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Properties of utility function for Barles and Soner model

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Abstract The nonlinear Black-Scholes equation has been increasingly attracting interest over the last two decades, because it provides more accurate values by considering transaction costs as a viable assumption. In this paper we review the fully nonlinear Black-Scholes equation with an adjusted volatility which is a function of the second derivative of the price and then we prove two new theorems in this realistic model.

Keywords. Nonlinear Black-Scholes equation, Option pricing, Transaction costs.

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1. Introduction

A derivative is a contract that derives its value from the performance of an underlying entity [15]. Financial derivatives enable parties to trade specific financial risks to other entities to manage these risks. The risk embodied in a derivatives contract can be traded by trading the contract itself, such as with options. As a matter of fact, option is one of the more common derivatives [10]. The value of the financial derivative derives from the reference price. Because the future reference price is not known, the value of the financial derivative at maturity can only be estimated. The assumptions made by Black and Scholes [3] when they derived their option pricing formula were volatility σ was constant and there were no transaction costs or taxes. The volatility σ here follows a stochastic differential equation of the geometric Brownian motion

$$dS = \rho S dt + \sigma S dW$$

with a drift ρ [14]. The standard Black-Scholes model has been widely accepted by academics and used by practitioners. Nevertheless, it has also attracted criticism because the essential model parameter, the volatility, is not constant. It is often determined by computing the implied volatility out of the observed option prices by inverting the Black-Scholes formula. This leads to a generalization of the Black-Scholes model replacing the constant volatility by a volatility function. This volatility

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function leads to a notable increase in the option price. There are several models of the generalized Black-Scholes equation like Leland's model [11], Barles' and Soner's model [2], etc. We focus on Barles' and Soner's model in this paper.

The contents of this paper are as follows: In the next section we review the Barles' and Soner's model. In section 3 we prove two new properties of the utility function in the Barles' and Soner's model. Finally, we sum up the conclusions in section 4.

2. Barles' and Soner's model

In 1998, Barles and Soner [2], derived a model assuming that investor's preferences are characterized by an exponential utility function. They used an exponential utility function and proved - using the theory of stochastic optimal control - that option price V is the unique viscosity solution of the Black-Scholes equation. Consider

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2.1}$$

where the nonlinear volatility is given by

$$\sigma^2 = \sigma_0^2(1 + \Psi[\exp(r(T-t))a^2 S^2 \frac{\partial^2 V}{\partial S^2}]), \tag{2.2}$$

where S is the price of the underlying asset, T is the maturity date, r is the risk-free interest rate, σ_0 is the asset volatility, and a is transaction cost. Function $\Psi(A)$ is the solution of the following nonlinear ordinary differential equation

$$\Psi'(A) = \frac{\Psi(A)+1}{2\sqrt{A\Psi(A)} - A}, \quad A \neq 0, \quad \Psi(0) = 0. \tag{2.3}$$

The European call option pricing with transaction costs by strike price E is the solution to Eq. (2.1) on $S \in [0,\infty)$ and $t \in [0,T]$, with the following terminal and boundary conditions:

$$V(S,T) = f(S) = \max(S-E, 0), \qquad V(0,t) = 0, \qquad \lim_{S\to\infty} V(S,t) = S. \tag{2.4}$$

In the appendix of [2], it is shown that the function $A \to A(1+\Psi(A))$ is nondecreasing in $\mathbb{R}$. This implies that the nonlinear Black-Scholes equation is a degenerate parabolic equation and the theory of viscosity solutions applies to this equation. We would like to refer the interested reader to [4, 8] for the viscosity solutions. The existence of a viscosity solution to (2.1) has been proved in [2].

The values of $\Psi(A)$ in (2.3) play an important role in the solution of (2.1). We get the solution of (2.3) in the Theorem 1.

Theorem 2.1. *The function Ψ, the solution of Eq. (2.3), defines*

$$A = (-\frac{\operatorname{arcsin} h(\sqrt{\Psi})}{\sqrt{\Psi+1}} + \sqrt{\Psi})^2, \quad \Psi > 0,$$

$$A = -(\frac{\arcsin(\sqrt{-\Psi})}{\sqrt{\Psi+1}} - \sqrt{-\Psi})^2, \quad -1 < \Psi < 0.$$

Proof. Consider $A > 0$, because of $\sqrt{A\Psi}$ in (2.3) we should have $\Psi > 0$. Since $\Psi' = \frac{d\Psi}{dA} = \frac{1}{\frac{dA}{d\Psi}} = \frac{1}{A'}$,

$$\frac{1}{A'} = \frac{\Psi+1}{2\sqrt{A}\sqrt{\Psi} - A},$$

therefore

$$A'(\Psi+1) = 2\sqrt{A}\sqrt{\Psi} - A,$$

so

$$A' + \frac{A}{\Psi+1} = \frac{2\sqrt{\Psi}}{\Psi+1}A^{\frac{1}{2}}. \tag{2.5}$$

Eq. (2.5) is a Bernoulli differential equation with the solution

$$A = \Big(\frac{1}{\sqrt{\Psi+1}}\int \frac{\sqrt{\Psi}}{\sqrt{\Psi+1}}d\Psi + c\Big)^2,$$

which c is constant. By considering $\frac{\Psi}{\Psi+1} = z^2$, we can see $d\Psi$ as

$$d\Psi = \frac{2z}{(1-z^2)^2}dz,$$

therefore

$$\int \frac{\sqrt{\Psi}}{\sqrt{\Psi+1}}d\Psi = \int \frac{2z^2}{(1-z)^2(1+z)^2}dz.$$

By decomposition of fractions we get

$$\frac{2z^2}{(1-z)^2(1+z)^2} = \frac{-1}{2(1-z)} + \frac{1}{2(1-z)^2} + \frac{-1}{2(1+z)} + \frac{1}{2(1+z)^2},$$

so

$$A = \left(\frac{1}{\sqrt{\Psi+1}}\left(\frac{1}{2}\ln(\frac{\sqrt{\Psi+1}-\sqrt{\Psi}}{\sqrt{\Psi+1}+\sqrt{\Psi}}) + \sqrt{\Psi}\sqrt{\Psi+1}\right) + c\right)^2.$$

Because $\Psi(0) = 0$, we obtain $c = 0$. On the other hand, we know

$$\arcsin h(\sqrt{\Psi}) = \ln(\sqrt{\Psi} + \sqrt{\Psi+1}).$$

Also

$$\ln(\frac{\sqrt{\Psi+1}+\sqrt{\Psi}}{\sqrt{\Psi+1}-\sqrt{\Psi}}) = \ln(\sqrt{\Psi+1}+\sqrt{\Psi})^2,$$

therefore

$$\frac{1}{2}\ln(\frac{\sqrt{\Psi+1}-\sqrt{\Psi}}{\sqrt{\Psi+1}+\sqrt{\Psi}}) = -\frac{1}{2}\ln(\sqrt{\Psi+1}+\sqrt{\Psi})^2,$$

so

$$A = \left(\sqrt{\Psi} - \frac{1}{\sqrt{\Psi+1}}\arcsin h(\sqrt{\Psi})\right)^2.$$

When $A < 0$, we follow [6] and use the change of variables

$$A = -z^2, \quad g(\Psi) = \sqrt{-\Psi},$$

so Eq. (2.3) becomes

$$2g^2 + gz = (1 - g^2)\frac{dz}{dg}.$$

Then

$$z = \frac{c + \arcsin g}{\sqrt{1 - g^2}} - g.$$

By initial condition $\Psi(0) = 0$, we get

$$\sqrt{-A} = \frac{\arcsin \sqrt{-\Psi}}{\sqrt{\Psi + 1}} - \sqrt{-\Psi}.$$

□

As said in [5], Theorem 2 will play an important role in studying the consistency of the numerical methods for solving the model.

Theorem 2.2. *Let $g(A) = A\Psi(A)$. Then $g(A)$ is a continuously differentiable function at $A = 0$ and satisfies*

$$|g'(A)| \le \max\{G, 2|A|\Psi'(A_2) + d_2\}, \quad A \in \mathcal{R}, \tag{2.6}$$

where $\mathcal{R}$ is real numbers, and

$$A_2 \simeq 9.58, \quad d_2 = \Psi(A_2) - \Psi'(A_2)A_2 \simeq 2.62, \tag{2.7}$$

and

$$G = \max\{|g'(A)|; A_1 \le A \le A_2\}, \quad A_1 = -\frac{(4\pi - 3\sqrt{3})^2}{36}. \tag{2.8}$$

Proof. see [5]. □

3. Properties of utility function Ψ

As a matter of fact the analysis of (2.3) implies some properties that we see in the next theorems.

Theorem 3.1. *The utility function Ψ satisfies the following properties*

$$\lim_{A\to\infty} \frac{\Psi(A)}{A} = 1, \qquad \lim_{A\to-\infty} \Psi(A) = -1. \tag{3.1}$$

Proof. See [2]. □

Theorem 3.2. *The utility function Ψ is a one to one increasing function mapping the real line onto the interval $(-1, \infty)$.*

Proof. See [6]. □

Theorem 5 and Theorem 6 are expressed and proved for the first time in this paper.

Theorem 3.3. *The function $\Psi^2(x)$ is a convex function (convex downward) in the domain $x < 0$.*

Proof. Consider $x < y$ two arbitrary numbers in $(-\infty, 0)$. We know

$$2\Psi(x)\Psi(y) \leq \Psi^2(x) + \Psi^2(y).$$

From [5], we get $\frac{d^2\Psi}{dx^2} > 0$ for $x < 0$. Therefore $\Psi(x)$ is a convex function (convex downward) for $x < 0$. Two positive constants α and β are chosen so that $\alpha + \beta = 1$. So

$$\Psi(\alpha x + \beta y) \leq \alpha\Psi(x) + \beta\Psi(y).$$

Also

$$\begin{aligned}
\Psi^2(\alpha x + \beta y) &= \Psi(\alpha x + \beta y)\Psi(\alpha x + \beta y) \\
&\leq \Big(\alpha\Psi(x) + \beta\Psi(y)\Big)\Big(\alpha\Psi(x) + \beta\Psi(y)\Big) \\
&= \alpha^2\Psi^2(x) + \alpha\beta\big(2\Psi(x)\Psi(y)\big) + \beta^2\Psi^2(y) \\
&\leq \alpha^2\Psi^2(x) + \alpha\beta\big(\Psi^2(x) + \Psi^2(y)\big) + \beta^2\Psi^2(y) \\
&= \Psi^2(x)\big(\alpha^2 + \alpha\beta\big) + \Psi^2(y)\big(\alpha\beta + \beta^2\big) \\
&= \alpha\Psi^2(x) + \beta\Psi^2(y).
\end{aligned}$$

□

Theorem 3.4. *The function $h(x) = x\Psi'(x) - \Psi(x)$ is a decreasing function for $x \in (-\infty, \infty)$.*

Proof. (i) : Consider $x < y$ two arbitrary negative constants. There is a constant c_1, $x < c_1 < y$, such that $\Psi(y) - \Psi(x) = \Psi'(c_1)(y - x)$. Therefore

$$\begin{aligned}
h(y) - h(x) &= y\Psi'(y) - \Psi(y) - \big(x\Psi'(x) - \Psi(x)\big) \\
&= \Psi(x) - \Psi(y) + (y - x)\Psi'(y) + x\big(\Psi'(y) - \Psi'(x)\big) \\
&= -\Psi'(c)(y - x) + (y - x)\Psi'(y) + x\big(\Psi'(y) - \Psi'(x)\big) \\
&= (y - x)\big(\Psi'(y) - \Psi'(c_1)\big) + x\big(\Psi'(y) - \Psi'(x)\big).
\end{aligned}$$

Since Ψ is a convex downward function in $(-\infty, 0)$, so Ψ' is increasing, $\Psi'(x) \leq \Psi'(c_1) \leq \Psi'(y)$ and

$$\Psi'(c_1) - \Psi'(y) \leq 0, \quad \Psi'(y) - \Psi'(x) \geq 0, \quad \Psi'(y) - \Psi'(c_1) \leq \Psi'(y) - \Psi'(x). \tag{3.2}$$

Therefore

$$\begin{aligned}
h(y) - h(x) &= (y - x)\big(\Psi'(y) - \Psi'(c_1)\big) + x\big(\Psi'(y) - \Psi'(x)\big) \\
&\leq (y - x)\big(\Psi'(y) - \Psi'(x)\big) + x\big(\Psi'(y) - \Psi'(x)\big) \\
&= y\big(\Psi'(y) - \Psi'(x)\big) \leq 0.
\end{aligned}$$

(ii) : Consider $x < y$ two arbitrary positive constants. There is a number c_2, $x < c_2 < y$, such that $\Psi(y) - \Psi(x) = \Psi'(c_2)(y - x)$. Therefore

$$h(y) - h(x) = (x - y)\big(\Psi'(c_2) - \Psi'(y)\big) - x\big(\Psi'(x) - \Psi'(y)\big).$$

Since Ψ is a concave downward function in $(0, \infty)$, so Ψ' is decreasing, $\Psi'(x) \geq \Psi'(y)$, and

$$\Psi'(c_2) - \Psi'(y) \leq \Psi'(x) - \Psi'(y). \tag{3.3}$$

Therefore

$$\begin{aligned} h(y) - h(x) &\leq (x - y)\big(\Psi'(x) - \Psi'(y)\big) - x\big(\Psi'(x) - \Psi'(y)\big) \\ &= -y\big(\Psi'(x) - \Psi'(y)\big) \leq 0. \end{aligned}$$

(iii) : When x and y are two arbitrary constants which $x < 0 < y$, the proof is the same of (ii), and thus the proof is complete. □

Several numerical methods for solving Barles' and Soner's model have already been proposed and we refer the interested reader to [1, 7, 5, 6, 12, 9, 13].

4. Conclusions

In this paper we considered a nonlinear Black-Scholes model for option pricing under variable transaction costs. The diffusion coefficient of the nonlinear parabolic equation for the price V is assumed to be a function of the underlying asset price. The main goal of this paper was to review the Barles' and Soner's model of the nonlinear Black-Scholes equation and proved two new theorems about this model.

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