Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 7, No. 1, 2019, pp. 1-15

Finite integration method with RBFs for solving time-fractional convection-diffusion equation with variable coefficients

Jafar Biazar*
Department of Applied Mathematics,
Faculty of Mathematical Science,
University of Guilan, Rasht, Iran.
E-mail: biazar@guilan.ac.ir

Mohammad Ali Asadi
Department of Applied Mathematics,
Faculty of Mathematical Science,
University of Guilan, Rasht, Iran.
E-mail: eng.asadi@gmail.com

Abstract In this paper, a modification of the finite integration method (FIM) is combined with the radial basis function (RBF) method to solve a time-fractional convection-diffusion equation with variable coefficients. The FIM transforms partial differential equations into integral equations and this creates some constants of integration. Unlike the usual FIM, the proposed method computes constants of integration by using initial conditions. This leads to fewer computations rather than the standard FIM. Also, a product Simpson method is used to overcome the singularity included in the definition of fractional derivatives, and an integration matrix is obtained. Some numerical examples are provided to show the efficiency of the method. In addition, a comparison is made between the proposed method and the previous ones.

Keywords. Time-fractional convection-diffusion equation, Radial basis functions, Finite integration method, Product Simpson integration method.

2010 Mathematics Subject Classification. 65D30, 45D05, 34K28.

1. Introduction

Fractional order differential equations are used in modeling of many phenomena in engineering, chemistry, physics, finance, and other disciplines, see [8, 14, 19] and the references therein. Note that most fractional order differential equations do not have solutions in a closed form. So it is important to propose new methods for finding numerical solutions of these equations. Recently, several numerical methods have been developed to solve fractional partial differential equations (FPDEs) such as generalized fractional-order Legendre functions [5], Finite Difference Method (FDM) [13], compact finite difference method [7], Chebyshev wavelets collocation method [30], operational method [18], homotopy perturbation method [28], and many others. One of the examples of FPDEs is time-fractional order convection-diffusion equations

Received: 25 October 2017 ; Accepted: 3 November 2018.
* Corresponding author.

(TFCDEs), which are obtained from the classical diffusion equations. It used to simulate super-diffusive flow processes [2].

Some techniques presented so far for numerical solutions of TFCDEs are compact difference method [20,23], Galerkin method [22], Chebyshev wavelets collocation method [30], Jacobi polynomials [3], and operational method [1,27].

In this paper, we consider the following time-fractional convection-diffusion equation with variable coefficients

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + a(x)\frac{\partial u(x,t)}{\partial x} + b(x)\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \qquad 0 < x < 1,\ 0 < t \leq 1,$$

and the following initial and boundary conditions

$$\begin{aligned} &u(x,0) = g(x), && 0 < x < 1,\\ &u(0,t) = h_0(t), \quad u(1,t) = h_1(t), && 0 \leq t \leq 1, \end{aligned}$$

where nonzero functions $a(x)$ and $b(x)$ are continuous. Here the time-fractional derivative of order $0 < \alpha < 1$ is defined in the Caputo sense as the following

$$\frac{\partial^\alpha f(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-\tau)^{m-1-\alpha}\frac{\partial^m f(x,\tau)}{\partial \tau^m}d\tau, \quad m = \lceil \alpha \rceil.$$

Radial basis function method (RBF) is an effective tool in multi-variable approximation theory that can be used in any arbitrary space dimensions [4, 11, 26]. The motivation for RBF method originates from R. Hardy [12]. The main advantage of RBF method as a meshless method, is to eliminate known deficiencies of the mesh-based approaches (see for example [4, p. 39] and [10, p. 1]).

The Finite Integration Method (FIM) is an integral based technique, and was first introduced by Weiland in 1977 [24]. In recent years, there has been an increasing amount of literature on developing FIM with RBFs for solving partial differntial equations (PDEs), see for example [15, 17, 25, 29]. The FIM converts a PDE into an integral equation that gives much higher accurate approximations than the Finite Difference Method (FDM), Finite Element Method (FEM), and Point Collocation Method (PCM) [15–17].

In this paper, an improved FIM is developed for solving time-fractional convection-diffusion equations with variable coefficients. The constants of integration are determined by using initial conditions, while in the previous studies of FIM, interpolation methods were used. Also, the integration matrix of the singular integral in the definition of the fractional derivative is obtained by product Simpson's rules. The field points are generated uniformly along the t coordinate and the roots of the Legendre polynomials are used along the x coordinate. To demonstrate the accuracy and efficiency of the improved FIM, numerical examples are given.

The outline of this paper is organized as follows: The basics of FIM are described in section 2. Section 3 is devoted to computation of matrix of integration for a singular integral. In section 4, combination of FIM and RBF is described. In section 5, we used FIM based on RBF to solve some fractional convectional-diffusion equations. Numerical examples are included in 6 and a conclusion is provided in 7.

2. Finite integration method

Consider an integral of one-dimension function $f(x)$ in the region $[0,x]$:

$$F^{(1)}(x) := \int_0^x f(t)dt, \quad x \in [0,b]. \tag{2.1}$$

By applying linear interpolation technique, which uses trapezoidal rule, we have

$$\begin{aligned} F^{(1)}(x_k) &= \int_0^{x_k} f(t)dt = \sum_{i=1}^{k} \int_{x_i}^{x_{i+1}} f(t)dt \\ &\approx \sum_{i=1}^{k} \frac{h}{2} \left(f(x_i) + f(x_{i+1})\right) = \sum_{i=1}^{k} a_{ki}^{(1)} f(x_i), \end{aligned} \tag{2.2}$$

where

$$a_{1i}^{(1)} = 0,$$
$$a_{ki}^{(1)} = \begin{cases} \frac{h}{2}, & i = 1, \\ h, & i = 2, \dots, k-1, \\ \frac{h}{2}, & i = k, \\ 0, & i > k, \end{cases}$$

and $x_i = (i-1)h$, $h = \frac{b-0}{N-1}$, $i = 1, 2, .., N$, are nodal points in the region $[0,b]$, and $x_1 = 0$, $x_N = b$.

Note that integration (2.2) can be written in a matrix form as

$$\mathbf{F}^{(1)} = \mathbf{A}^{(1)}\mathbf{f}, \tag{2.3}$$

where

$$\mathbf{F}^{(1)} = [F_1^{(1)}, F_2^{(1)}, \dots F_k^{(1)}]^T, \qquad F_i^{(1)} = F^{(1)}(x_i),$$
$$\mathbf{f} = [f_1, f_2, \dots f_k]^T, \qquad f_i = f(x_i),$$

and the first order integration matrix

$$\mathbf{A}^{(1)} := [a_{ki}^{(1)}] = h \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1 & 1 & 1/2 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 1/2 & 1 & 1 & \dots & 1 & 1/2 \end{bmatrix}_{N \times N}. \tag{2.4}$$

Similarly, consider a double-layer integral of $f(x)$ from 0 to x:

$$F^{(2)}(x) := \int_0^x \int_0^\eta f(\xi)d\xi d\eta, \quad x \in [0,b], \tag{2.5}$$

and apply the trapezoidal rule again for integral function $F^{(2)}(x)$, we have

$$\begin{aligned} F^{(2)}(x_k) &= \int_0^{x_k}\int_0^{\eta} f(\xi)d\xi d\eta \approx \sum_{i=1}^{k} a_{ki}^{(1)}\int_0^{x_i} f(\eta)d\eta \\ &\approx \sum_{i=1}^{k}\sum_{j=1}^{i} a_{ki}^{(1)}a_{ij}^{(1)} f(x_j) = \sum_{i=1}^{k} a_{ki}^{(2)} f(x_i), \end{aligned} \tag{2.6}$$

This double-layer integral can be written again in a matrix form as

$$\mathbf{F}^{(2)} = \mathbf{A}^{(2)}\mathbf{f} = \left(\mathbf{A}^{(1)}\right)^2\mathbf{f}, \tag{2.7}$$

where $\mathbf{F}^{(2)} = [F_1^{(2)}, F_2^{(2)}, \ldots F_k^{(2)}]^T, \quad F_i^{(2)} = F^{(2)}(x_i)$, and $\mathbf{A}^{(2)}$ is the double layer integration matrix and can be derived from $\mathbf{A}^{(1)}$.

For two-dimensional case, consider a uniform distribution of $M = N_1 \times N_2$ collocation points in the problem domain, where N_1 and N_2 are the total number of columns and rows, respectively.

Similar to Eqs. (2.1) and (2.5), we define

$$F_x^{(1)}(x,y) := \int_0^{x} f(t,y)dt, \tag{2.8}$$

$$F_x^{(2)}(x,y) := \int_0^{x}\int_0^{\eta} f(\xi,y)d\xi d\eta, \tag{2.9}$$

where $k = N_1 \times (j-1) + i$ is the total number of points at the ith column and the jth row of the domain. The subscript x is used to show that the integration is with respect to x and the variable y is considered to be constant.

Similarly, the values of integration in Eqs. (2.8) and (2.9), in nodal points (x_k, y_k), i.e.

$$\begin{aligned} F_x^{(1)}(x_k,y_k) &:= \int_0^{x_k} f(t,y_k)dt, \\ F_x^{(2)}(x_k,y_k) &:= \int_0^{x_k}\int_0^{\eta} f(\xi,y_k)d\xi d\eta, \end{aligned}$$

can be expressed in matrix form as

$$\mathbf{F}_x^{(1)} = \mathbf{A}_x^{(1)}\mathbf{f}, \quad \mathbf{F}_x^{(2)} = \left(\mathbf{A}_x^{(1)}\right)^2\mathbf{f},$$

where

$$\begin{aligned} \mathbf{F}_x^{(m)} &= [F_x^{(m)}(x_1,y_1), F_x^{(m)}(x_2,y_2), \ldots F_x^{(m)}(x_M,y_M)]^T, \quad m = 1,2, \\ \mathbf{f} &= [f(x_1,y_1), f(x_2,y_2), \ldots f(x_M,y_M)]^T, \end{aligned}$$

$$\mathbf{A}_x^{(1)} = \begin{bmatrix} \mathbf{A}^{(1)} & 0 & \dots & 0 \\ 0 & \mathbf{A}^{(1)} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathbf{A}^{(1)} \end{bmatrix}_{N_1 \times N_2}, \tag{2.10}$$

and $\mathbf{A}^{(1)}$ is the first order integration matrix given in (2.4).

Similar relations for the first and the second order integration along y axis can be computed.

3. First order integration matrix for singular integrals

In this section, the first order integration matrix is constructed for computing a class of singular integrals as following

$$I(t) = \int_0^t \frac{u(t,s)}{(t-s)^\alpha} ds, \quad 0 \le t \le T, \tag{3.1}$$

where $u(t,s)$ is a well-behaved function. Here we use a high order product integration method, based on Simpson's method to approximate singular integral (3.1). Sufficient conditions for the convergence of the method is investigated in [9].

Let $N \ge 1$ be an integer, $h = (T-0)/N$, and consider the grid points $t_i = ih$, $i = 0, 1, ..., N$. The method, approximate the integral (3.1) over $[0, t_i]$, by using repeatedly the product Simpson's rule, if i is even. When i is odd, we use the product Simpson's rule over $[0, t_{i-3}]$ and the product $\frac{3}{8}$ rule over $[t_{i-3}, t_i]$. Thus, integral (3.1) is approximated by a quadrature formula in the following form

$$\int_0^{t_i} \frac{u(t_i,s)}{(t_i-s)^\alpha} ds \approx \sum_{j=0}^{\frac{i-2}{2}} \sum_{k=0}^{2} b_k(2j) u(t_i, s_{2j+k}),$$

when i is even, and

$$\int_0^{t_i} \frac{u(t_i,s)}{(t_i-s)^\alpha} ds \approx \sum_{j=0}^{\frac{i-5}{2}} \sum_{k=0}^{2} b_k(2j) u(t_i, s_{2j+k}) + \sum_{k=0}^{3} d_k u(t_i, s_{i+k-3}),$$

when i is odd, in which $s_j = t_j$ for $j = 0, ..., N$, and weights are computed by

$$b_0(j) = \frac{1}{2h^{\alpha-1}} \int_0^2 \frac{(v-1)(v-2)}{(i-j-v)^\alpha} dv,$$

$$b_1(j) = -\frac{1}{h^{\alpha-1}} \int_0^2 \frac{v(v-2)}{(i-j-v)^\alpha} dv,$$

$$b_2(j) = \frac{1}{2h^{\alpha-1}} \int_0^2 \frac{v(v-1)}{(i-j-v)^\alpha} dv,$$

and

$$
\begin{aligned}
d_0 &= -\frac{1}{6h^{\alpha-1}} \int_0^3 \frac{(v-1)(v-2)(v-3)}{(3-v)^\alpha} dv, \\
d_1 &= \frac{1}{2h^{\alpha-1}} \int_0^3 \frac{v(v-2)(v-3)}{(3-v)^\alpha} dv, \\
d_2 &= -\frac{1}{2h^{\alpha-1}} \int_0^3 \frac{v(v-1)(v-3)}{(3-v)^\alpha} dv, \\
d_3 &= \frac{1}{6h^{\alpha-1}} \int_0^3 \frac{v(v-1)(v-2)}{(3-v)^\alpha} dv.
\end{aligned}
$$

For $i=1$, we use the product trapezoidal rule, and we get

$$
\int_0^{t_1} \frac{u(t_i, s)}{(t_i - s)^\alpha} ds \approx w_0 u(t_1, s_0) + w_1 u(t_1, s_1),
$$

where

$$
w_0 = \frac{1}{h^{\alpha-1}} \int_0^1 \frac{1-v}{(1-v)^\alpha} dv, \qquad\qquad w_1 = \frac{1}{h^{\alpha-1}} \int_0^1 \frac{v}{(1-v)^\alpha} dv.
$$

Therefore, the first order integration matrix for singular integral (3.1), has the form

$$
\mathbf{A}^\alpha = \begin{bmatrix}
0 & & & & & & & & \\
w_0 & w_1 & & & & & & & \\
b_0(0) & b_1(0) & b_2(0) & & & & & & \\
d_0 & d_1 & d_2 & d_3 & & & & & \\
b_0(0) & b_1(0) & b_2(0)+b_0(2) & b_1(2) & b_2(2) & & & & \\
b_0(0) & b_1(0) & b_2(0)+d_0 & d_1 & d_2 & d_3 & & & \\
b_0(0) & b_1(0) & b_2(0)+b_0(2) & b_1(2) & b_2(2)+b_0(4) & b_1(4) & b_2(4) & & \\
b_0(0) & b_1(0) & b_2(0)+b_0(2) & b_1(2) & b_2(2)+d_0 & d_1 & d_2 & d_3 & \\
\vdots & & & & & & & & \ddots
\end{bmatrix}.
$$

4. The FIM with radial basis functions

Since RBF is a meshless method, we replace the linear interpolation in section 2 by RBF interpolation.

Based on RBF method, a function $u(x,t)$ at a point $\mathbf{x} = (x,t)$, is approximated by:

$$
u(\mathbf{x}) \approx \sum_{i=1}^{M} \alpha_i R_i(\mathbf{x}) + \sum_{j=1}^{Q} \beta_j P_j(\mathbf{x}) = \mathbf{R}^T(\mathbf{x})\boldsymbol{\alpha} + \mathbf{P}^T(\mathbf{x})\boldsymbol{\beta}, \tag{4.1}
$$

where $\mathbf{R}(\mathbf{x}) = [R_1(\mathbf{x}), R_2(\mathbf{x}), \ldots, R_M(\mathbf{x})]^T$ is a set of RBFs with centers $\mathbf{x}_i = (x_i, t_i)$, $i = 1, 2, \ldots, M$, $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_M]^T$ and $\boldsymbol{\beta} = [\beta_1, \beta_2, \ldots, \beta_M]^T$ are unknown coefficients to be determined, $P_j(\mathbf{x})$, $j = 1, 2, \ldots, Q$, are monomials in the space coordinate $\mathbf{x} = (x,t)$, and Q is the number of polynomial basis functions. For the linear basis functions, $\mathbf{P}^T(\mathbf{x})$ in Eq. (4.1) is

$$
\mathbf{P}^T(\mathbf{x}) = \{1, x, t\}, \quad Q = 3,
$$

and for the quadratic basis functions, we have

$$
\mathbf{P}^T(\mathbf{x}) = \{1, x, t, x^2, t^2, xt\}, \quad Q = 6.
$$

Since in Eq. (4.1) there are $M+Q$ unknown variables, to guarantee the uniqueness of the approximation, the polynomial terms $\mathbf{P}^T(\mathbf{x})$ satisfies the following additional requirements

$$\sum_{i=1}^{M} \alpha_i P_j(\mathbf{x}_i) = 0, \quad j = 1, 2, \ldots, Q. \tag{4.2}$$

It can be written in matrix form

$$\mathbf{P}_0^T \boldsymbol{\alpha} = \mathbf{0}. \tag{4.3}$$

By collocating Eq. (4.1) at points $\mathbf{x}_i = (x_i, t_i)$, $i = 1, 2, \ldots, M$, we have

$$\mathbf{R}_0^T \boldsymbol{\alpha} + \mathbf{P}_0^T \boldsymbol{\beta} = \mathbf{u}, \tag{4.4}$$

where

$$\mathbf{R}_0^T = \begin{bmatrix} R_1(\mathbf{x}_1) & R_2(\mathbf{x}_1) & \ldots & R_M(\mathbf{x}_1) \\ R_1(\mathbf{x}_2) & R_2(\mathbf{x}_2) & \ldots & R_M(\mathbf{x}_2) \\ \vdots & \vdots & \ldots & \vdots \\ R_1(\mathbf{x}_M) & R_2(\mathbf{x}_M) & \ldots & R_M(\mathbf{x}_M) \end{bmatrix}, \tag{4.5}$$

$$\mathbf{P}_0^T = \begin{bmatrix} P_1(\mathbf{x}_1) & P_2(\mathbf{x}_1) & \ldots & P_Q(\mathbf{x}_1) \\ P_1(\mathbf{x}_2) & P_2(\mathbf{x}_2) & \ldots & P_Q(\mathbf{x}_2) \\ \vdots & \vdots & \ldots & \vdots \\ P_1(\mathbf{x}_M) & P_2(\mathbf{x}_M) & \ldots & P_Q(\mathbf{x}_M) \end{bmatrix}, \tag{4.6}$$

and $\mathbf{u} = [u(\mathbf{x}_1), u(\mathbf{x}_2), \ldots, u(\mathbf{x}_M)]$. Solving linear system (4.3) and (4.4) gives

$$\boldsymbol{\alpha} = \mathbf{R}_0^{-1}\left(\mathbf{I} - \mathbf{P}_0(\mathbf{P}_0^T\mathbf{R}_0^{-1}\mathbf{P}_0)^{-1}\mathbf{P}_0^T\mathbf{R}_0^{-1}\right)\mathbf{u}, \tag{4.7}$$

$$\boldsymbol{\beta} = (\mathbf{P}_0^T\mathbf{R}_0^{-1}\mathbf{P}_0)^{-1}\mathbf{P}_0^T\mathbf{R}_0^{-1}\mathbf{u}, \tag{4.8}$$

where $\mathbf{I}$ denotes the identity matrix.

By substituting the coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ from (4.7) and (4.8) into (4.1), we have the approximation of $u(\mathbf{x})$, in terms of the nodal values u_i as follow

$$u(\mathbf{x}) \approx \Big(\mathbf{R}^T(\mathbf{x})\mathbf{R}_0^{-1}\left(\mathbf{I} - \mathbf{P}_0(\mathbf{P}_0^T\mathbf{R}_0^{-1}\mathbf{P}_0)^{-1}\mathbf{P}_0^T\mathbf{R}_0^{-1}\right) + \mathbf{P}^T(\mathbf{x})(\mathbf{P}_0^T\mathbf{R}_0^{-1}\mathbf{P}_0)^{-1}\mathbf{P}_0^T\mathbf{R}_0^{-1}\Big)\mathbf{u} = \sum_{i=1}^{M} \phi_i(\mathbf{x}) u_i, \tag{4.9}$$

where $\phi_i(\mathbf{x})$ is the shape function. By differentiating from Eq. (4.1) and using Eq. (4.9), we have

$$\frac{\partial u(\mathbf{x})}{\partial x} \approx \sum_{i=1}^{M} \alpha_i \frac{\partial R_i(\mathbf{x})}{\partial x} + \sum_{j=1}^{Q} \beta_j \frac{\partial P_j(\mathbf{x})}{\partial x} = \sum_{i=1}^{M} \phi_{i,x}(\mathbf{x}) u_i, \tag{4.10}$$

or in matrix form, $\mathbf{u}_x = \mathbf{D}_x \mathbf{u}$.

For the Multi-Quadric RBF (MQ), $R_i(x,t) = \sqrt{(x-x_i)^2+(t-t_i)^2+c^2}$, where c is the shape parameter, we can easily obtain the first derivative as

$$\frac{\partial R_i(x,t)}{\partial x} = \frac{x-x_i}{\sqrt{(x-x_i)^2+(t-t_i)^2+c^2}}. \tag{4.11}$$

Also, the first order integration of $u(\mathbf{x})$ is approximated as

$$U_x^{(1)}(\mathbf{x}) := \int_0^x u(\mathbf{x})dx \approx \sum_{i=1}^{M} \alpha_i \bar{R}_{i,x}(\mathbf{x}) + \sum_{j=1}^{Q} \beta_j \bar{P}_{j,x}(\mathbf{x}) = \sum_{i=1}^{M} \bar{\phi}_{i,x}^{(1)}(\mathbf{x}) u_i, \tag{4.12}$$

where

$$\begin{aligned}
\bar{R}_{i,x}(\mathbf{x}) &= \int_0^x R_i(\mathbf{x})dx, \quad \bar{P}_{j,x}(\mathbf{x}) = \int_0^x P_j(\mathbf{x})dx,\\
\bar{\phi}_{i,x}^{(1)}(\mathbf{x}) &= \int_0^x \phi_{i,x}(\mathbf{x})dx,
\end{aligned}$$

and the first order integration formula with respect to x, for the MQ-RBF is computed by

$$\begin{aligned}
\int_0^x R_i(\eta,t)d\eta = {}& \frac{1}{2}\left(c^2+(t-t_i)^2\right) \times \\
& \ln\left(x-x_i+\sqrt{(x-x_i)^2+(t-t_i)^2+c^2}\right) \\
& -\frac{1}{2}\left(c^2+(t-t_i)^2\right)\ln\left(-x_i+\sqrt{{x_i}^2+c^2+(t-t_i)^2}\right) \\
& \quad +\frac{x_i}{2}\sqrt{{x_i}^2+c^2+(t-t_i)^2}+\frac{x-x_i}{2}\sqrt{(x-x_i)^2+(t-t_i)^2+c^2}.
\end{aligned} \tag{4.13}$$

Furthermore, for the polynomial basis functions we have

$$\begin{aligned}
\mathbf{P}(\mathbf{x}) &= \{1, x, t, x^2, t^2, xt, \dots\}^T, \quad \mathbf{x} = (x,t),\\
\frac{\partial \mathbf{P}(\mathbf{x})}{\partial x} &= \{0, 1, 0, 2x, 0, t, \dots\}^T,\\
\int_0^x \mathbf{P}(\mathbf{x})dx &= \{x, \frac{x^2}{2}, xt, \frac{x^3}{3}, xt^2, \frac{x^2}{2}t, \dots\}^T.
\end{aligned}$$

Similarly, a double-layer integral of $u(\mathbf{x})$ is defined as

$$\begin{aligned}
U_x^{(2)}(\mathbf{x}) := \int_0^x \int_0^x u(\mathbf{x})dxdx &= \sum_{i=1}^{M} \alpha_i \bar{\bar{R}}_{i,x}(\mathbf{x}) + \sum_{j=1}^{Q} \beta_j \bar{\bar{P}}_{j,x}(\mathbf{x}) \\
&= \sum_{i=1}^{M} \bar{\bar{\phi}}_{i,x}^{(2)}(\mathbf{x}) u_i,
\end{aligned} \tag{4.14}$$

in which

$$\bar{\bar{R}}_{i,x}(\mathbf{x}) = \int_0^x \int_0^x R_i(\mathbf{x})dxdx, \quad \bar{\bar{P}}_{j,x}(\mathbf{x}) = \int_0^x \int_0^x P_j(\mathbf{x})dxdx,$$
$$\bar{\bar{\phi}}^{(2)}_{i,x}(\mathbf{x}) = \int_0^x \int_0^x \phi_{i,x}(\mathbf{x})dxdx.$$

Therefore, Eqs. (4.12) and (4.14) can be written in matrix form as:

$$\mathbf{U}_x^{(1)}(\mathbf{x}) := \mathbf{A}_x^{(1)}\mathbf{u}, \quad \mathbf{U}_x^{(2)}(\mathbf{x}) := \mathbf{A}_x^{(2)}\mathbf{u}, \tag{4.15}$$

where

$$\mathbf{A}_x^{(r)} = \begin{bmatrix} \bar{\phi}^{(r)}_{1,x}(\mathbf{x}_1) & \bar{\phi}^{(r)}_{2,x}(\mathbf{x}_1) & \dots & \bar{\phi}^{(r)}_{M,x}(\mathbf{x}_1) \\ \bar{\phi}^{(r)}_{1,x}(\mathbf{x}_2) & \bar{\phi}^{(r)}_{2,x}(\mathbf{x}_2) & \dots & \bar{\phi}^{(r)}_{M,x}(\mathbf{x}_2) \\ \vdots & \vdots & \dots & \vdots \\ \bar{\phi}^{(r)}_{1,x}(\mathbf{x}_M) & \bar{\phi}^{(r)}_{2,x}(\mathbf{x}_M) & \dots & \bar{\phi}^{(r)}_{M,x}(\mathbf{x}_M) \end{bmatrix}, \quad r = 1, 2. \tag{4.16}$$

Analogous relations is hold for derivatives and integrals with respect to t.

5. FIM based on RBFs for Solving Time-Fractional convection-diffusion Equation

The FIM can be easily extend for solving high dimensional problems. For illustration, consider the following time-fractional convection-diffusion equation with variable coefficients

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + a(x)\frac{\partial u(x,t)}{\partial x} + b(x)\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \qquad 0 < x < 1, \ 0 < t \leq 1, \tag{5.1}$$

and the following initial and boundary conditions

$$u(x,0) = g(x), \quad 0 < x < 1, \tag{5.2}$$
$$u(0,t) = h_0(t), \quad 0 \leq t \leq 1, \tag{5.3}$$
$$u(1,t) = h_1(t), \quad 0 \leq t \leq 1, \tag{5.4}$$

where nonzero functions $a(x)$ and $b(x)$ are continuous and $g(x)$, $h_0(t)$, and $h_1(t)$ are given functions in $L^2[0,1)$ and $f(x,t)$ is a given function in $L^2([0,1) \times [0,1))$. Furthermore, the time-fractional derivative of order $0 < \alpha < 1$ is defined in the Caputo sense.

At first, by using the definition of fractional Caputo derivative in Eq. (5.1) we get

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{1}{(t-s)^\alpha}ds + a(x)\frac{\partial u(x,t)}{\partial x} + b(x)\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \tag{5.5}$$

By twice integrating of Eq. (5.1) with respect to x, and using integration by part, and then rearranging the terms, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)}\int_0^x\int_0^\eta\int_0^t \frac{\partial u(\xi,s)}{\partial s}\frac{1}{(t-s)^\alpha}dsd\xi d\eta\\
&-\int_0^x\int_0^\eta (a'(\xi)-b''(\xi))u(\xi,t)d\xi d\eta\\
&+\int_0^x (a(\eta)-2b'(\eta))u(\eta,t))\,d\eta + b(x)u(x,t) - xb(0)\frac{\partial u(x,t)}{\partial x}\bigg|_{x=0}\\
&\qquad\qquad = \int_0^x\int_0^\eta f(\xi,t)d\xi d\eta + (b(0)+xa(0)-xb'(0))\,u(0,t). \qquad (5.6)
\end{aligned}$$

Based on RBF method, the solution of Eq. (5.5), $u(x,t)$, at a point $\mathbf{x}_i=(x_i,t_i)$, $i=1,2,\ldots,M$, is approximated by

$$u(x,t)=u(\mathbf{x})\approx\sum_{i=1}^M\phi_i(\mathbf{x})u_i=\sum_{i=1}^M\phi_i(x,t)u_i. \qquad (5.7)$$

Now, by substituting (5.7) in (5.6) and applying the finite integration method, we get

$$\begin{aligned}
&\left(\frac{1}{\Gamma(1-\alpha)}\mathbf{A}_x^{(2)}\mathbf{A}_t^\alpha\mathbf{D}_t-\mathbf{A}_x^{(2)}(\mathbf{a}'-\mathbf{b}'')+\mathbf{A}_x^{(1)}(\mathbf{a}-2\mathbf{b}')+\mathbf{b}-\mathbf{X}b(0)\mathbf{D}_x^0\right)\mathbf{u}\\
&\qquad\qquad = \mathbf{A}_x^{(2)}\mathbf{f}+(\mathbf{b}(0)+\mathbf{X}\mathbf{a}(0)-\mathbf{X}\mathbf{b}'(0))\,\mathbf{h}_0, \qquad (5.8)
\end{aligned}$$

where

$$\begin{aligned}
[\mathbf{D}_x^0]_{i,j}&=[\phi_{i,x}(0,t_j)]^T,\\
\mathbf{f}&=[f(x_1,t_1),f(x_2,t_2),\ldots,f(x_M,t_M)]^T,\\
\mathbf{h}_0&=[h(t_1),h(t_2),\ldots,h(t_M)]^T,\\
\mathbf{X}&=diag[x_1,x_2,\ldots,x_M],\\
\mathbf{a}&=diag[a(x_1),a(x_2),\ldots,a(x_M)],\\
\mathbf{a}'&=diag[a'(x_1),a'(x_2),\ldots,a'(x_M)],
\end{aligned}$$

and $\mathbf{b}$, $\mathbf{b}'$, and $\mathbf{b}''$ are defined similar to $\mathbf{a}$ and $\mathbf{a}'$.
Moreover, we can write the initial and boundary conditions (5.2) and (5.4) in a matrix form as follow

$$g(x)=u(x,0)\approx\sum_{i=1}^M\phi_i(x,0)u_i, \qquad \text{or}\quad \boldsymbol{\phi}^{(0)}\mathbf{u}=\mathbf{g}, \qquad (5.9)$$

$$h_1(x)=u(1,t)\approx\sum_{i=1}^M\phi_i(1,t)u_i, \qquad \text{or}\quad \boldsymbol{\phi}_1\mathbf{u}=\mathbf{h_1}. \qquad (5.10)$$

Eq. (5.8) with Eqs. (5.9) and (5.10), creates a linear system of equations and can be solved in terms of unknown values u_i, $i=1,2,\ldots,M$.

6. Numerical Examples

In this section, numerical examples are provided to illustrate the efficiency of this approach. In all examples, the value of the shape parameter is chosen to be $c = 1/N_x$.

Example 6.1. Consider the following time-fractional convection-diffusion equation [30]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + x\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} = 2t^\alpha + 2x^2 + 2, \tag{6.1}$$

where $0 < \alpha < 1$, with the initial condition $u(x,0) = 0$, $0 < x < 1$, and the boundary conditions

$$u(0,t) = \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}, \quad u(1,t) = 1 + \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}, \quad 0 \le t \le 1,$$

The exact solution is $u(x,t) = x^2 + \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}$.
Table 1 shows the absolute errors of numerical results, achieved by FIM with MQ-RBFs, for $N_x = N_t = 10$ and $\alpha = 0.3, 0.5, 0.7$. Also, Figure 1 shows the graph of the absolute errors for $N_x = 30$, $N_t = 15$, and $\alpha = 0.5$. A comparison is made between the absolute errors obtained by the present method with the Harr wavelet method [6] and Sinc-Legendre method [21] for $\alpha = 0.5$ in Table 2.

Table 1. Absolute Errors for Example 6.1, at $t = 1$ by $N_x = N_t = 10$.

x_i	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$
0.0130	6.8226e-04	7.5436e-06	5.7748e-05
0.0675	1.3844e-03	2.2100e-06	9.7319e-04
0.1603	2.4514e-04	1.5987e-05	1.3594e-03
0.2833	6.9608e-04	2.0179e-05	2.2230e-03
0.4256	7.2284e-04	1.4552e-05	2.6108e-03
0.5744	6.1914e-04	1.6345e-05	2.5043e-03
0.7167	3.6897e-04	2.4712e-05	1.9489e-03
0.8397	7.5934e-05	1.0328e-05	1.1247e-03
0.9325	2.3393e-04	2.8730e-05	4.6176e-04
0.9870	2.9617e-04	7.9615e-06	1.9783e-05

Example 6.2. Consider the following time-fractional convection-diffusion equation [30]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} = \left(\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + 4\pi^2 t^2\right)\sin(2\pi x), \tag{6.2}$$

where $0 < \alpha < 1$, with the initial condition $u(x,0) = 0$, $0 < x < 1$, and the boundary conditions $u(0,t) = 0$, $u(1,t) = 0$, $0 \le t \le 1$, The exact solution is $u(x,t) = t^2 \sin(2\pi x)$.

Table 2. Comparison of Absolute Errors for Example 6.1, for $\alpha = 0.5$, $t = 0.5$.

x_i	Harr wavelet [6] $m = 64$	Sinc-Legendre [21] $m = 25$	Present method $N_x = N_t = 10$	Present method $N_x = 20, N_t = 15$
0.1	1.210e-3	6.462e-6	5.5127e-05	1.7417e-05
0.2	1.259e-3	1.578e-5	6.3034e-05	9.1361e-06
0.3	1.865e-3	2.272e-5	2.5286e-05	3.6982e-06
0.4	7.412e-3	2.674e-5	9.7841e-06	5.5253e-08
0.5	1.000e-6	2.759e-5	2.3230e-06	2.0309e-06
0.6	7.460e-3	2.534e-5	6.5798e-06	2.1323e-06
0.7	1.724e-3	2.035e-5	3.6040e-06	4.3969e-07
0.8	4.990e-3	1.320e-5	1.8435e-06	2.6442e-06
0.9	1.678e-2	4.653e-6	6.3126e-05	5.3220e-06

Figure 1. Absolute Error for Example 6.1 by $N_x = 30$, $N_t = 15$, $\alpha = 0.5$.

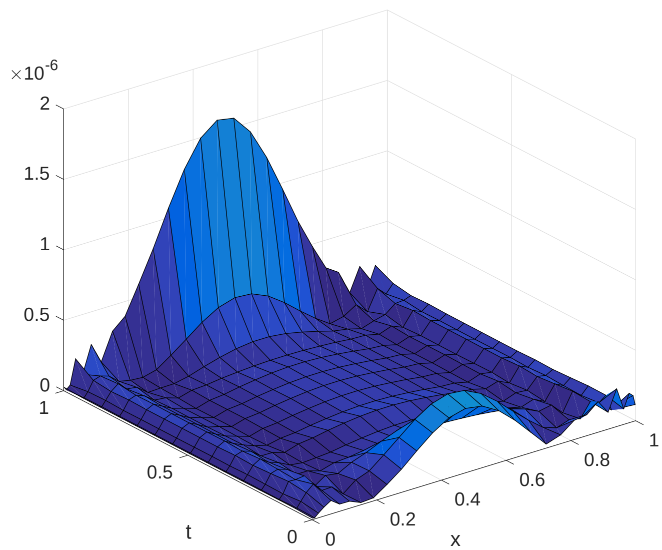


Table 3 shows the absolute errors of numerical results, achieved by FIM with MQ-RBFs, for $N_x = N_t = 15$ and $\alpha = 0.3, 0.5, 0.7$. Also, Figure 2 shows the graph of the absolute error for $N_x = N_t = 15$, and $\alpha = 0.5$. Table 4 shows a comparison between the absolute errors obtained by the present method with the Chebyshev wavelets [30] for $\alpha = 0.5$.

Table 3. Absolute Errors for Example 6.2, by FIM-RBF method.

x_i	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$
0.0060	5.2870e-03	5.2375e-03	5.1375e-03
0.0314	6.6530e-03	7.0803e-03	7.9649e-03
0.0759	1.0664e-03	1.3431e-04	1.7973e-03
0.1378	9.6358e-04	2.5440e-03	5.8306e-03
0.2145	7.9222e-04	2.8262e-03	7.0697e-03
0.3029	9.6562e-04	2.9346e-03	7.0582e-03
0.3994	1.0528e-03	2.2557e-03	4.8016e-03
0.5000	1.2604e-03	1.1870e-03	1.0918e-03
0.6006	1.4816e-03	1.3947e-04	2.5852e-03
0.6971	1.6544e-03	4.3188e-04	4.7010e-03
0.7855	1.8245e-03	2.9531e-04	4.6375e-03
0.8622	2.1498e-03	5.2249e-04	2.8054e-03
0.9241	2.5276e-03	1.5729e-03	3.7269e-04
0.9686	2.8134e-03	2.4291e-03	1.6473e-03
0.9940	3.0375e-03	2.9790e-03	2.8591e-03

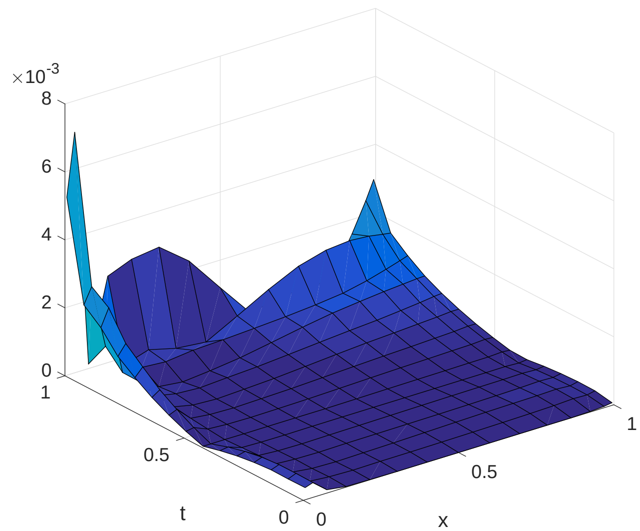
Table 4. Comparison of Absolute Errors for Example 6.2, for $\alpha = 0.5$.

(x_i, t_i)	Chebyshev wavelets [30] $k = 2, M = 6$	Present method $N_x = N_t = 20$	$N_x = N_t = 30$
(0.1,0.1)	2.3839e-5	2.3961e-05	2.8260e-06
(0.2,0.2)	3.6105e-5	5.7397e-07	4.0035e-07
(0.3,0.3)	2.6326e-5	3.8013e-06	2.1145e-07
(0.4,0.4)	1.4284e-5	2.8072e-06	4.1308e-07
(0.5,0.5)	1.9179e-15	1.3523e-05	2.4121e-06
(0.6,0.6)	5.8484e-6	5.4555e-05	1.0361e-05
(0.7,0.7)	1.3371e-5	1.1569e-04	2.1877e-05
(0.8,0.8)	1.0486e-5	1.9818e-04	3.5520e-05
(0.9,0.9)	4.8384e-6	3.3039e-04	7.1919e-05

7. Conclusion

In this paper, a modification of the finite integration method in combination with the radial basis function method is extended to solve time-fractional convection-diffusion equation with variable coefficients. By transforming PDEs into integral equations, by the finite integration method, constants of integration are appeared. So, the proposed method used initial conditions for determining these constants. This leads to fewer computations rather than the usual FIM. Also, to overcome the singularity included in the definition of fractional derivatives, an integration matrix is obtained by the product Simpson method. The main advantage of FIM is that it gives much higher accurate approximations than the finite difference method, finite

FIGURE 2. Absolute Error for Example 6.2 by $N_x = N_t = 15$, $\alpha = 0.5$.



element method, and point collocation method [15–17]. As the results obtained from the proposed approach show in Tables, the presented approach is efficient and reliable. Also, the comparisons with the previous methods show that the proposed method have enough accurate solutions.

REFERENCES

[1] S. Abbasbandy, S. Kazem, M. S. Alhuthali, and H. H. Alsulami, *Application of the operational matrix of fractional-order legendre functions for solving the time-fractional convection-diffusion equation*, Appl. Math. Comput., *266* (2015), 31–40.

[2] O. P. Agrawal, *Solution for a fractional diffusion-wave equation defined in a bounded domain*, Nonlinear. Dyn., *29*(1) (2002), 145–155.

[3] M. Behroozifar and A. Sazmand, *An approximate solution based on jacobi polynomials for time-fractional convection-diffusion equation*, Appl. Math. Comput., *296* (2017), 1–17.

[4] M. D. Buhmann, *Radial Basis Functions: Theory and Implementations*, Cambridge University Press, 2004.

[5] Y. Chen, Y. Sun, and L. Liu, *Numerical solution of fractional partial differential equations with variable coefficients using generalized fractional-order legendre functions*, Appl. Math. Comput., *244*(Supplement C) (2014), 847–858.

[6] Y. Chen, Y. Wu, Y. Cui, Z. Wang, and D. Jin, *Wavelet method for a class of fractional convection-diffusion equation with variable coefficients*, J. Comput. Sci., *1*(3) (2010), 146–149.

[7] M. Cui, *Compact finite difference method for the fractional diffusion equation*, J. Comput. Phys., *228*(20) (2009), 7792–7804.

[8] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 2010.

[9] T. Diogo, N. B. Franco, and P. Lima, *High order product integration methods for a Volterra integral equation with logarithmic singular kernel*, Commun. Pure Appl. Anal., *3*(2) (2004), 217–235.
[10] G. E. Fasshauer, *Meshfree methods*, In M. Rieth and W. Schommers, editors, *Handbook of Theoretical and Computational Nanotechnology*, page 33-97, American Scientific Publishers, 2006.
[11] G. E. Fasshauer, *Meshfree Approximation Methods with MATLAB*, Word Scientific Publishing, 2007.
[12] R. L. Hardy, *Multiquadric equations of topography and other irregular surfaces*, J. Geophys. Res.,*76*(8) (1971), 1905–1915.
[13] M. M. Khader and S. Kumar, *An accurate numerical method for solving the linear fractional klein-gordon equation*, Math. Meth. Appl. Sci., *37*(18) (2014), 2972–2979.
[14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Volume 204 (North-Holland Mathematics Studies), Elsevier Science Inc., New York, USA, 2006.
[15] M. Li, C. S. Chen, Y. C. Hon, and P. H. Wen, *Finite integration method for solving multi-dimensional partial differential equations*, Appl. Math. Model., *39*(17) (2015), 4979–4994.
[16] M. Li, Z. L. Tian, Y. C. Hon, C. S. Chen, and P. H. Wen, *Improved finite integration method for partial differential equations*, Eng. Anal. Bound. Elem., *64*(Supplement C) (2016), 230–236.
[17] Y. Li and Y. C. Hon, *Finite integration method with radial basis function for solving stiff problems*, Eng. Anal. Bound. Elem., *82*(Supplement C) (2017), 32–42.
[18] Y. Li and W. Zhao, *Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations*, Appl. Math. Comput., *216*(8) (2010), 2276–2285.
[19] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*. Mathematics in Science and Engineering. Elsevier Science, 1998.
[20] L. Ren and Y. M. Wang, *A fourth-order extrapolated compact difference method for time-fractional convection-reaction-diffusion equations with spatially variable coefficients*, Appl. Math. Comput., *312* (2017), 1–22.
[21] A. Saadatmandi, M. Dehghan, and M. R. Azizi, *The sinc-legendre collocation method for a class of fractional convection-diffusion equations with variable coefficients*, Commun Nonlinear Sci. Numer. Simulat., *17*(11) (2012), 4125–4136.
[22] J. Wang, T. Liu, H. Li, Y. Liu, and S. He, *Second-order approximation scheme combined with -Galerkin {MFE} method for nonlinear time fractional convection-diffusion equation*, Comput. Math. Appl., *73*(6) (2017), 1182–1196.
[23] Y. M. Wang, *A high-order compact finite difference method and its extrapolation for fractional mobile/immobile convection-diffusion equations*, Calcolo, *54*(3) (2017), 733–768.
[24] T. Weiland, *A discretization model for the solution of Maxwell's equations for six-component fields*, Archiv Elektronik und Uebertragungstechnik, *31* (1977), 116–120.
[25] P. H. Wen, Y. C. Hon, M. Li, and T. Korakianitis, *Finite integration method for partial differential equations*, Appl. Math. Model., *37*(24) (2013), 10092–10106.
[26] H. Wendland, *Scattered Data Approximation*, Cambridge University Press, 2004.
[27] J. Xie, Q. Huang, and X. Yang, *Numerical solution of the one-dimensional fractional convection diffusion equations based on chebyshev operational matrix*, SpringerPlus, *5*(1) (2016), Article number: 1149.
[28] A. Yıldırım and H. Koçak, *Homotopy perturbation method for solving the space-time fractional advection-dispersion equation*, Adv. Water Res., *32*(12) (2009), 1711–1716.
[29] Y. Yu, D. Xu, and Y.C. Hon, *Reconstruction of inaccessible boundary value in a sideways parabolic problem with variable coefficients-forward collocation with finite integration method*, Eng. Anal. Bound. Elem., *61*(Supplement C) (2015), 78–90.
[30] F. Zhou and X. Xu, *The third kind chebyshev wavelets collocation method for solving the time-fractional convection diffusion equations with variable coefficients*, Appl. Math. Comput., *280*(Supplement C) (2016), 11–29.