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A numerical approach for variable-order fractional unified chaotic systems with time-delay

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Abstract This paper proposes a new computational scheme for approximating variable-order fractional integral operators by means of finite element scheme. This strategy is extended to approximate the solution of a class of variable-order fractional nonlinear systems with time-delay. Numerical simulations are analyzed in the perspective of the mean absolute error and experimental convergence order. To illustrate the effectiveness of the proposed scheme, dynamical behaviors of the variable-order fractional unified chaotic systems with time-delay are investigated in the time domain.

Keywords. Variable-order fractional calculus, Finite element scheme, Unified chaotic systems.

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1. Introduction

The growth of practical applications in the use of fractional calculus has attracted the attention of many engineers and researchers. The advantages of using fractional operators have been shown in the literature [5, 7, 25, 27, 28, 33]. For instance, it has been shown that the behavior of viscoelastic materials can be correctly described by a fractional model with a small number of model parameters compared to using conventionals integer-order models with a large number of model parameters [9, 11, 16]. In addition, it has been shown that the fractional-order controllers are more sophisticated than regular integer-order controllers due to employing more control parameters [12].

In the past two decades, many scientists have been aware of the potential use of

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chaotic dynamics in engineering applications, such as electrical engineering, information processing, secure communications. In [15], the existence of chaos in fractional-order systems is studied. Following this idea, many other fractional order chaotic systems have been introduced, such as fractional order Chua [40], Lorenz [43], Chen [24], financial [38], Qi [52], Rabinovich-Fabrikant [45], Rössler [26], a chaotic fractional damped-driven pendulum [8] and Liu [13] systems. One of the most important chaotic systems is the fractional unified chaotic system. Unified chaotic system is a chaotic system which changes between Lorenz, Lü and Chen chaotic systems family with parameter changes [53].

Fractional delay chaotic systems as a generalization of the delay chaotic systems belong to a wider class of the fractional functional systems that the derivative of the unknown function at a certain time is expressed in terms of the value of the function at the previous times. Moreover, the next state of the systems depends not only upon their present state but also upon all of their past states. Recently, variable-order (VO) fractional calculus has been used to describe hereditary processes in a more efficient way in comparison to fixed order (FO) fractional calculus. Furthermore, the advantages of using VO fractional ordinary and partial differential equations compared to FO fractional calculus have been revealed in different applications such as modeling of many mechanical, electrical, physiological and hormonal control systems [6, 14, 19, 22, 31, 32, 34, 39, 46]. Moreover, new numerical techniques for the FO and VO fractional dynamic systems with time-delay have become a strong topic to be considered [1, 2, 3, 4, 10, 21, 30, 35, 37, 48, 49].

The paper is organized as follows. In Section 2, a brief review of the VO fractional calculus including the different types of VO fractional operators is presented. In Section 3, we design an efficient approach for the VO fractional integral based on finite element scheme. In Section 4, we apply the proposed scheme to the VO fractional unified chaotic systems with time-delay. In Section 5, we outline the main conclusions.

2. Preliminaries and Problem Statement

We assume that the function $y(t)$ is $\kappa-1$ times continuously differentiable and that $y^{(\kappa)}(t)$ is once integrable, where $t \in \mathbb{R}^+$ and κ is an arbitrary positive integer. Let us consider $0 \leq \kappa - 1 < \alpha(t) \leq \kappa$. We consider the following definitions.

Definition 2.1 ([44])**.** Let $y(t)$ be a piecewise continuous function in $(0,\infty)$ and integrable in any finite subinterval of $[0,\infty)$. For any $t>0$ and any complex function $\alpha(\cdot,\cdot)$ with $\mathrm{Re}\{\alpha(\cdot,\cdot)\}>0$, the left-sided VO fractional Riemann-Liouville integral of $y(t)$ is defined by

$$ {}^{\nu(\cdot)}_{0+}\mathcal{J}_t^{\alpha(\cdot,\cdot)} y(t) = \int_{0+}^{t} \frac{1}{\Gamma(\alpha(t,\zeta))}(t-\zeta)^{\alpha(t,\zeta)-1} y(\zeta)d\zeta, \tag{2.1} $$

where $\zeta > 0$ and $\Gamma(.)$ denotes the Gamma function.

Definition 2.2 ([44])**.** The κ-th derivative of (2.1) defines the left-sided VO fractional derivative of $y(t)$, given by

$$ {}^{\nu(\cdot)}_{0+}\mathscr{D}_t^{\alpha(\cdot,\cdot)} y(t) = \int_{0+}^{t} \frac{(t-\zeta)^{\kappa-\alpha(t,\zeta)-1}}{\Gamma(\kappa-\alpha(t,\zeta))} y^{(\kappa)}(\zeta)d\zeta, \tag{2.2} $$

where $0 \leq \kappa - 1 < \alpha(\cdot,\cdot) \leq \kappa$ and $\zeta > 0$.

There are two main types of VO fractional derivative formulations that emerge from various choices of the arguments of $\alpha(t,\zeta)$ as follows in the two definitions.

Definition 2.3. The VO fractional derivative type 1 denoted by $(\nu 1)$ is defined by

$$ {}^{\nu 1}_{0+}\mathscr{D}^{\alpha(\cdot)}_t y(t) = \frac{1}{\Gamma(\kappa - \alpha(t))} \int_{0+}^{t} \frac{y^{(\kappa)}(\zeta)\mathrm{d}\zeta}{(t-\zeta)^{\alpha(t)+1-\kappa}}, \qquad \kappa - 1 < \alpha(t) \leq \kappa. \quad (2.3) $$

Definition 2.4. The VO fractional derivative type 2 denoted by $(\nu 2)$ is defined by

$$ {}^{\nu 2}_{0+}\mathscr{D}^{\alpha(\cdot)}_t y(t) = \int_{0+}^{t} \frac{1}{\Gamma(\kappa - \alpha(\zeta))} \frac{y^{(\kappa)}(\zeta)\mathrm{d}\zeta}{(t-\zeta)^{\alpha(\zeta)+1-\kappa}}, \qquad \kappa - 1 < \alpha(t) \leq \kappa. \quad (2.4) $$

The underlying concepts, the characteristics of these operators and their application are discussed in [29, 47].

3. Discretization of the VO fractional integrals

Thought the paper, we always assume $y(t)$ a smooth function defined on $[a,T]$, along with the notations $t_j = a + jh$, $y(t_j) = y_j$ and $\alpha(t_j) = \alpha_j$, $j = 0, 1, \ldots, \left[\frac{T-a}{h}\right]$, where h denotes the uniform step size and $[x]$ takes the integer part of x, being the maximum integer that does not exceed x.
In this section, we present discretization of the VO fractional integral. Then, we apply proposed scheme to solve a class of VO fractional differential equation with time-delay.

3.1. **Discretization of the VO Fractional Integral.** For the time instant t_j, $j = 1, ..., N-1$, we need to calculate

$$ {}^{\nu(\cdot)}_{a+}\mathcal{J}^{\alpha(\cdot,\cdot)}_{t_j} y(t) = \int_{a+}^{t_j} \frac{1}{\Gamma(\alpha(\cdot,\cdot))} (t_j - \zeta)^{\alpha(\cdot,\cdot)-1} y(\zeta) d\zeta, \quad (3.1) $$

where ζ is an auxiliary variable belong to the interval $[a, t_j]$.
If $\alpha(t,\zeta) = \alpha(t)$, then

$$ \begin{aligned} {}^{\nu 1}_{a+}\mathcal{J}^{\alpha(\cdot)}_{t_j} y(t) &= \int_{a+}^{t_j} \frac{1}{\Gamma(\alpha_j)} (t_j - \zeta)^{\alpha_j - 1} y(\zeta) d\zeta \\ &= \frac{1}{\Gamma(\alpha_j)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t_k - \zeta)^{\alpha_j - 1} y(\zeta) d\zeta. \end{aligned} $$

Alternatively, taking compound trapezoidal quadrature will yield

$$ \begin{aligned} {}^{\nu 1}_{a+}\mathcal{J}^{\alpha(\cdot)}_{t_j} y(t) &\approx \frac{1}{\Gamma(\alpha_j)} \sum_{k=0}^{j-1} \frac{y_k + y_{k+1}}{2} \int_{t_k}^{t_{k+1}} (t_k - \zeta)^{\alpha_j - 1} d\zeta \\ &= \sum_{k=0}^{j-1} \frac{y_k + y_{k+1}}{2\Gamma(\alpha_j + 1)} \left((t_{j+1} - t_k)^{\alpha_j} - (t_{j+1} - t_{k+1})^{\alpha_j} \right). \end{aligned} $$

Therefore

$$ {}^{\nu 1}_{0+}\mathcal{J}^{\alpha(\cdot)}_{t_j} y(t) \approx \sum_{k=0}^{j-1} \frac{y_k + y_{k+1}}{2\Gamma(\alpha_j+1)} \left((t_{j+1}-t_k)^{\alpha_j} - (t_{j+1}-t_{k+1})^{\alpha_j} \right). \tag{3.2} $$

Moreover, if $\alpha(t,\zeta)=\alpha(\zeta)$, then

$$ \begin{aligned} {}^{\nu 2}_{a+}\mathcal{J}^{\alpha(\cdot)}_{t_j} y(t) &= \int_{a+}^{t_j} \frac{1}{\Gamma(\alpha(\zeta))}(t_j-\zeta)^{\alpha(\zeta)-1} y(\zeta) d\zeta \\ &= \sum_{k=0}^{j-1} \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t_{k+1}} (t_k-\zeta)^{\alpha_k -1} y(\zeta) d\zeta. \end{aligned} $$

Alternatively, taking compound trapezoidal quadrature will yield

$$ \begin{aligned} {}^{\nu 2}_{a+}\mathcal{J}^{\alpha(\cdot)}_{t_j} y(t) &\approx \sum_{k=0}^{j-1} \frac{y_k + y_{k+1}}{2\Gamma(\alpha_k)} \int_{t_k}^{t_{k+1}} (t_k-\zeta)^{\alpha_k -1} d\zeta \\ &= \sum_{k=0}^{j-1} \frac{y_k + y_{k+1}}{2\Gamma(\alpha_k+1)} \left((t_{j+1}-t_k)^{\alpha_k} - (t_{j+1}-t_{k+1})^{\alpha_k} \right), \end{aligned} $$

hence

$$ {}^{\nu 2}_{0+}\mathcal{J}^{\alpha(\cdot)}_{t_j} y(t) \approx \sum_{k=0}^{j-1} \frac{y_k + y_{k+1}}{2\Gamma(\alpha_k+1)} \left((t_{j+1}-t_k)^{\alpha_k} - (t_{j+1}-t_{k+1})^{\alpha_k} \right). \tag{3.3} $$

3.2. **Indirect algorithm for VO fractional differential equation with time-delay.** In this subsection, we consider the following initial value problem (IVP) for VO fractional differential equation with time-delay:

$$ \begin{cases} {}^{\nu(\cdot)}_{0+}\mathscr{D}^{\alpha(t)}_{t} y(t) = f(t, y(t), y(t-\tau)), & t>0, \\ y(t) = \phi(t), & -\tau \le t \le 0, \end{cases} \tag{3.4} $$

where $\kappa - 1 < \alpha(t) \le \kappa$, τ is the delay time and $\phi(t)$ is the history function defined on the interval $t \in [-\tau, 0]$. The existence and uniqueness of solution for Eq. (3.4) investigated by Parsa et al. [30].

It is easily noted that Eq. (3.4) can be transformed to Abel-Volterra integral equation

$$ y(t) = T_{\kappa-1}[y;0](t) + {}^{\nu(\cdot)}_{0+}\mathcal{J}^{\alpha(\cdot,\cdot)}_{t} f(t, y(t), y(t-\tau)), \tag{3.5} $$

where

$$ T_{\kappa-1}[y;0](t) = \sum_{j=0}^{\kappa-1} \phi^{(j)}(0)\frac{t^j}{j!}, $$

and

$$ {}^{\nu(\cdot)}_{0+}\mathcal{J}^{\alpha(\cdot,\cdot)}_{t} f(t, y(t), y(t-\tau)) = \int_{0+}^{t} \frac{1}{\Gamma(\alpha(\cdot,\cdot))}(t-\zeta)^{\alpha(\cdot,\cdot)-1} f(\zeta, y(\zeta), y(\zeta-\tau)) \mathrm{d}\zeta. $$

For solving (3.5) on $[0,T]$, the interval $[-\tau, T]$ is divided to $m+n$ subintervals, where m and n are integers such that $m = \frac{\tau}{h}$, $n = \frac{T}{h}$ and $t_j = jh$, $j = -m, -m+$

$1, \ldots, -1, 0, 1, , \ldots, n$.
The discretized version of (3.5) is given as

$$y_n = T_{\kappa-1}[y;0](t_n) + \int_{0+}^{t_n} \frac{1}{\Gamma(\alpha(\cdot,\cdot))}(t-\zeta)^{\alpha(\cdot,\cdot))-1} f(\zeta, y(\zeta), y(\zeta-\tau))\mathrm{d}\zeta. \quad (3.6)$$

Applying the approximation formulas, Eq. (3.2) and Eq. (3.3), to the above equation yields, respectively

$$y_n = T_{\kappa-1}[y;0](t_n) + \sum_{j=0}^{n-1} \frac{f(t_j, y_j, y_{j-m}) + f(t_{j+1}, y_{j+1}, y_{j+1-m})}{2\Gamma(\alpha_n+1)} \left((t_{n+1}-t_j)^{\alpha_n} - (t_{n+1}-t_{j+1})^{\alpha_n}\right), \quad (3.7)$$

and

$$y_n = T_{\kappa-1}[y;0](t_n) + \sum_{j=0}^{n-1} \frac{f(t_j, y_j, y_{j-m}) + f(t_{j+1}, y_{j+1}, y_{j+1-m})}{2\Gamma(\alpha_j+1)} \left((t_{n+1}-t_j)^{\alpha_j} - (t_{n+1}-t_{j+1})^{\alpha_j}\right). \quad (3.8)$$

For these schemes, since both side of Eqs. (3.7) and (3.8) include the unknown variable y_n, and due to the non-linearity of the function f, it is often difficult to derive y_n. Therefore, to achieve a better approximate solution, we substitute a predicted value y_n into the righthand side of (3.4).
Let y_n^p be the predicted value. We use VO fractional Adams-Bashforth scheme [30] to drive the predicted value

$$y_n^p = \sum_{j=0}^{\kappa-1} \phi^{(j)}(0)\frac{t_n^j}{j!} + \sum_{j=0}^{n} c_{j,n} f(t_j, y_j, y_{j-m}), \quad (3.9)$$

where

$$c_{j,n} = \frac{h^{\alpha_j}}{\Gamma(\alpha_j+1)}((n-j)^{\alpha_j} - (n-j-1)^{\alpha_j}), \qquad 0 \le j \le n.$$

Ultimately, by replacing y_n^p in the righthand side of (3.4) by (3.5) gives

$$\begin{aligned} y_n = {} & \sum_{j=0}^{\kappa-1} \phi^{(j)}(0)\frac{t_n^j}{j!} \\ & + \frac{f(t_{n-1}, y_{n-1}, y_{n-1-m}) + f(t_n, y_n^p, y_{n-m})}{2\Gamma(\alpha_n+1)} \left((t_{n+1}-t_{n-1})^{\alpha_n} - (t_{n+1}-t_n)^{\alpha_n}\right) \\ & + \sum_{j=0}^{n-2} \frac{f(t_j, y_j, y_{j-m}) + f(t_{j+1}, y_{j+1}, y_{j+1-m})}{2\Gamma(\alpha_n+1)} \left((t_{n+1}-t_j)^{\alpha_n} - (t_{n+1}-t_{j+1})^{\alpha_n}\right), \end{aligned} \quad (3.10)$$

and

$$y_n = \sum_{j=0}^{\kappa-1} \phi^{(j)}(0)\frac{t_n^j}{j!}$$
$$+ \frac{f(t_{n-1}, y_{n-1}, y_{n-1-m}) + f(t_n, y_n^p, y_{n-m})}{2\Gamma(\alpha_{n-1}+1)} \left((t_{n+1}-t_{n-1})^{\alpha_{n-1}} - (t_{n+1}-t_n)^{\alpha_{n-1}}\right)$$
$$+ \sum_{j=0}^{n-2} \frac{f(t_j, y_j, y_{j-m}) + f(t_{j+1}, y_{j+1}, y_{j+1-m})}{2\Gamma(\alpha_j+1)} \left((t_{n+1}-t_j)^{\alpha_j} - (t_{n+1}-t_{j+1})^{\alpha_j}\right). \quad (3.11)$$

4. Numerical illustrative examples

In this section, the computational efficiency and the accuracy of the proposed scheme are analyzed in the view point of the mean absolute error (MAE) and the experimental convergence order (ECO) defined as

$$MAE = \frac{1}{N}\sum_{i=1}^{N} |y_i^N - y_{2i}^{2N}|, \quad (4.1)$$

and

$$ECO = \log_h(MAE), \quad (4.2)$$

where y_i^N and y_{2i}^{2N} are the approximate values of $y(t_i)$, N represents the number of interior mesh points and h denotes the uniform step size. The numeric computations are performed using Maple v18 and the results generated with a desktop PC including one AMD Athlon 64 X2 Dual Core Processor 5200+@2.6 GHz. For measuring the computational cost, we compute the CPU time with units in seconds.

Example 4.1. Let $y(t) = J_0(t)$ be a Bessel function of the first kind. Then we get:

$$^{\nu 1}_{0+}\mathcal{J}_t^{\alpha(t)} y(t) = \frac{t^{\alpha(t)}{}_1F_2([\frac{1}{2}],[1+\frac{1}{2}\alpha(t), \frac{1}{2}+\frac{1}{2}\alpha(t)]; -\frac{1}{4}t^2)}{\Gamma(\alpha(t)+1)}, \quad Re(\alpha(t)) \geq 0, \quad (4.3)$$

where ${}_pF_q(a_1,\ldots,a_p; b_1,\ldots,b_q; t)$ denotes the generalized hypergeometric function.

In Example 4.1, it is assumed $T = 1$. The performance of proposed algorithms are studied by integration of $y(t) = J_0(t)$ which is shown in Tables 1 and 2. The results show that reducing h increases the accuracy but increases CPU time which was expected. The expressions of the FO fractional integrals of $y(t)$ with $\alpha_k = 3 - t_k$, $t_k = 0.1k$, $k = 0,\ldots,10$, and the $\nu 1$- and $\nu 2$-type fractional integrals $y(t)$ with $\alpha(t) = 3 - t$. The expression of $\nu 1$-type fractional integral can be constructed by the expressions of the FO fractional integral for different values of α_k.
Fig. 1 shows that the expression of $\nu 1$-type fractional integral at time t_k intersects the equivalent expression given by FO fractional integral when $\alpha_k = 3 - t_k$.

Figure 1. Comparison of the exact expression (23) with $\alpha_k = 3 - 0.1t_k$, $k = 0, \ldots, 10$, and the numerical approximations of the $\nu1$- and $\nu2$-types of VO fractional integrals of $y(t) = J_0(t)$, with proposed algorithms, for $\alpha(t) = 2 - t$ and step size $h = 0.01$.

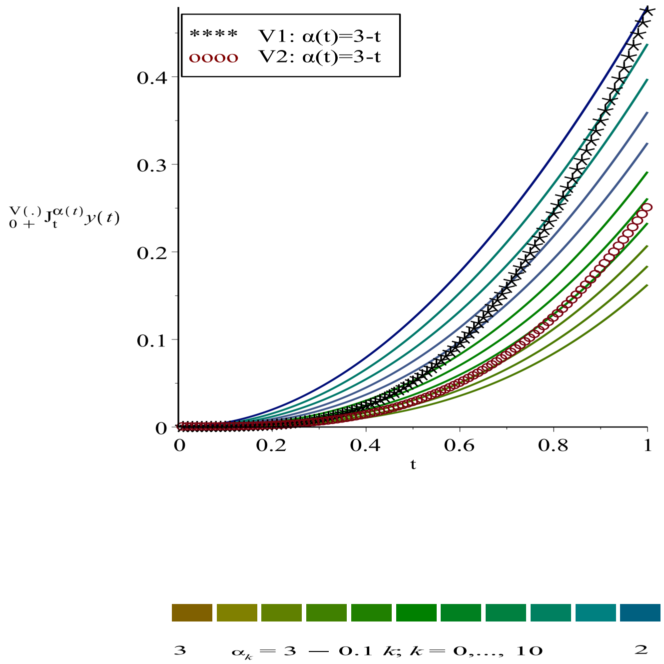


Table 1. Example 4.1: The MAE†, ECO† and CPU time (sec) for the VO ($\nu1 - type$) fractional integral of the function $y(t) = J_0(t)$ by using the proposed algorithm, Eq.(6), for several values of $\alpha(t)$ and different step sizes h in $[0, 1]$.

	$\alpha(t) = 3 - 0.5t$			$\alpha(t) = 4 - 0.25\cos(\pi t)$		
Step size	*MAE*†	*ECO*†	*CPU time*	*MAE*†	*ECO*†	*CPU time*
$h = 0.01$	4.94×10^{-4}	1.653	1.342	4.29×10^{-5}	2.184	1.544
$h = 0.005$	2.45×10^{-4}	1.569	5.054	2.12×10^{-5}	2.031	5.195
$h = 0.0025$	1.22×10^{-4}	1.504	20.233	1.05×10^{-5}	1.913	20.872

4.1. **Application of the proposed algorithms for VO fractional unified chaotic systems with time-delay.** Recently, the Lorenz chaotic system has been extended to the unified chaotic systems which includes the Chen and Lü systems [50]. These

Table 2. Example 4.1: The MAE, ECO and CPU time (sec) for the VO ($\nu 2 - type$) fractional integral of the function $y(t) = J_0(t)$ by using the proposed algorithm , Eq.(7), for several values of $\alpha(t)$ and different step sizes h in $[0, 1]$.

	$\alpha(t) = 3 - 0.5t$			$\alpha(t) = 4 - 0.25\cos(\pi t)$		
Step size	*MAE*	*ECO*	*CPU time*	*MAE*	*ECO*	*CPU time*
$h = 0.01$	1.90×10^{-4}	1.861	1.669	3.24×10^{-5}	2.245	1.872
$h = 0.005$	9.40×10^{-5}	1.751	6.521	1.63×10^{-5}	2.082	7.270
$h = 0.0025$	4.67×10^{-5}	1.664	32.011	8.16×10^{-6}	1.956	32.058

three chaotic systems, corresponding to different sets of system parameter values, are topologically different. Moreover, these systems are described by the following differential equations:

$$\begin{cases} {}_{0+}^{\nu(\cdot)}\mathscr{D}_t^{\alpha(t)} x(t) = (25\lambda + 10)(y(t) - x(t - \tau_x)), \\ {}_{0+}^{\nu(\cdot)}\mathscr{D}_t^{\beta(t)} y(t) = (28 - 35\lambda)x(t) - x(t)z(t) + (29\lambda - 1)y(t - \tau_y), \\ {}_{0+}^{\nu(\cdot)}\mathscr{D}_t^{\gamma(t)} z(t) = x(t)y(t) - \frac{(8+\lambda)}{3} z(t - \tau_z), \end{cases} \tag{4.4}$$

with initial conditions

$$x(t_1) = x_0, \qquad y(t_2) = y_0, \qquad z(t_3) = z_0,$$

where $t_1 \in [-\tau_x, 0]$, $t_2 \in [-\tau_y, 0]$, $t_3 \in [-\tau_z, 0]$, $0 < \alpha(t), \beta(t), \gamma(t) \leq 1$, $\lambda \in [0, 1]$ is a parameter, and $\tau_x, \tau_y, \tau_z \geq 0$ represent the delay terms for the x, y and z variables, respectively. When $\lambda = 0$, it is reduced to the VO fractional Lorenz system with time-delay; while when $\lambda = 1$, it becomes the VO fractional Chen system with time-delay, and when $\lambda = 0.8$, it is the VO fractional Lü system with time-delay. Further, when $0 < \lambda < 0.8$, system (4.4) is classified as generalized VO fractional Lorenz system with time-delay, and when $0.8 < \lambda < 1$, it is called generalized VO fractional Chen system with time-delay.

A comparison of the three chaotic attractors shows a significant difference between the VO fractional Lorenz, Chen and Lü systems with time-delay. It can be seen the chaotic systems can be transformed into limit cycles or stable orbits with suitable choice of delay and variable-order parameters.

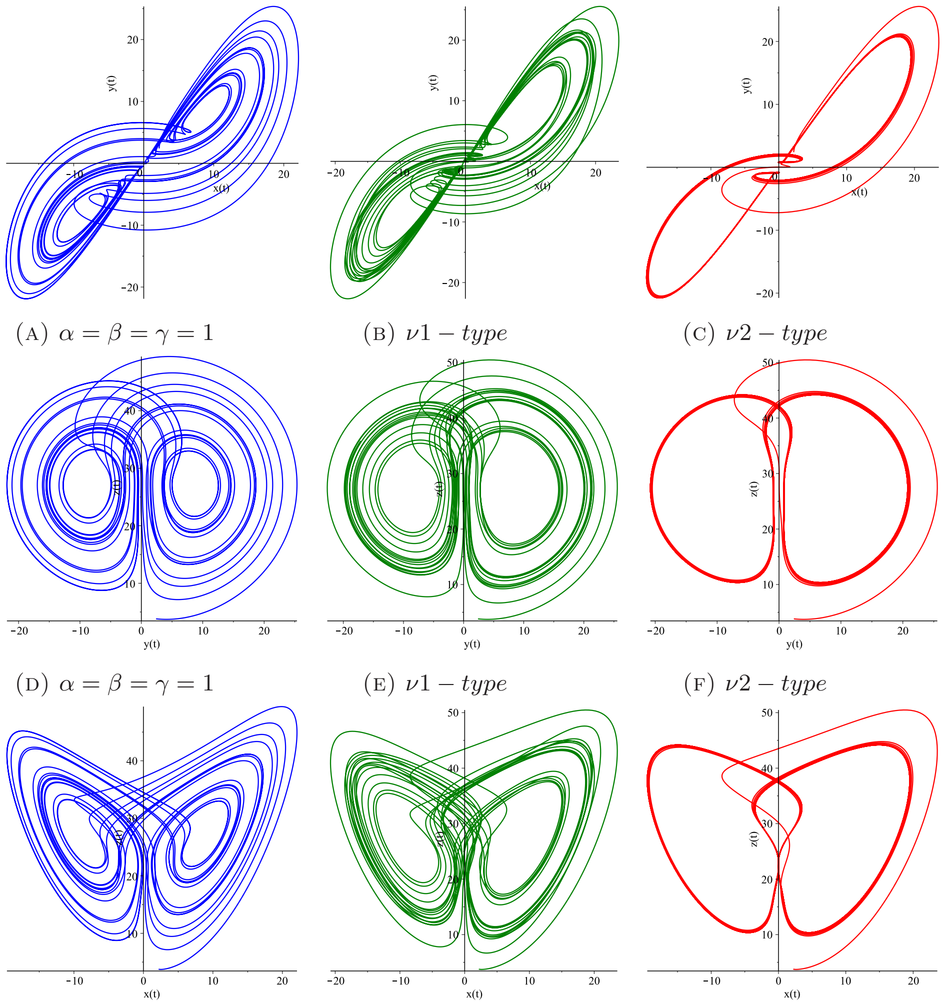
Model 4.1. The Lorenz oscillator is a three-dimensional dynamical system that exhibits chaotic flow.The FO Lorenz system is described in [41, 54]. In this subsection, we consider the following VO fractional Lorenz system with time-delay as

$$\begin{cases} {}_{0+}^{\nu(\cdot)}\mathscr{D}_t^{\alpha(t)} x(t) = 10(y(t) - x(t - \tau_x)), \\ {}_{0+}^{\nu(\cdot)}\mathscr{D}_t^{\beta(t)} y(t) = 28x(t) - x(t)z(t) - y(t - \tau_y), \\ {}_{0+}^{\nu(\cdot)}\mathscr{D}_t^{\gamma(t)} z(t) = x(t)y(t) - \frac{8}{3} z(t - \tau_z), \end{cases} \tag{4.5}$$

with initial conditions

$$x(t_1) = 2.2, \qquad y(t_2) = 2.4, \qquad z(t_3) = 3.8,$$

FIGURE 2. Phase portrait of VO fractional Lorenz system with time-delay, with $T = 20$, $\tau_x = \tau_y = \tau_z = 0.05$, $\alpha(t) = 0.9 + 0.005t$, $\beta = \gamma = 1$ and step size $h = 0.004$.



(A) $\alpha = \beta = \gamma = 1$ (B) $\nu 1 - type$ (C) $\nu 2 - type$

(D) $\alpha = \beta = \gamma = 1$ (E) $\nu 1 - type$ (F) $\nu 2 - type$

(G) $\alpha = \beta = \gamma = 1$ (H) $\nu 1 - type$ (I) $\nu 2 - type$

where $t_1 \in [-\tau_x, 0]$, $t_2 \in [-\tau_y, 0]$, $t_3 \in [-\tau_z, 0]$ and $0 < \alpha(t), \beta(t), \gamma(t) \leq 1$.

In Figure 2, we depict the phase solutions of the Eq. (4.5) using proposed scheme for showing dynamical behavior of system with $\tau_x = \tau_y = \tau_z = 0.05$, $T = 20$ and step size $h = 0.004$ for $\alpha(t) = 0.9 + 0.005t$ and $\beta = \gamma = 1$.

Model 4.2. The Chen's system is another a simple three-dimensional autonomous system, that is not topologically equivalent to Lorenzs system and which has a chaotic

attractor as well [23, 51]. The FO Chen's system is described in [17, 36].
We Consider the following VO fractional Chen system with time-delay as

$$\begin{cases} {}^{\nu(\cdot)}_{0+}\mathscr{D}_t^{\alpha(t)}x(t) = 35(y(t) - x(t-\tau_x)), \\ {}^{\nu(\cdot)}_{0+}\mathscr{D}_t^{\beta(t)}y(t) = -7x(t) - x(t)z(t) + 28y(t-\tau_y), \\ {}^{\nu(\cdot)}_{0+}\mathscr{D}_t^{\gamma(t)}z(t) = x(t)y(t) - 3z(t-\tau_z), \end{cases} \tag{4.6}$$

initial conditions are same as Model 4.1.
In model 4.2, we assume that $\alpha(t) = 1$, $\beta(t) = 0.9 + 0.005t$, $\gamma(t) = 0.8 + 0.01t$ and $T = 20$. Furthermore, we let $\tau_x = 0.005$, $\tau_y = 0.025$ and $\tau_z = 0.25$. The phase portraits of the numerical solutions of (4.6) using proposed schemes are shown in Figure 3.

Model 4.3. The Lü system is known as a bridge between the Lorenz and Chen systems. Its fractional version is studied in [18, 20, 42]. We Consider the following VO fractional Lü system with time-delay as

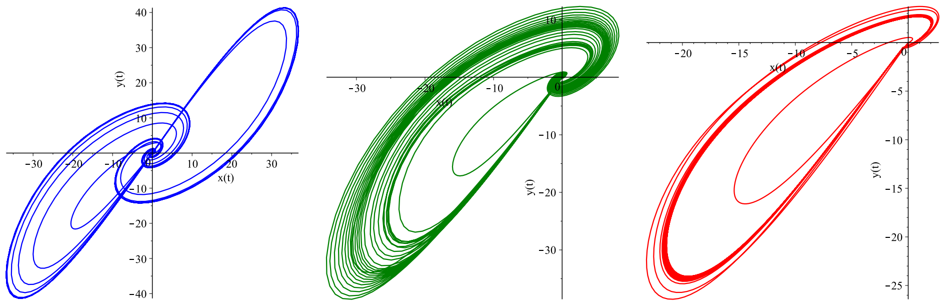
$$\begin{cases} {}^{\nu(\cdot)}_{0+}\mathscr{D}_t^{\alpha(t)}x(t) = 30(y(t) - x(t-\tau_x)), \\ {}^{\nu(\cdot)}_{0+}\mathscr{D}_t^{\beta(t)}y(t) = -x(t)z(t) + 22.2y(t-\tau_y), \\ {}^{\nu(\cdot)}_{0+}\mathscr{D}_t^{\gamma(t)}z(t) = x(t)y(t) - \frac{8.8}{3}z(t-\tau_z), \end{cases} \tag{4.7}$$

With initial conditions similar to Model 4.1. In Figure 4, we depict the phase solutions of the Eq. (4.7) using proposed algorithms with $\tau_x = 0.008$, $\tau_y = 0.01$, and $\tau_z = 0.1$, $T = 20$ and step size $h = 0.004$ for $\alpha(t) = 0.95 + 0.0025t$, $\beta(t) = 1$ and $\gamma(t) = 0.85 + 0.0075t$.

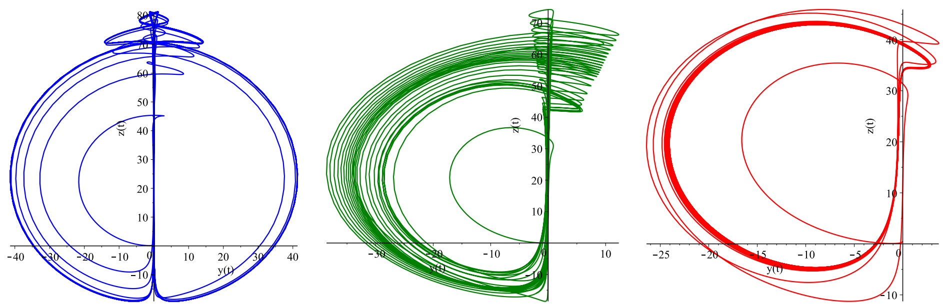
5. Conclusion

In this paper, a new numerical discretization formula was introduced and implemented for approximating the VO fractional integral based on a finite element scheme. By adopting the approximation formula, we obtained a predictor-corrector scheme for the numerical solution of a class of VO fractional differential equations with time-delays. The VO fractional unified chaotic systems with time-delay were solved by proposed algorithms and the results were analyzed using phase portraits. Moreover, the results revealed that VO fractional operators can act as a modulation parameter that are useful for a better describing and chaos controlling of dynamic systems with time-delay.

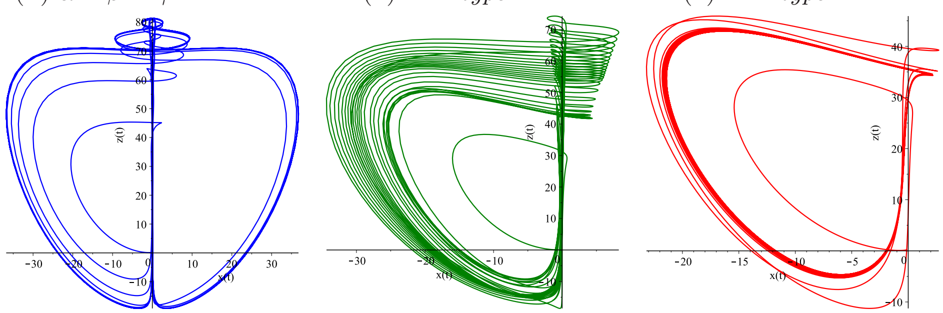
FIGURE 3. Phase portrait of VO fractional Chen system with time-delay, with $T = 20$, $\tau_x = 0.005$, $\tau_y = 0.025$, $\tau_z = 0.25$, $\alpha(t) = 1$, $\beta(t) = 0.9 + 0.005t$, $\gamma(t) = 0.8 + 0.01t$ and step size $h = 0.004$.



(A) $\alpha = \beta = \gamma = 1$ (B) $\nu 1 - type$ (C) $\nu 2 - type$

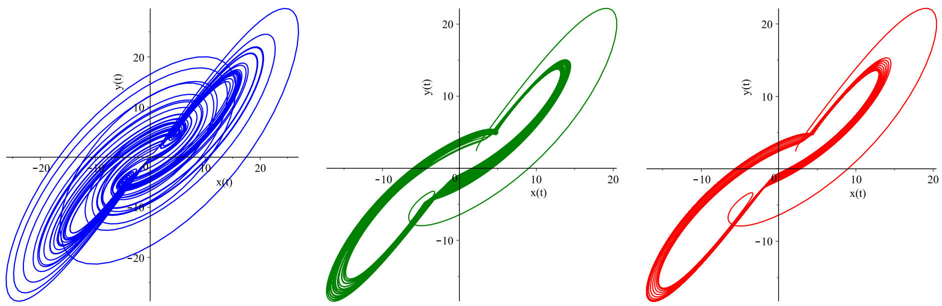


(D) $\alpha = \beta = \gamma = 1$ (E) $\nu 1 - type$ (F) $\nu 2 - type$

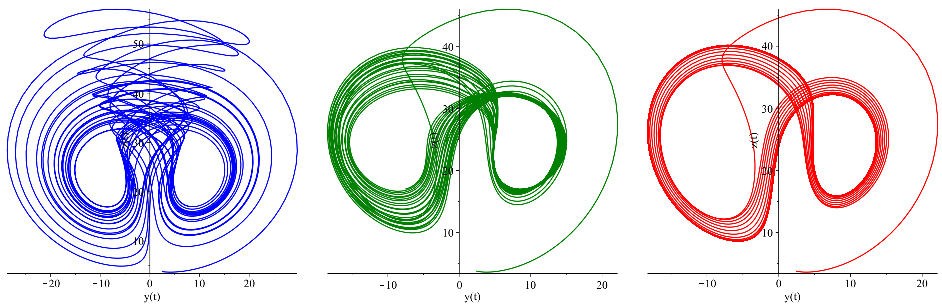


(G) $\alpha = \beta = \gamma = 1$ (H) $\nu 1 - type$ (I) $\nu 2 - type$

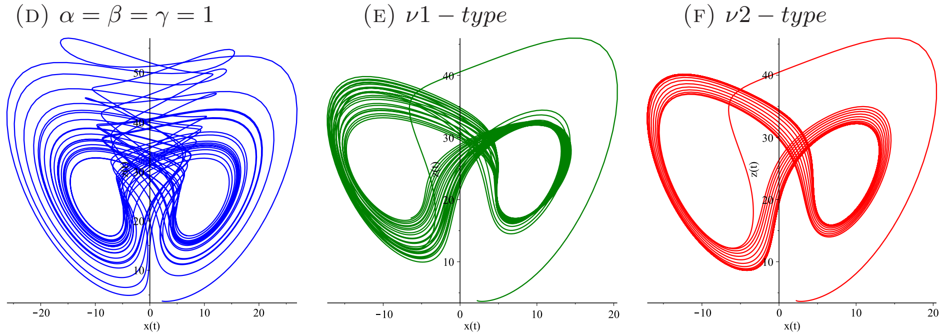
FIGURE 4. Phase portrait of VO fractional Lü system with time-delay, with $T = 20$, $\tau_x = 0.008$, $\tau_y = 0.001$, $\tau_z = 0.1$, $\alpha(t) = 0.95 + 0.0025t$, $\beta(t) = 1$, $\gamma(t) = 0.85 + 0.0075t$ and step size $h = 0.004$.



(A) $\alpha = \beta = \gamma = 1$ (B) $\nu 1 - type$ (C) $\nu 2 - type$



(D) $\alpha = \beta = \gamma = 1$ (E) $\nu 1 - type$ (F) $\nu 2 - type$



(G) $\alpha = \beta = \gamma = 1$ (H) $\nu 1 - type$ (I) $\nu 2 - type$

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