - k_1^3 t)})^2, \tag{3.3}$$

and the corresponding potential field reads as $v(x,t) = -u_x(x,t)$.

To achieve two solitons solution, we assume

$$u(x,t) = 2\frac{\Upsilon_1}{\Upsilon_2} - 2(\frac{\Upsilon_3}{\Upsilon_2})^2, \tag{3.4}$$

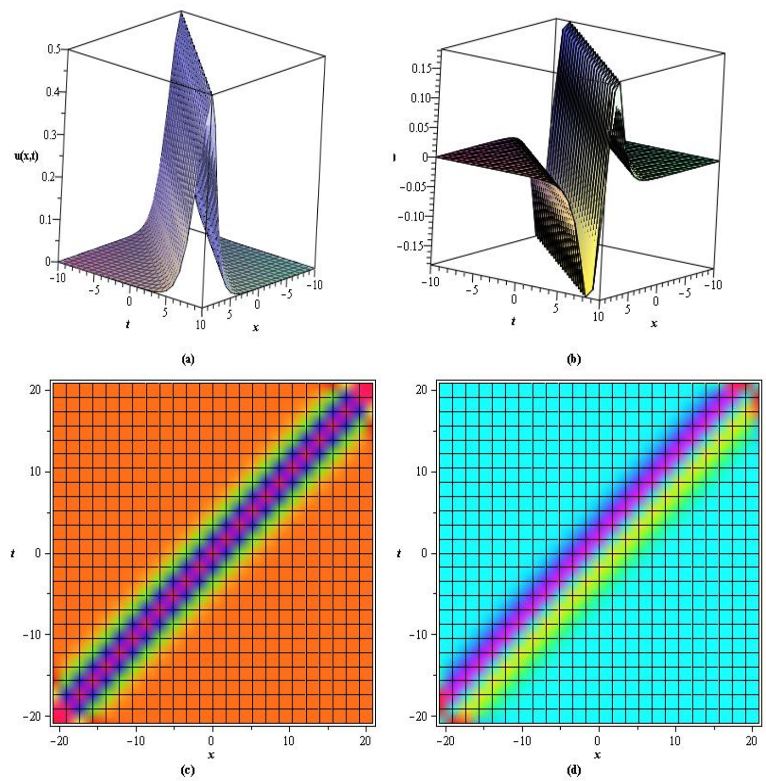
where $\Upsilon_1 = k_1^2 c_1 exp(k_1 x + w_1 t) + k_2^2 c_2 exp(k_2 x + w_2 t) + a_{12}(k_1 + k_2)^2 c_1 c_2 exp((k_1 + k_2)x + (w_1 + w_2)t)$, $\Upsilon_2 = a_0 + c_1 exp(k_1 x + w_1 t) + c_2 exp(k_2 x + w_2 t) + a_{12} c_1 c_2 exp((k_1 + k_2)x + (w_1 + w_2)t)$ and $\Upsilon_3 = k_1 c_1 exp(k_1 x + w_1 t) + k_2 c_2 exp(k_2 x + w_2 t) + a_{12}(k_1 + k_2) c_1 c_2 exp((k_1 + k_2)x + (w_1 + w_2)t)$.

Using Eq. (3.4) into the Eq. (3.1) and applying the same procedure used in finding two wave solutions in CBS equations with the help of commercial software *Maple 13* and solving for a_0, w_1, w_2 and a_{12} we have $a_0 = const.$, $a_{12} = \frac{(k_1-k_2)^2}{a_0(k_1+k_2)^2}$, $w_1 = -k_1^3$, $w_2 = -k_2^3$.
Now inserting these parameters in the Eq. (3.4), we get the required two waves soliton solution and thus the solution is

$$u(x,t) = 2\frac{\Upsilon_1}{\Upsilon_2} - 2(\frac{\Upsilon_3}{\Upsilon_2})^2, \tag{3.5}$$

where $\Upsilon_1 = k_1^2 c_1 exp(k_1 x - k_1^3 t) + k_2^2 c_2 exp(k_2 x - k_2^3 t) + \frac{(k_1-k_2)^2}{a_0} c_1 c_2 exp((k_1 + k_2)x - (k_1^3 + k_2^3)t)$, $\Upsilon_2 = a_0 + c_1 exp(k_1 x - k_1^3 t) + c_2 exp(k_2 x - k_2^3 t) + \frac{(k_1-k_2)^2}{a_0(k_1+k_2)^2} c_1 c_2 exp((k_1 + k_2)x -$

FIGURE 4. (a) Profile of the single solitary wave solution Eq. (3.3) of KdV equation, (b) Potential field with $c_1 = k_1 = l_1 = a_0 = 1$, along $z = 0$, (c) density plot of (a) and (d) density plot of (b).



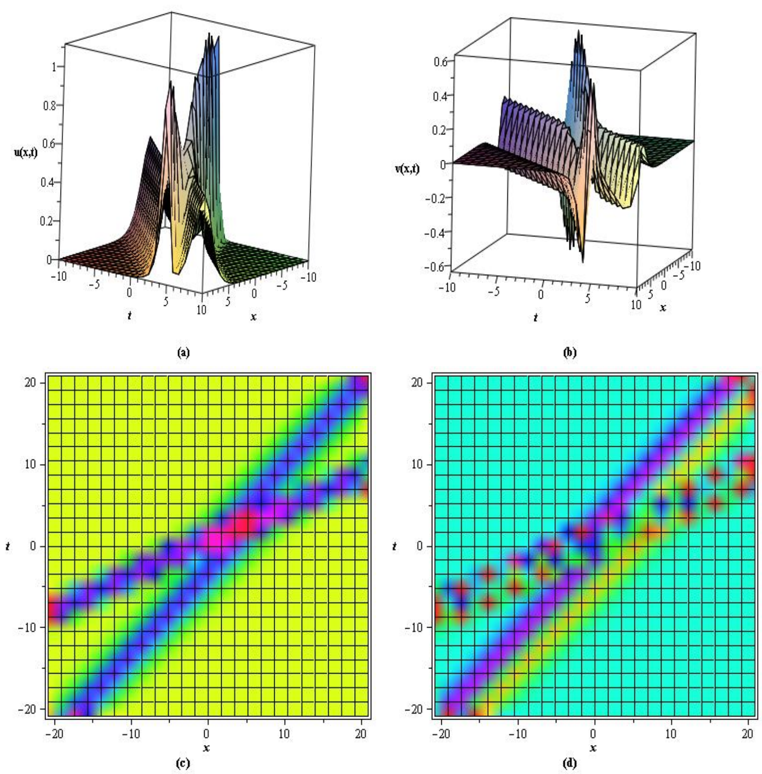
$(k_1^3+k_2^3)t)$ and $\Upsilon_3 = k_1c_1exp(k_1x-k_1^3t)+k_2c_2exp(k_2x-k_2^3t)+\frac{(k_1-k_2)^2}{a_0(k_1+k_2)}c_1c_2exp((k_1+k_2)x-(k_1^3+k_2^3)t)$; a_0, k_1, k_2, c_1, c_2 are constants.

Similarly, we can obtain the three wave soliton solution which is also retaining the nature of soliton and getting no new phenomena and so we avoid these.

Observing figures and density plot of the multi-solutions of the KdV equations, we see that the solutions are completely elastic after collision. That is, before ($t < 0$) and after ($t > 0$) collision of the wave density and direction remain same except at phase shift $t = 0$.

Remark: All of the solutions available in this paper have been checked with the help of *Maple 13* and we observe that they satisfy the corresponding original equation.

Figure 5. (a) Profile of three solitary wave fusion solution Eq. (3.5) of KdV equation, (b) Corresponding potential field with $c_1 = k_1 = c_2 = a_0 = 1$, $k_2 = -1.5$, along $z = 0$, (c) density plot of (a) and (d) density plot of (b).



4. Conclusion

We used direct rational exponential scheme to investigate elastic and non-elastic multi-soliton solution for the Calogero Bogoyavlenskii Schiff equation and KdV equation. Elastic solitons are found after and before collision between two or three solitons of the Calogero Bogoyavlenskii Schiff equation and KdV equation when $a_{ij} \neq 0, (i, j = 1, \cdots, 3)$. The three dimensional surfaces are provided to visualize the real shape of the elastic solutions and corresponding potential energies.

Acknowledgment

I am grateful to the Ministry of science and technology-Government of People's republic of Bangladesh for their support in this research work.

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