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Homotopy perturbation method for eigenvalues of non-definite Sturm-Liouville problem

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Abstract In this paper, we consider the application of the homotopy perturbation method (HPM) to compute the eigenvalues of the Sturm-Liouville problem (SLP) which is called non-definite SLP. Two important Examples show that HPM is reliable method for computing the eigenvalues of SLP.

Keywords. Turning point, Sturm-Liouville, Homotopy perturbation method, Eigenvalues.

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1. Introduction

We study the indefinite Sturm-Liouville spectral problem

$$y'' + (\lambda\, r(x) - q(x))y = 0, \;\; a \le x \le b, \tag{1.1}$$
$$y(a) = y(b) = 0,$$

defined on the interval $[a, b]$ where λ is a real parameter, $r(x), q(x)$ are real and integrable on $[a, b]$; moreover,

$$\int_a^b \sqrt{r_+(t)}dt > 0, \qquad where\ r_+(x) = max\{r(x), 0\}. \tag{1.2}$$

It follows from [5] that the spectrum of this problem is discrete and has no finite accumulation points; moreover, only finitely many eigenvalues lie the outside the real and imaginary axes. In what follows, we shall assume that λ is a positive parameter. This paper focuses on the Homotopy perturbation analysis (HPM), which has been introduced by He to solve approximately the differential equations [9] and [10]. Among numerical methods, the finite difference methods [3, 4], the variational methods and recently Homotopy perturbation analysis [2] are commonly referred as some traditional and powerful methods for solving classic Sturm-Liouville problem. Of course, many new developments and improvements are often introduced [15, 17]. Scott [16] presented an initial-value method for SLP with non separated boundary conditions. Paine [14], Pryce [15] and Andrew [3] used finite difference scheme and asymptotic

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correction technique to solve classic SLP with Dirichlet boundary conditions. Anderson and Hoog [4] extended the method of Paine to the general separated boundary conditions. Çelik [7] investigated the collocation method for approximating computation of classic SLP eigenvalues by truncated Chebyshev series. Çelik and Gokmer [8] also applied the collocation method for computation of periodic SLP. Yücel [18] applied the polynomial -based differential quadrature (PDQ) and Fourier expansion-based differential quadrature (FDQ) methods have been to compute the eigenvalues of periodic SLP. Chen and Ping Ma [6] proposed the Legendre-Galerkin-Chebyshev collocation method (LGCC) to compute the eigenvalues of SLP with many different boundary conditions. An improvement for Chebyshev collocation method in solving certain SLP is proposed by Yuan and et al [19]. They investigated SLP with two turning points and semi periodic boundary conditions.
At first, we mention the theory of higher order distribution of positive eigenvalues associated with problem (1.1), on the assumption that turning point is of arbitrary order. In particular, where the end points a or b is a zero of the weight function $r(x)$. At the end, we present numerical method both classic and non classic Sturm-Liouville problem. In this paper, we present Homotopy perturbation analysis for approximate computation of eigenvalues of SLP with Dirichlet boundary condition by focusing on a very important special case i.e. $r(x) = x^{\alpha}$ or $r(x) = (x - x_{\nu})^{\alpha}$, in which α must be of odd order given the assumption of non-definiteness.

2. Eigenvalues of SLP:Theory and HPM

The theory of boundary eigenvalue problem (1.1) dates back to the pioneering research of R.G.D. Richardson and O. Haupt (see [13] and the references therein for a brief history and survey). The leading term in the asymptotic expansion of the real eigenvalues was the subject of the Jörgens conjecture dating from 1964, a conjecture that was finally proved and extended in [5]. The thrust of this conjecture is that, once suitably relabeled, the positive λ_n^+ eigenvalues admit the asymptotic estimate

$$\lambda_n^+ \sim \frac{n^2\pi^2}{(\int_a^b \sqrt{r_+(x)}dx)^2},\ n \to \infty.$$

Mingarelli and Jodayree [16, 17] considered the case $r(x) = x^{\alpha}$ on $[a, b]$. They showed following estimation

$$\sqrt{\lambda_n^+} = \frac{n\pi - \frac{\pi}{4}}{\int_0^b x^{\alpha/2}} - \frac{1}{n\pi}\{\frac{4\nu^2 - 1}{8\int_0^b x^{\alpha/2}} - \frac{1}{2}\int_0^b \frac{q(x)}{x^{\alpha/2}}\} + O(\frac{1}{n^2}),$$

where $\nu = \frac{1}{\alpha+2}$.

Also if we assume that $r(a) = 0$ and $r(x) > 0$ on $(a, b]$ then

$$\sqrt{\lambda_n^+} = \frac{n\pi + (\frac{\nu\pi}{2} - \frac{\pi}{4})}{\int_a^b x^{\alpha/2}} - \frac{1}{n\pi}\{\frac{4\nu^2 - 1}{8\int_a^b x^{\alpha/2}} - \frac{1}{2}H(b)\} + O(\frac{1}{n^2}), \tag{2.1}$$

where

$$H(b)=\int_{x_\nu}^{b}\left(\frac{q(x)}{\tilde{r}(x)}-\frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}(\tilde{r}^{-1/4})\right)\frac{\tilde{r}}{r^{\frac{1}{2}}}dx,$$

and

$$\tilde{r}=\left(\frac{d\xi}{dx}\right)^2=\frac{4r(x)}{(\alpha+2)^2(\xi(x))^\alpha},$$

$$\xi(x)=\begin{cases}-\left(\int_x^{x_\nu}(-r(t))^{1/2}dt\right)^{\frac{2}{\alpha+2}}, & x\le x_\nu,\\ \left(\int_{x_\nu}^{x}(r(t))^{1/2}dt\right)^{\frac{2}{\alpha+2}}, & x_\nu\le x.\end{cases}$$

Without loss of generality, we consider problem

$$y''+(\lambda\, r(x)-q(x))y=0,\ 0\le x\le 1, \tag{2.2}$$

$$y(0)=y(1)=0,$$

where $r(x)=x^\alpha$ or $r(x)=(x-x_\nu)^\alpha$.

Since the homotopy perturbation method usually defines the given differential equation in an operator form we will rewrite (1.1) in following form

$$A(y)=L(y)+N(y)=f(x).$$

Here $L=\frac{d^2}{dx^2}$, $N(y)=-(\lambda r(x)-q(x))y$ and $f(x)=0$. Now we construct a homotopy $\upsilon(x,p):\Omega\times[0,1]\to R$ which satisfies

$$H(\upsilon,p)=(1-p)[L(y)-L(y_0)]+p[N(\upsilon)-f(x)]=0,\ p,x\in[0,1],$$

where $p\in[0,1]$ is embedding parameter, y_0 is an initial approximation, which satisfies the boundary conditions. Obviously we have

$$H(\upsilon,0)=L(y)-L(y_0)],$$

$$H(\upsilon,1)=A(y)-f(x).$$

Changing process of p from zero to unity is just that of $\upsilon(x,p)$ from y_0 to $y(x)$. We assume that the solution of equation

$$H(\upsilon,p)=(1-p)[L(y)-L(y_0)]+p[N(\upsilon)-f(x)]=0,\ p,x\in[0,1], \tag{2.3}$$

can be written as a power series in p,$\upsilon=\upsilon_0+p\upsilon_1+p^2\upsilon_2+\dots$.

By setting $p=1$, the approximation solution of $A(y)-f(x)=0$ is obtained.

3. Numerical examples and conclusions

In this section of the paper, for demonstrate the efficiency and accuracy of the HPM method, we give several numerical examples. Numerical results show that the HPM method is effective method for non-definite SLP.

Example 3.1. Consider the boundary value problem

$$y'' + \lambda x^{\alpha} y = 0, \ y(0) = y(1) = 0,$$

Let $L(y) = y''$ and $N(y) = \lambda x^{\alpha} y$. We also assume that

$$Y(x,p) = y_0(x) + py_1(x) + p^2 y_2(x) + p^3 y_3(x) + \dots .$$

By substituting of above in the differential equation and equating the coefficients of p we obtain

coefficients of $p^0 : y_0''(x) = 0, \ y_0(0) = 0,$
coefficients of $p^1 : y_1''(x) + \lambda x^{\alpha} y_0(x) = 0, \ y_1(0) = y_1'(0) = 0,$
coefficients of $p^2 : y_2''(x) + \lambda x^{\alpha} y_1(x) = 0, \ y_2(0) = y_2'(0) = 0,$

$$\vdots$$

If we solve the above equations we get

$$\begin{aligned} y_0(x) &= ax, \\ y_1(x) &= -a\lambda x^{\alpha+3}/(\alpha+2)(\alpha+3), \\ y_2(x) &= a\lambda^2 x^{\alpha+6}/(\alpha+2)(\alpha+3)(\alpha+5)(\alpha+6), \\ y_3(x) &= -a\lambda^3 x^{\alpha+8}/(\alpha+2)(\alpha+3)(\alpha+5)(\alpha+6)(\alpha+8)(\alpha+9), \\ &\vdots \end{aligned}$$

Therefore the solution of the problem is

$$\begin{aligned} y(x,\lambda) = a(x &- \lambda x^{\alpha+3}/(\alpha+2)(\alpha+3) + \lambda^2 x^{\alpha+6}/(\alpha+2)(\alpha+3)(\alpha+5)(\alpha+6) \\ &-\lambda^3 x^{\alpha+8}/(\alpha+2)(\alpha+3)(\alpha+5)(\alpha+6)(\alpha+8)(\alpha+9) + \cdots). \end{aligned}$$

If we consider special case by choosing $a = \lambda^{5/6}$ and $\alpha = 1$ then we have

$$y(x) = x^{1/2} J_{1/3}(2/3\lambda^{1/2} x^{3/2}).$$

To satisfy the other boundary condition we have $y(1) = 0$, which implies the eigenvalues are the roots of $J_{1/3}(2/3\lambda^{1/2}) = 0$. On the other hand from [1] one can see that the roots of $J_{\nu}(z)$ are

$$j_m \sim \beta - \frac{\alpha - 1}{8\beta} - \frac{4(\alpha-1)(7\alpha-31)}{3(8\beta)^3} - \dots ,$$

where

$$\beta = (m + \nu/2 - 1/4)\pi, \ \alpha = 4\nu^2.$$

By inserting $z = 2/3\lambda^{1/2}x^{3/2}$ and $\nu = 1/3$ we get

$$\sqrt{\tilde{\lambda_n}} = \frac{3}{2}(n\pi - \frac{\pi}{12}) + \frac{5}{72(n\pi - \frac{\pi}{12})} + O(\frac{1}{n^3}).$$

On the other hand, by relation (2.1), the asymptotic distribution of eigenvalues satisfies

$$\sqrt{\lambda_n} = \frac{3}{2}(n\pi - \frac{\pi}{12}) + \frac{5}{48n\pi} + O(\frac{1}{n^2}).$$

From above we conclude that error of approximation satisfies

$$|\sqrt{\lambda_n} - \sqrt{\tilde{\lambda_n}}| = \frac{5}{48n\pi} - \frac{5}{72(n\pi - \frac{\pi}{12})} + O(\frac{1}{n^2}).$$

Table 1 shows comparison the eigenvalues for HPM method and asymptotic distribution of eigenvalues.

Table 1. Approximate solutions and error in Example 3.1.

k	λ_k	$\lambda_k^{HPM(19)}$	$\lambda_k^{HPM(39)}$	$\lambda_k^{HPM(49)}$	$\frac{\lvert\lambda_k - \lambda_k^{HPM(49)}\rvert}{\lambda_k}$
1	18.94736582	18.95626559	18.95626559	18.95626559	0.0004697
2	81.87858442	81.88658338	81.88658338	81.88658338	0.0000977
3	189.2152218	189.2209333	189.2209333	189.2209333	0.0000302
4	340.9632159	340.9678986	340.9669591	340.9669591	0.0000012
5	537.1237981	528.4613037	537.1257454	537.1257454	0.0000011
6	777.6973851		777.6975694	777.6975694	0.0000000
7	1062.684157		1062.682527	1062.682527	0.0000000
8	1392.084204		1392.080661	1392.080659	0.0000000
9	1765.897577		1765.851673	1765.891983	0.0000000
10	2184.124306			2184.116511	0.0000003
11	2646.764411			2646.752214	0.0000004
12	3153.817905			3174.832813	0.0006663

Example 3.2. Consider the following SLP

$$y'' + (\lambda x^{\alpha} - x^{\beta})y = 0, \ y(0) = y(1) = 0,$$

where $\alpha > 0$ and $\beta > \frac{\alpha}{2}$. Note that the weight function vanishes at the left endpoint. By (2.1), we have

$$\sqrt{\lambda_n} = k(n\pi + \frac{\nu\pi}{2} - \frac{\pi}{4}) - \frac{1}{n\pi}\{\frac{k(4\nu^2-1)}{8} - \frac{1}{2(\beta - \nu + 2)}\} + O(\frac{1}{n^2}).$$

In the special case, $\alpha = 1, \ \beta = 2$, the results are given in the Table 2. The relative error shows that position is better for large eigenvalues.

$$y'' + (\lambda x - x^2)y = 0, \ y(0) = y(1) = 0, \tag{3.1}$$

We also assume that

$$Y(x,p) = y_0(x) + py_1(x) + p^2y_2(x) + p^3y_3(x) + \dots.$$

By substituting of above in (3.1) and equating the coefficients of p we obtain

coefficients of $p^0 : y_0''(x) = 0,\ y_0(0) = 0,$
coefficients of $p^1 : y_1''(x) + (\lambda x - x^2)y_0(x) = 0,\ y_1(0) = y_1'(0) = 0,$
coefficients of $p^2 : y_2''(x) + (\lambda x - x^2)y_1(x) = 0,\ y_2(0) = y_2'(0) = 0,$

$$\vdots$$

If we solve the above equations we get

$$y_0(x) = ax,$$

$$y_1(x) = ax^5/4.5 - a\lambda x^4/3.4,$$

$$y_2(x) = ax^9/4.5.8.9 + a\lambda^2x^7/3.4.6.7 - 5a\lambda x^8/60.42,$$

$$\vdots$$

Therefore the solution of the problem is

$$y(x,\lambda) = a(x + x^5/4.5 - \lambda x^4/3.4 + x^9/4.5.8.9 + \lambda^2x^7/3.4.6.7 - 5\lambda x^8/60.42 + \cdots).$$

Table 2 shows comparison the eigenvalues for HPM and asymptotic distribution of eigenvalues.

Table 2. Approximate solutions and error in Example 3.2.

k	λ_k	$\lambda_k^{HPM(19)}$	$\lambda_k^{HPM(39)}$	$\lambda_k^{HPM(49)}$	$\frac{\lvert\lambda_k-\lambda_k^{HPM(49)}\rvert}{\lambda_k}$
1	18.94736582	19.55584713	19.55584713	18.95626559	0.0004697
2	81.87858442	82.48671257	82.48671257	81.88658338	0.0000977
3	189.2152218	189.8210080	189.8210080	189.2209333	0.0000302
4	340.9632159	341.5679438	341.5670043	340.9669591	0.0000012
5	537.1237981	529.0613346	537.7257753	537.1257454	0.0000011
6	777.6973851		778.2975905	777.6975694	0.0000000
7	1062.684157		1063.282543	1062.682527	0.0000000
8	1392.084204		1392.680673	1392.080659	0.0000000
9	1765.897577		1766.451682	1765.891983	0.0000000
10	2184.124306			2184.116511	0.0000003
11	2646.764411			2646.752214	0.0000004
12	3153.817905			3174.832813	0.0006663

4. Conclusions

In this paper, we investigate the HPM for approximation of eigenvalues of non-definite SLP with Dirichlet boundary conditions. One of the main advantage of this method is that the approximate trivial solution ($y_0(x)$) will spontaneously be satisfy in Dirichlet boundary conditions. The numerical examples showed that the HPM is efficient and considerable.

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