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An extended complete Chebyshev system of 3 Abelian integrals related to a non-algebraic Hamiltonian system

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Abstract In this paper, we study the Chebyshev property of the 3-dimentional vector space $E = \langle I_0, I_1, I_2 \rangle$, where $I_k(h) = \int_{H=h} x^k y\, dx$ and $H(x,y) = \frac{1}{2}y^2 + \frac{1}{2}(e^{-2x}+1) - e^{-x}$ is a non-algebraic Hamiltonian function. Our main result asserts that E is an extended complete Chebyshev space for $h \in (0, \frac{1}{2})$. To this end, we use the criterion and tools developed by Grau et al. in [5] to investigate when a collection of Abelian integrals is Chebyshev.

Keywords. Non-algebraic Hamiltonian, Abelian integral, Chebyshev property, ECT-system.

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1. Introduction

The problem considered in this work is closely related to the second part of the Hilbert's 16th problem. The second part of the Hilbert's 16th problem asks about the maximum number of limit cycles and their relative locations in planar polynomial vector fields. A weaker version of this problem, weak Hilbert's 16th problem, is proposed by Arnold [1] to study the zeros of Abelian integrals. To state this problem, consider a small perturbation of a Hamiltonian vector field $X_\varepsilon = X_H + \varepsilon Y$, where

$$X_H(x,y) = (-H_y(x,y), H_x(x,y)), \qquad Y(x,y) = (P(x,y), Q(x,y)),$$

and ε is a small positive parameter. Also suppose that the level curves $\{\gamma_h\}_{h\in(a,b)}$, $\gamma_h \subset \{(x,y)|H(x,y) = h\}$, form a continuous family of ovals. By Poincaré-Pontryagin Theorem [9], the first approximation in ε of the displacement function of the Poincaré

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map of X_ε is given by $I(h) = \int_{\gamma_h} Pdy - Qdx$. As an immediate consequence of Poincaré-Pontryagin Theorem, the number of isolated zeros of $I(h)$, counted with multiplicity, gives an upper bound for the number of limit cycles of X_ε for $\varepsilon \to 0^+$. If P and Q are polynomials which their coefficients are considered as parameters, then we can decompose $I(h)$ into a linear combination of the form

$$I(h) = c_0 I_0(h) + c_1 I_1(h) + \cdots + c_{n-1} I_{n-1}(h),$$

where each c_k depends on the parameters, and each I_k is an Abelian integral of form $\int_{\gamma_h} x^i y^j dx$ or $\int_{\gamma_h} x^i y^j \, dy$. So the problem is converted to finding the maximum number of isolated zeros of any function in the space $\langle I_0, I_1, \cdots, I_{n-1}\rangle$. This problem becomes easier when the basis of this vector space is a Chebyshev system. We recall that a real vector space E of real analytic functions on a real interval J is called Chebyshev, provided that each $f \in E$ has at most $(\dim E) - 1$ isolated zeros (counted with multiplicity) on J. One of the simplest example is when $P = 0$, $Q = (\alpha_0 + \alpha_1 x + \alpha_2 x^2)\, y$ and H being a polynomial, for which $I(h)$ reduces to

$$I(h) = \alpha_0 \int_{H=h} y \, dx + \alpha_1 \int_{H=h} x \, y \, dx + \alpha_2 \int_{H=h} x^2 \, y \, dx.$$

But in the non-polynomial setting, especially when the exponential function appears in the Hamiltonian function H, it makes things very complicated and difficult. There are very few results on this area, when the Hamiltonian function H is not a polynomial. Recently, the authors of [4] studied the number of limit cycles for perturbed pendulum-like equations on the cylinder, in which the perturbation terms are defined by trigonometric polynomials of degree n, and the associated Hamiltonian is given by $H(x, y) = \frac{y^2}{2} + 1 - \cos(x)$. An excellent work is also done by Villadelprat et al. in [3] based on a "computer assisted proof" using interval arithmetic. They applied this approach to prove a conjecture expressed by Dumortier and Roussarie in [2]. We recall it from [3] in the following.

Conjecture 1.1. For each non-negative integer k, define $J_k(h) = \int_{\gamma_h} y^{2k-1} \, dx$, in which $\gamma_h \subset \{(x, y) | A(x) + B(x)y^2 = h\}$ with $A(x) = \frac{1}{2} - e^{-2x}(x + \frac{1}{2})$ and $B(x) = e^{-2x}$. Then $(J_0, J_1, \cdots, J_n)$ is an extended complete Chebyshev system on $[0, \frac{1}{2})$ for $n \geq 0$.

In [3], the authors showed that this conjecture is true for $n = 0, 1, 2$, or equivalently, (J_0, J_1, J_2) is an extended complete Chebyshev system on $[0, \frac{1}{2})$.

In this paper we consider the non-polynomial potential system

$$\dot{x} = y, \qquad \dot{y} = e^{-x}(e^{-x} - 1), \tag{1.1}$$

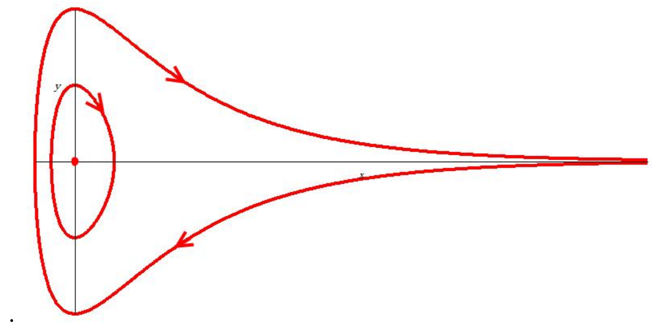
with Hamiltonian function

$$H(x, y) = \frac{1}{2} y^2 + \frac{1}{2}(e^{-2x} + 1) - e^{-x}.$$

The only equilibrium point of system (1.1) is a center at the origin whose periodic orbits lie in the interior of an annulus encircled by $H(x, y) = \frac{1}{2}$. The phase portrait of system (1.1) is shown in Figure 1. We define

$$\bar{J}_i(h) = \int_{\gamma_h} x^i y \, dx, \qquad i = 0, 1, 2, \tag{1.2}$$

Figure 1. Phase portrait of system (1.1)



where $\gamma_h = \{(x,y)|A(x)+B(x)y^2 = h, h \in (0, \frac{1}{2})\}$ is the period annulus of system (1.1) with $A(x) = \frac{1}{2}(e^{-2x}+1) - e^{-x}$ and $B(x) = \frac{1}{2}$. Our main result is as follows:

Theorem 1.2. *$(\bar{J}_0, \bar{J}_1, \bar{J}_2)$ is an extended complete Chebyshev system on $(0, \frac{1}{2})$.*

The above result shows that the cyclicity of the period annulus of system (1.1) under the small perturbation $\varepsilon(c_0 + c_1\, x + c_2\, x^2)y\frac{\partial}{\partial y}$ is at most two. Our proof is based on a criterion appearing in [5], which greatly simplifies the investigation of the Chebyshev property. This algebraic criterion is briefly explained in [3] with a good manner.

The paper is organized as follows. To prove our main result, we need some definitions and preliminaries, which will be given in Section 2. At last, Section 3 is devoted to the proof of the main result (Theorem 1.2).

2. Definitions and preliminaries

In this section we review some definitions and results from [5] that will be used in this paper.

Definition 2.1. Let $f_0, f_1, \cdots, f_{n-1}$ be analytic functions on an interval I.

- The ordered set of functions $(f_0, f_1, \cdots, f_{n-1})$ is a complete Chebyshev system (in short, CT-system) on I if for all $k = 1, 2, \cdots, n$, any nontrivial linear combination

  $$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x),$$

  has at most $k-1$ isolated zeros on I.
- The ordered set of functions $(f_0, f_1, \cdots, f_{n-1})$ is an extended complete Chebyshev system (in short, ECT-system) on I if for all $k = 1, 2, \cdots, n$, any nontrivial linear combination

  $$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x),$$

  has at most $k-1$ isolated zeros on I counted with multiplicity.

Definition 2.2. Let $f_0, f_1, \cdots, f_{k-1}$ be analytic functions on I. The continuous Wronskian of $(f_0, f_1, \cdots, f_{k-1})$ at $x \in I$ is defined as

$$W[f_0, f_1, \cdots, f_{k-1}](x) = \begin{vmatrix} f_0(x) & \cdots & f_{k-1}(x) \\ f_0'(x) & \cdots & f_{k-1}'(x) \\ & \vdots & \\ f_0^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

Lemma 2.3. *(see [6, 8]) The ordered set of functions $(f_0, f_1, \cdots, f_{n-1})$ is an ECT-system on I if and only if for each $k = 1, 2, \cdots, n$,*

$$W[f_0, f_1, \cdots, f_{k-1}](x) \neq 0, \qquad \textit{for all } \ x \in I.$$

Suppose that $H(x, y) = A(x) + B(x)y^2$ is an analytic function in some open subset of the plane that has a local minimum at the origin with $H(0, 0) = 0$. Then there exists a punctured neighbourhood P of $(0, 0)$ foliated by ovals $\gamma_h \subset \{H(x, y) = h\}$. Let us parameterize the set of ovals γ_h inside P by the energy levels $h \in (0, h_0)$ for some positive number h_0. The projection of P on the x-axis is an interval (x_l, x_r) with $x_l < 0 < x_r$. Under these assumptions, $A(x)$ has a zero of even multiplicity at $x = 0$, and there exists an analytic involution σ such that

$$A(x) = A(\sigma(x)), \qquad \textit{for all } \ x \in (x_l, x_r). \tag{2.1}$$

Recall that a map σ is an involution if $\sigma^2 = Id$ and $\sigma \neq Id$. Given a function κ defined on $(x_l, x_r)\backslash\{0\}$, we define its σ-balance as

$$\mathcal{B}_\sigma(\kappa)(x) = \kappa(x) - \kappa(\sigma(x)).$$

Following these notations we now restate Theorem B in [5].

Theorem 2.4. *Let $f_0, f_1, \cdots, f_{n-1}$ be analytic functions on (x_l, x_r), and consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x) y^{2s-1}\, dx, \qquad i = 0, 1, \cdots, n-1.$$

Let σ be the involution associated to A and define $\ell_i := \mathcal{B}_\sigma(\frac{f_i}{A' B^{\frac{2s-1}{2}}})$. If $(\ell_0, \ell_1, \cdots, \ell_{n-1})$ is a CT-system on $(0, x_r)$ and $s > (n-2)$, then $(I_0, I_1, \cdots, I_{n-1})$ is an ECT-system on $(0, h_0)$.

In some cases the condition $s > (n-2)$ in previous Theorem is not fulfilled, but it is possible to overcome this problem using Lemma 4.1 in [5]. So we restate it here.

Lemma 2.5. *(see [5]) Let $\gamma_h \subset \{(x, y)|A(x) + B(x)y^2 = h\}$ be an oval. If F/A' is analytic at $x = 0$, then for each integer k*

$$\int_{\gamma_h} F(x) y^{k-2}\, dx = \int_{\gamma_h} G(x) y^k\, dx,$$

where $G(x) = \frac{2}{k} B^{k/2}(x) \left(\frac{F B^{1-k/2}}{A'}\right)'(x)$.

3. Proof of Theorem 1.2

First note that the Abelian integrals $\bar{J}_0$, $\bar{J}_1$ and $\bar{J}_2$ defined in (1.2) do not fulfilled the hypothesis $s > (n-2)$ in Theorem 2.4. So, we first prove:

Lemma 3.1. *For $i = 0, 1, 2$, $\bar{J}_i(h) = \frac{1}{h^2}\int_{\gamma_h} f_i(x)y^5\,dx$, where*

$$\begin{aligned}
f_0(x) =& \frac{1}{4} + \frac{7}{60}\,\mathrm{e}^{x} + \frac{1}{30}\,\mathrm{e}^{2\,x},\\
f_1(x) =& \frac{1}{60}\,\mathrm{e}^{2\,x}\,(2\,x+3) + \frac{1}{60}\,\mathrm{e}^{x}\,(5+7\,x) - \frac{2}{15} + \frac{1}{4}\,x,\\
f_2(x) =& \frac{1}{30}\,\mathrm{e}^{2\,x}\left(x^2+3\,x+1\right) + \frac{1}{60}\,\mathrm{e}^{x}\left(10\,x+7\,x^2-4\right)\\
&+ \frac{1}{30} - \frac{4}{15}\,x + \frac{1}{4}\,x^2.
\end{aligned}$$

Proof. We set $J(h) = a\bar{J}_0(h) + b\bar{J}_1(h) + c\bar{J}_2(h)$. Then

$$\begin{aligned}
h^2J(h) &= \int_{\gamma_h} (A(x)+B(x)y^2)^2(a+bx+cx^2)y\,dx\\
&= \int_{\gamma_h}\left(\sum_{i=1}^{3} g_i(x)y^{2i-1}\right)dx,
\end{aligned}$$

where $g_1(x) = A^2(x)(a+bx+cx^2)$, $g_2(x) = 2A(x)B(x)(a+bx+cx^2)$ and $g_3(x) = B^2(x)(a+bx+cx^2)$. Now, using Lemma 2.5 with $F = g_1$ and $k = 3$ we get

$$h^2J(h) = \int_{\gamma_h} (\hat{g}_2y^3 + g_3y^5)\,dx,$$

where

$$\begin{aligned}
\hat{g}_2(x) =& -\frac{1}{12}\,\mathrm{e}^{x}\,\big[(-8\,bx-8\,cx^2-8\,a+2\,cx+b)\mathrm{e}^{-3\,x}\\
&+(15\,bx+15\,cx^2+15\,a-6\,cx-3\,b)\mathrm{e}^{-2\,x}\\
&+(-6\,bx-6\,cx^2-6\,a+3\,b+6\,cx)\mathrm{e}^{-x}\\
&-a-bx-cx^2-b-2\,cx\,\big].
\end{aligned}$$

Again, Lemma 2.5 with $F = \hat{g}_2$ and $k = 5$ implies that

$$h^2J(h) = \int_{\gamma_h} \hat{g}_3y^5\,dx,$$

where

$$\begin{aligned}
\hat{g}_3(x) =& \frac{1}{60}\mathrm{e}^{2\,x}\,\big[\left(15\,a+15\,bx-8\,b+15\,cx^2+2\,c-16\,cx\right)\mathrm{e}^{-2\,x}\\
&+\left(7\,a-4\,c+5\,b+7\,cx^2+10\,cx+7\,bx\right)\mathrm{e}^{-x}\\
&+2\,c+2\,bx+2\,a+3\,b+2\,cx^2+6\,cx\,\big].
\end{aligned}$$

Note that g_1/A' and $\hat{g}_2/A'$ are analytic at $x = 0$ because it is easy to see that $g_1(0) = 0$ and $\hat{g}_2(0) = 0$, so that we could apply Lemma 2.5. Finally computations show that $\hat{g}_3 = af_0 + bf_1 + cf_2$. This ends the proof. □

Now by setting $h^2\bar{J}_i(h) = \tilde{J}_i(h)$, we get

$$\tilde{J}_i(h) := \int_{\gamma_h} f_i(x)y^5\,dx \qquad for\ i = 0, 1, 2.$$

So, $(\bar{J}_0, \bar{J}_1, \bar{J}_2)$ is an ECT-system on $(0, \frac{1}{2})$ if and only if $(\tilde{J}_0, \tilde{J}_1, \tilde{J}_2)$ is an ECT-system on $(0, \frac{1}{2})$. By Theorem 2.4, the latter is also equivalent to that $(\mathcal{B}_\sigma(g_0), \mathcal{B}_\sigma(g_1), \mathcal{B}_\sigma(g_2))$ being an ECT-system on $(0, x_r)$ where $g_i := \frac{f_i}{A'B^{5/2}}$ for $i = 0, 1, 2$.

On the other hand that by Lemma 2.6 of [7], if $\varphi : I \to J$ is a diffeomorphism, then $(f_0, f_1, ..., f_{n-1})$ is an ECT-system on J if and only if $(f_0 \circ \varphi, f_1 \circ \varphi, \cdots, f_{n-1} \circ \varphi)$ is an ECT-system on I. Consequently, since σ is a C^1 diffeomorphism from $(0, x_r)$ to $(x_l, 0)$ and $\mathcal{B}_\sigma(g_i \circ \sigma) = -\mathcal{B}_\sigma(g_i)$, it is enough to prove that $(\mathcal{B}_\sigma(g_0), \mathcal{B}_\sigma(g_1), \mathcal{B}_\sigma(g_2))$ is an ECT-system on $(x_l, 0)$. Setting $\ell_i = \mathcal{B}_\sigma(g_i)$, a direct computation shows that

$$\begin{aligned}
\ell_0(x) =& \frac{\sqrt{2}\mathrm{e}^{x+\sigma(x)}}{15(\mathrm{e}^{-x}-1)(\mathrm{e}^{-\sigma(x)}-1)}(-7\,\mathrm{e}^{x-2\,\sigma(x)} + 7\,\mathrm{e}^{x-\sigma(x)} - 15\,\mathrm{e}^{-2\sigma(x)}\\
&+ 15\,\mathrm{e}^{-\sigma(x)} - 2\,\mathrm{e}^{2\,x-2\,\sigma(x)} + 2\,\mathrm{e}^{2\,x-\sigma(x)} + 7\,\mathrm{e}^{\sigma(x)-2\,x} - 7\,\mathrm{e}^{\sigma(x)-x}\\
&+ 15\,\mathrm{e}^{-2\,x} - 15\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{2\,\sigma(x)-2\,x} - 2\,\mathrm{e}^{2\sigma(x)-x}),
\end{aligned}$$

$$\begin{aligned}
\ell_1(x) =& \frac{\sqrt{2}\mathrm{e}^{x+\sigma(x)}}{15(\mathrm{e}^{-x}-1)(\mathrm{e}^{-\sigma(x)}-1)}(-2\,x\mathrm{e}^{2\,x-2\,\sigma(x)} + 2\,x\mathrm{e}^{2\,x-\sigma(x)} - 8\,\mathrm{e}^{-2\,x}\\
&+ 8\,\mathrm{e}^{-2\,\sigma(x)} - 8\,\mathrm{e}^{-\sigma(x)} - 5\,\mathrm{e}^{x-2\,\sigma(x)} + 5\,\mathrm{e}^{x-\sigma(x)} - 15\,x\mathrm{e}^{-2\,\sigma(x)}\\
&+ 15\,x\mathrm{e}^{-\sigma(x)} - 7\,\mathrm{e}^{x-2\,\sigma(x)}x + 7\,\mathrm{e}^{x-\sigma(x)}x - 3\,\mathrm{e}^{2\,x-2\,\sigma(x)}\\
&+ 3\,\mathrm{e}^{2\,x-\sigma(x)} + 2\,\sigma\left(x\right)\mathrm{e}^{2\,\sigma(x)-2\,x} - 2\,\sigma\left(x\right)\mathrm{e}^{2\,\sigma(x)-x}\\
&+ 8\,\mathrm{e}^{-x} + 5\,\mathrm{e}^{\sigma(x)-2\,x} - 5\,\mathrm{e}^{\sigma(x)-x} + 15\,\sigma\left(x\right)\mathrm{e}^{-2\,x}\\
&- 15\,\sigma\left(x\right)\mathrm{e}^{-x} + 7\,\mathrm{e}^{\sigma(x)-2\,x}\sigma\left(x\right) - 7\,\mathrm{e}^{\sigma(x)-x}\sigma\left(x\right)\\
&+ 3\,\mathrm{e}^{2\,\sigma(x)-2\,x} - 3\,\mathrm{e}^{2\,\sigma(x)-x}),
\end{aligned}$$

$$\begin{aligned}
\ell_2(x) =& \frac{\sqrt{2}\mathrm{e}^{x+\sigma(x)}}{15\left(\mathrm{e}^{-\sigma(x)}-1\right)}(2\,\mathrm{e}^{-2\,x} - 2\,\mathrm{e}^{-x} - 15\,x^2\mathrm{e}^{-2\,\sigma(x)} + 15\,x^2\mathrm{e}^{-\sigma(x)}\\
&+ 15\,\sigma^2\left(x\right)\mathrm{e}^{-2\,x} - 15\,\sigma^2\left(x\right)\mathrm{e}^{-x} + 16\,x\mathrm{e}^{-2\,\sigma(x)} - 16\,x\mathrm{e}^{-\sigma(x)}\\
&- 16\,\sigma\left(x\right)\mathrm{e}^{-2\,x} + 16\,\sigma\left(x\right)\mathrm{e}^{-x} - 10\,\mathrm{e}^{x-2\,\sigma(x)}x + 10\,\mathrm{e}^{x-\sigma(x)}x\\
&+ 6\,\sigma\left(x\right)\mathrm{e}^{2\,\sigma(x)-2\,x} - 6\,\sigma\left(x\right)\mathrm{e}^{2\,\sigma(x)-x} + 10\,\mathrm{e}^{\sigma(x)-2\,x}\sigma\left(x\right)\\
&- 10\,\mathrm{e}^{\sigma(x)-x}\sigma\left(x\right) - 6\,x\mathrm{e}^{2\,x-2\,\sigma(x)} + 6\,x\mathrm{e}^{2\,x-\sigma(x)} - 2\,\mathrm{e}^{-2\,\sigma(x)}\\
&+ 2\,\mathrm{e}^{-\sigma(x)} + 4\,\mathrm{e}^{x-2\,\sigma(x)} + 2\,\mathrm{e}^{2\,\sigma(x)-2\,x} - 2\,\mathrm{e}^{2\,\sigma(x)-x} - 4\,\mathrm{e}^{x-\sigma(x)}\\
&- 2\,\mathrm{e}^{2\,x-2\,\sigma(x)} + 2\,\mathrm{e}^{2\,x-\sigma(x)} - 4\,\mathrm{e}^{\sigma(x)-2\,x} - 2\,x^2\mathrm{e}^{2\,x-2\,\sigma(x)}\\
&+ 4\,\mathrm{e}^{\sigma(x)-x} + 2\,x^2\mathrm{e}^{2\,x-\sigma(x)} - 7\,\mathrm{e}^{x-2\,\sigma(x)}x^2 + 7\,\mathrm{e}^{x-\sigma(x)}x^2\\
&+ 2\,\sigma^2\left(x\right)\mathrm{e}^{2\,\sigma(x)-2\,x} - 2\,\sigma^2\left(x\right)\mathrm{e}^{2\,\sigma(x)-x} + 7\,\mathrm{e}^{\sigma(x)-2\,x}\sigma^2\left(x\right)\\
&- 7\,\mathrm{e}^{\sigma(x)-x}\sigma^2\left(x\right)).
\end{aligned}$$

Rest of this section is devoted to proving that $W[\ell_0]$, $W[\ell_0,\ell_1]$ and $W[\ell_0,\ell_1,\ell_2]$ are non-vanishing on $(x_l, 0)$.

Proposition 3.2. *Wronskian $W[\ell_0]$ is nonvanishing on $(-\ln(2),0)$.*

Proof. Recall that $g_0 = \frac{f_0}{A'B^{5/2}} = \frac{2\sqrt{2}f_0}{A'}$. Since $xA'(x) > 0$ and $f_0(x) > 0$ for all $x \in \mathbb{R}$ one concludes that $xg_0(x) > 0$ for all $x \in (-\ln(2),+\infty)\setminus\{0\}$. Taking this into account, since $\sigma(x) > 0$ for all $x \in (-\ln(2),0)$, it turns out that $g_0(\sigma(x)) > 0 > g_0(x)$ for all $x \in (-\ln(2),0)$. Thus, $\mathcal{B}_\sigma(g_0)(x) = g_0(x) - g_0(\sigma(x)) < 0$ for all $x \in (-\ln(2),0)$, completing the proof. □

Proposition 3.3. *Wronskian $W[\ell_0,\ell_1]$ is nonvanishing on $(-\ln(2),0)$.*

Proof. Let $y = e^x$, $z = \sigma(x)$ and $w = e^{\sigma(x)}$. Then straight-forward computations show that

$$W[\ell_0,\ell_1](x) = \frac{2D_1(x,y,z,w)}{225y^2(w-1)^3(y-1)^2}, \tag{3.1}$$

in which D_1 is a polynomial of x, y, z and w with long expression. Also we can write $A(x) = A(\sigma(x))$ as

$$w = \frac{y}{2y-1}, \qquad z = x - \ln(2y-1). \tag{3.2}$$

Taking these relations into account, we obtain

$$D_1(x,y,z,w) = -\frac{16y^7}{(2y-1)^{10}} S_1(y, \ln(2y-1)), \tag{3.3}$$

where $S_1(y,\ln(2y-1)) = a_0(y) + a_1(y)\ln(2y-1)$ and

$$\begin{aligned} a_0(y) =& 2\,y\,(y-1)^4\,(16\,y^8 + 116\,y^7 + 578\,y^6 - 68\,y^5 - 325\,y^4 \\ & - 310\,y^3 + 481\,y^2 - 184\,y + 23), \\ a_1(y) =& 3\,\left(5 - 20\,y + 9\,y^2 + 22\,y^3 + 19\,y^4\right)(2\,y-1)^3\,(y-1)^3. \end{aligned}$$

By Sturm's Theorem $a_0(y)$ and $a_1(y)$ have no real zeros in $(\frac{1}{2},1)$ and $a_0(y) > 0$ and $a_1(y) < 0$ for all $y \in (\frac{1}{2},1)$. Since $\ln(2y-1) < 0$ for all $y \in (\frac{1}{2},1)$, this implies that $S_1(y,\ln(2y-1)) > 0$ for all $y \in (\frac{1}{2},1)$. □

Proposition 3.4. *$W[\ell_0,\ell_1,\ell_2]$ is nonvanishing on $(-\ln(2),0)$.*

Proof. Using the notation y, z and w as defined in the proof of Proposition 3.3 and easy calculations show that

$$W[\ell_0,\ell_1,\ell_2](x) = -\frac{4\sqrt{2}D_2(x,y,z,w)}{3375y^6(w-1)^6(y-1)^3} \tag{3.4}$$

in which D_2 being a polynomial. Taking (3.2) into account, we obtain

$$D_2(x,y,z,w) = -\frac{16y^{13}(y-1)^6}{(2y-1)^{18}} S_2(y,\ln(2y-1)),$$

where $S_2(y, \ln(2y-1)) = \bar{a}_0(y) + \bar{a}_1(y)\ln(2y-1) + \bar{a}_2(y)\ln^2(2y-1)$ and

$$\begin{aligned}\bar{a}_0(y) =& 4\,y^2\,(y-1)\,(512\,y^{14} + 7744\,y^{13} + 87936\,y^{12} + 25040\,y^{11}\\ &- 227632\,y^{10} - 456268\,y^9 + 1307434\,y^8 - 889832\,y^7\\ &+ 66397\,y^6 + 36944\,y^5 + 155304\,y^4 - 151544\,y^3\\ &+ 57704\,y^2 - 10402\,y + 743),\\ \bar{a}_1(y) =& 4\,y\,(2\,y-1)^3\,(5700\,y^{10} + 8210\,y^9 - 11213\,y^8 - 74108\,y^7\\ &+ 80623\,y^6 + 3472\,y^5 - 12824\,y^4 - 16448\,y^3 + 15616\,y^2\\ &- 4520\,y + 452),\\ \bar{a}_2(y) =& 9\,(y-1)\,(2\,y-1)^3\,(608\,y^{10} + 2256\,y^9 + 3880\,y^8 - 808\,y^7\\ &- 2856\,y^6 - 434\,y^5 + 763\,y^4 + 904\,y^3 - 863\,y^2 + 250\,y - 25).\end{aligned}$$

Applying Sturm's Theorem, one concludes that $\bar{a}_0(y) < 0$ and $\bar{a}_2(y) < 0$ for all $y \in (\frac{1}{2}, 1)$. Again Sturm's Theorem implies that $\bar{a}_1(y)$ has one real zero in $(\frac{1}{2}, 1)$. Straight-forward computations show that this root occurs at $\alpha \cong 0.725242989570266$ such that $\bar{a}_1(y) > 0$ for all $y \in (\frac{1}{2}, \alpha)$ and $\bar{a}_1(y) < 0$ for all $y \in (\alpha, 1)$. Therefore,

(i): If $y \in (\frac{1}{2}, \alpha)$, then $S_2(y, \ln(2y-1)) < 0$.

(ii): If $y = \alpha$, then $S_2(\alpha, \ln(2\alpha-1)) = \bar{a}_0(\alpha) + \bar{a}_2(\alpha)\ln^2(2\alpha-1) < 0$.

(iii): If $y \in (\alpha, 1)$, then considering $S_2(y, \ln(2y-1))$ as a quadratic polynomial of $\ln(2y-1)$, one has $\Delta = -16y^2(2y-1)^3 q(y)$ where Δ is the discriminant of S_2 with

$$\begin{aligned}q(y) =& 37129 - 965354\,y + 11165036\,y^2 - 74890064\,y^3 + 317108876\,y^4\\ &- 857859464\,y^5 + 1415617764\,y^6 - 1375973104\,y^7\\ &+ 2084318926\,y^8 - 8731832924\,y^9 + 22589674980\,y^{10}\\ &- 29342465280\,y^{11} + 17122834684\,y^{12} - 18317729096\,y^{13}\\ &+ 83992403612\,y^{14} - 179849986224\,y^{15} + 189613221889\,y^{16}\\ &- 86937594730\,y^{17} - 7935120272\,y^{18} + 13925579120\,y^{19}\\ &+ 10882988320\,y^{20} - 8012034944\,y^{21} - 1816850112\,y^{22}\\ &+ 669421440\,y^{23} + 553559040\,y^{24} + 47167488\,y^{25} + 2801664\,y^{26}.\end{aligned}$$

Sturm's Theorem implies that $q(y)$ has one real zero in $(\alpha, 1)$. Simple computations show that this root occurs at the $\beta \cong 0.846287457125663$ such that $q(y) > 0$ for all $y \in (\alpha, \beta)$ and $q(y) < 0$ for all $y \in (\beta, 1)$. Then

(i): If $y \in (\alpha, \beta)$, then $S_2(y, \ln(2y-1)) < 0$.

(ii): If $y = \beta$, then $\Delta = 0$ and $S_2(y, \ln(2y-1))$ has a double root in $-\frac{\bar{a}_1(y)}{4\bar{a}_2(y)}$. However, the relation $\ln(2y-1) = -\frac{\bar{a}_1(y)}{4\bar{a}_2(y)}$ gives $y \cong 1.372293500$, contradicting with $y = \beta$.

(iii): If $y \in (\beta, 1)$, then $\Delta > 0$ and in this case, we can not achieve any result.

Now, since $\ln(2y-1)$ is increasing and $\ln^2(2y-1)$ is decreasing in $(\beta,1)$, for $\beta \le x_1 < y < x_2 < 1$ we can write

$$S_2(y,\ln(2y-1)) \le \bar{a}_0(y)+\bar{a}_1(y)\ln(2x_1-1)+\bar{a}_2(y)\ln^2(2x_2-1) = Q(y). \quad (3.5)$$

By (3.5), in the interval $(\beta, 0.88)$ we can choose

$$Q_1(y) = \bar{a}_0(y) + \bar{a}_1(y)\ln(2\beta-1) + \bar{a}_2(y)\ln^2(2(0.88)-1),$$

such that by Sturm's Theorem $Q_1(y) < 0$ for all $y \in (\beta, 0.88)$, and hence $S_2(y,\ln(2y-1)) < 0$ for all $y \in (\beta, 0.88)$.

Similarly, in the interval $(0.88, 0.9)$ we can choose

$$Q_2(y) = \bar{a}_0(y) + \bar{a}_1(y)\ln(2(0.88)-1) + \bar{a}_2(y)\ln^2(2(0.9)-1),$$

such that by Sturm's Theorem $Q_2(y) < 0$ for all $y \in (0.88, 0.9)$, and hence $S_2(y,\ln(2y-1)) < 0$ for all $y \in (0.88, 0.9)$.

In the same way, for all $y \in (0.9, 0.91)$ we can let

$$Q_3(y) = \bar{a}_0(y) + \bar{a}_1(y)\ln(2(0.9)-1) + \bar{a}_2(y)\ln^2(2(0.91)-1),$$

such that in this interval by Sturm's Theorem, we have $Q_3(y) < 0$, and then $S_2(y,\ln(2y-1)) < 0$ for all $y \in (0.9, 0.91)$.

Also, for all $y \in (0.91, 0.92)$ we can let

$$Q_4(y) = \bar{a}_0(y) + \bar{a}_1(y)\ln(2(0.91)-1) + \bar{a}_2(y)\ln^2(2(0.92)-1),$$

such that in this interval by Sturm's Theorem, we have $Q_4(y) < 0$, and this leads to $S_2(y,\ln(2y-1)) < 0$ for all $y \in (0.91, 0.92)$.

Similarly, in the interval $(0.92, 0.925)$ we can choose

$$Q_5(y) = \bar{a}_0(y) + \bar{a}_1(y)\ln(2(0.92)-1) + \bar{a}_2(y)\ln^2(2(0.925)-1),$$

such that by Sturm's Theorem $Q_5(y) < 0$ for all $y \in (0.92, 0.925)$, and hence $S_2(y,\ln(2y-1)) < 0$ for all $y \in (0.92, 0.925)$.

Moreover, for all $y \in (0.925, 0.93)$ we can choose

$$Q_6(y) = \bar{a}_0(y) + \bar{a}_1(y)\ln(2(0.925)-1) + \bar{a}_2(y)\ln^2(2(0.93)-1),$$

such that in this interval by Sturm's Theorem, we have $Q_6(y) < 0$, and then $S_2(y,\ln(2y-1)) < 0$ for all $y \in (0.925, 0.93)$.

Finally, in the interval $(0.93, 0.935)$ we can choose

$$Q_7(y) = \bar{a}_0(y) + \bar{a}_1(y)\ln(2(0.93)-1) + \bar{a}_2(y)\ln^2(2(0.935)-1),$$

such that by Sturm's Theorem $Q_7(y) < 0$ for all $y \in (0.93, 0.935)$, and hence $S_2(y,\ln(2y-1)) < 0$ for all $y \in (0.93, 0.935)$.

To determine the sign of $S_2(y,\ln(2y-1))$ in the remaining interval, set $y = 1+\tau$ where $\tau \in (-0.065, 0)$. Due to convergence of the Taylor expansion of S_2 at $\tau = 0$, we consider this Taylor expansion up to order n denoted by $P_n(\tau)$. By Sturm's Theorem, we see that the polynomial $P_n(\tau)$ has no real root in $(-0.065, 0)$ for any given $n \in \mathbb{N}$. It then follows that $S_2(1+\tau,\ln(2\tau+1))$ is negative for all $\tau \in (-0.065, 0)$. □

4. Conclusion

In this paper, we considered a non-polynomial potential system that resulted from the conjecture expressed by Dumortier and Roussarie. By a criterion which greatly simplifies the investigation of the Chebyshev property, we showed that the cyclicity of the period annulus of this system under the small perturbation is at most two.

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