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A total variation diminishing high resolution scheme for nonlinear conservation laws

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Abstract In this paper, we propose a novel high resolution scheme for scalar nonlinear hyperbolic conservation laws. The aim of high resolution schemes is to provide at least second order accuracy in smooth regions and produce sharp solutions near the discontinuities. We prove that the proposed scheme that is derived by utilizing an appropriate flux limiter is nonlinear stable in the sense of total variation diminishing (TVD). The TVD schemes are robust against the spurious oscillations and preserve the sharpness of the solution near the sharp discontinuities and shocks. We also, prove the positivity and maximum-principle properties for this scheme. The numerical results are presented for both of the advection and Burger's equation. A comparison of numerical results with some classical limiter functions is also provided.

Keywords. High resolution schemes, Flux limiter, Total variation diminishing, Nonlinear conservation laws.

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1. Introduction

Mathematical modeling of many problems in various fields of engineering such as fluid dynamics, aerodynamics, hydrodynamics, Magnetohydrodynamics (MHD),... lead to the nonlinear hyperbolic conservation laws [11, 19]. In this class, it is not surprisingly to know that, even in presence of smooth initial conditions we observe forming shock waves and discontinuities in the solution [8, 14, 17]. Therefore, the numerical solution of nonlinear conservation laws is undoubtedly one of the most challenging problems. The opinion of the numerical methods in this context is to provide nonoscillatory solution, and to capture the correct shock speed and shock location. In fact, the high resolution schemes are developed to evaluate high order numerical solution in smooth regions and sharp results in the vicinity of shocks and discontinuities [6, 14]. There are a great deal of publication in the literature with the

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aim of introducing novel nonoscillatory schemes which are nonlinearly stable in the sense of total variation diminishing (TVD). The major portion of the theory is due to the pioneering work of Harten [6] in developing a benchmark to specify a CFL number that guarantee the TVD stability of numerical scheme. In this paper we develop a total variation diminishing high resolution scheme for nonlinear conservation law

$$u_t + f(u)_x = 0, \quad u(x, 0) = u_0(x), \ \ x \in \mathbb{R}, \tag{1.1}$$

where f is the nonlinear flux function. We develop a novel TVD limiter function that eliminates the spurious oscillations in the solution and guarantees TVD stability and positivity of the scheme. A comparison of the new limiter function with its counterparts is also provided in numerical results.

In recent decades, the study of high resolution schemes and TVD limiter functions have had increasingly rate. Many researchers have studied and developed a large class of TVD schemes based on different TVD flux limiter functions. For an extensive review of these scheme we refer to [20] and references their in. On the other hand, some authors have developed the maximum principle satisfying numerical schemes [12, 21]. We show that the presented scheme in this paper satisfies the maximum-principle. For more details on the classification and a comparative study of TVD schemes and limiter functions we refer to [4, 10]. The rest of the paper is organized as follows: In section 2 we introduce the basic concepts of TVD schemes, in section 3 we explain the construction of the novel high resolution TVD scheme and we prove the positivity and maximum-principle-satisfying properties of the scheme. The numerical results and demonstrations is given in section 4.

2. Preliminaries

In this section we review the main properties of the total variation diminishing scheme. We explain shortly the idea of switching from a high order numerical flux to a low order dissipative numerical flux along with an appropriate TVD flux limiter.

Let $u_j^n \approx u(x_j, t_n)$ be the numerical approximation of the true solution at the typical point (x_j, t_n) with step lengthes Δx and Δt in space and time variables, respectively. The total variation of the vector of point values u^n is defined by

$$TV(u^n) = \sum_j \mid u_{j+1}^n - u_j^n \mid .$$

Accordingly, a numerical scheme is called total variation diminishing (TVD) if $\{TV(u^n)\}$ is a nonincreasing sequence [1, 7, 14]. In other words,

$$TV(u^{n+1}) \leq TV(u^n) \quad n = 0, 1, 2, \ldots . \tag{2.1}$$

The TVD property (2.1) is a tool to verify the nonlinear stability of the numerical solutions. The generation of the spurious near the discontinuities and shocks is the Achilles' heel of the numerical methods in this context. Therefore, TVD scheme try to prevent the unexpected nonphysical oscillations throughout the solution. Traditionally, this is done by adding an amount of dissipation to the solution [8, 14, 17]. The classical upwind method is an instance of a typical dissipative method that is just a first order method. The enforcing the monotonicity property for a linear scheme, i.e., does not generate a new extreme, leads to a scheme that is at most first order. In fact, the order reduction of linear monotone schemes is a common drawback of such schemes. This result is known as Godunov theorem that we give a descriptive version of it [5].

Theorem 2.1. *(**Godunov theorem**) Monotone behavior of a numerical solution cannot be assured for a linear finite-difference methods with more than first-order accuracy.*

Therefor, the high order TVD schemes fall into the class of nonlinear numerical schemes. The idea of construction of a typical high order TVD scheme is to make a nonlinear combination of a low order flux with a high order one [2, 3, 6, 9, 13, 14, 15, 16]. This is accomplished with the aid of an appropriate flux function that near a shock it degenerates to the first order dissipative scheme. More precisely, let us consider the following advection equation

$$u_t + (au)_x = 0, \quad x \in I, \tag{2.2}$$

where I is an interval and we suppose that $a > 0$ and consider the initial data $u(x,0) = u_0(x)$. The conservative scheme for solving (1.1) and (2.2) is as follow

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(f^n_{j+\frac{1}{2}} - f^n_{j-\frac{1}{2}}), \tag{2.3}$$

where, $f^n_{j\pm\frac{1}{2}}$ are numerical flux functions that can be specified by several methods such as Riemann solvers. In high resolution schemes the numerical fluxes are written in the following form [14],

$$f^n_{j+\frac{1}{2}} = f^n_{L_{j+\frac{1}{2}}} + \phi_j^n[f^n_{H_{j+\frac{1}{2}}} - f^n_{L_{j+\frac{1}{2}}}], \tag{2.4}$$

where the so-called limiter function ϕ is a function of the quotient θ_j^n,

$$\theta_j^n = \frac{u_j^n - u_{j-1}^n}{u_{j+1}^n - u_j^n},$$

and f_L and f_H are low-order and high-order numerical flux functions, respectively.

In a special case, the first and second order fluxes are given by

$$f^n_{L_{j+\frac{1}{2}}} = au_j^n, \;\; f^n_{H_{j+\frac{1}{2}}} = \frac{a}{2}(u_{j+1}^n + u_j^n), \tag{2.5}$$

where the first order flux is upwind flux. Therefore, the corresponding high resolution scheme is given as

$$u_j^{n+1} = u_j^n - \nu\Big((u_j^n - u_{j-1}^n) + \frac{1}{2}(\phi_j^n(u_{j+1}^n - u_j^n) - \phi_{j-1}^n(u_j^n - u_{j-1}^n))\Big), \tag{2.6}$$

where $\nu = a\frac{\Delta t}{\Delta x}$ is the Courant number, and ϕ_j^n is a limiter function. With a third-order correction we find the following novel scheme

$$f^n_{L_{j+\frac{1}{2}}} = au_j^n, \;\; f^n_{H_{j+\frac{1}{2}}} = \frac{a}{12}(5u_{j+1}^n + 8u_j^n - u_{j-1}^n), \tag{2.7}$$

that can be written

$$f^n_{L_{j+\frac{1}{2}}} = au_j^n, \quad f^n_{H_{j+\frac{1}{2}}} = a\Big(u_j^n + (\frac{5}{12} + \frac{1}{12}\theta_j^n)(u_{j+1}^n - u_j^n)\Big), \tag{2.8}$$

with $\phi(\theta_j) = (\frac{5}{12} + \frac{1}{12}\theta_j)$, this reduces to the following numerical scheme

$$u_j^{n+1} = u_j^n + \nu\Big(1 - \phi_{j-1}^n + \frac{1}{\theta_j^n}\phi_j^n\Big)(u_{j-1}^n - u_j^n). \tag{2.9}$$

With a fourth-order correction

$$f_{L_{j+\frac{1}{2}}}^n = au_j^n, \quad f_{H_{j+\frac{1}{2}}}^n = \frac{a}{24}(9u_{j+1}^n + 19u_j^n - 5u_{j-1}^n + u_{j-2}^n), \tag{2.10}$$

that can be written

$$f_{L_{j+\frac{1}{2}}}^n = au_j^n, \quad f_{H_{j+\frac{1}{2}}}^n = a\Big(u_j^n + \frac{1}{24}(9 - \theta_j^n\theta_{j-1}^n + 4\theta_j^n)(u_{j+1}^n - u_j^n)\Big), \tag{2.11}$$

with $\phi(\theta_j) = \frac{1}{24}(9 - \theta_j\theta_{j-1} + 4\theta_j)$, now, the scheme can be represented in the following incremental form

$$u_j^{n+1} = u_j^n + \nu\Big(1 - \phi_{j-1}^n + \frac{1}{\theta_j^n}\phi_j^n\Big)(u_{j-1}^n - u_j^n). \tag{2.12}$$

In next section we study and prove the TVD and positivity of (2.6) and (2.9), (2.12).

3. Construction of high resolution TVD schemes

In this section we prove the TVD and positivity of the schemes (2.6) and (2.9), (2.12). The TVD benchmark of Harten [7] is a tool to verify the TVD property of the numerical methods in incremental form.

Lemma 3.1. *(Harten). An explicit scheme in the following incremental form*

$$u_j^{n+1} = u_j^n + C_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n) - D_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n), \tag{3.1}$$

is TVD only if $C_{j+\frac{1}{2}} \geq 0$, $D_{j-\frac{1}{2}} \geq 0$, $0 \leq C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1$.

The following theorem determines the conditions on the limiter function that the method (2.6) to be a TVD scheme.

Theorem 3.2. *The scheme (2.6) is TVD under restrictions* $0 \leq \phi \leq 2$, $0 \leq \phi \leq 2\theta$.

Proof. First we note that the scheme (2.6) can be represented in the following incremental form

$$u_j^{n+1} = u_j^n - \Big(\nu + \frac{\nu}{2}\frac{\phi_j^n}{\theta_j^n} - \frac{\nu}{2}\phi_{j-1}^n\Big)(u_j^n - u_{j-1}^n),$$

where, in accordance to the Lemma 3.1 we have,

$$D_{j-\frac{1}{2}} = \Big(\nu(1 + \frac{\phi_j^n}{2\theta_j^n}) - \frac{\nu}{2}\phi_{j-1}^n\Big),$$

$$C_{j+\frac{1}{2}} = 0.$$

It is sufficient to verify the following condition

$$0 \leq D_{j-\frac{1}{2}} \leq 1,$$

or,

$$0 \le \nu(1+\frac{\phi_j^n}{2\theta_j^n}) - \frac{\nu}{2}\phi_{j-1}^n \le 1,$$

therefore, we find

$$-2 \le \frac{\phi}{\theta} - \phi \le \frac{2(1-\nu)}{\nu},$$

it turns out that under the CFL condition $0 < \nu \le \frac{1}{2}$ we have following restiriction

$$0 \le \mid \frac{\phi}{\theta} - \phi \mid \le 2,$$

therefore the conditions $0 \le \phi \le 2$ and $0 \le \phi \le 2\theta$ on ϕ fulfill the TVD property of the scheme (2.6). □

Similarly, the following theorem determines the conditions under which the scheme (2.9) is a TVD method.

Theorem 3.3. *The scheme (2.9) is TVD under the constraints* $0 \le \phi \le 1$, $0 \le \phi \le \theta$.

Proof. The scheme (2.9) can be written as

$$u_j^{n+1} = u_j^n - \nu\Big(1 - \phi_{j-1}^n + \frac{1}{\theta_j^n}\phi_j^n\Big)(u_j^n - u_{j-1}^n),$$

the scheme (2.9) can be represented in the form (3.1) with

$$D_{j-\frac{1}{2}} = \nu\Big(1 - \phi_{j-1}^n + \frac{1}{\theta_j^n}\phi_j^n\Big),$$

$$C_{j+\frac{1}{2}} = 0,$$

can be easily seen that $D_{j-\frac{1}{2}} \ge 0$. Furthermore, satisfy condition $0 \le C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \le 1$, if

$$0 \le D_{j-\frac{1}{2}} \le 1,$$

therefore, we have

$$0 \le \nu\Big(1 - \phi + \frac{1}{\theta}\phi\Big) \le 1, \ -1 \le \frac{1}{\theta}\phi - \phi \le \frac{1-\nu}{\nu}$$

under the CFL condition $1 \le \frac{1-\nu}{\nu}$, $0 < \nu \le \frac{1}{2}$ we have following restriction

$$-1 \le \frac{1}{\theta}\phi - \phi \le 1, \ 0 \le \mid \frac{1}{\theta}\phi - \phi \mid \le 1,$$

therefore

$$0 \le \phi \le 1, \ 0 \le \phi \le \theta,$$

hence scheme (2.9) under this constraints is TVD. □

Recently, the maximum-principle satisfying schemes become an active research field in developing numerical methods for nonlinear conservation laws [12, 21]. A numerical scheme is said to be maximum-principle-satisfying if for some constant values of m, M we have

$$m \le u_j^n \le M, \quad \forall j, \tag{3.2}$$

then

$$m \leq u_j^{n+1} \leq M, \quad \forall j, \tag{3.3}$$

In the following theorem we verify the positivity and maximum-principle property of the scheme (2.9).

Theorem 3.4. *The scheme (2.9) is positivity preserving and maximum-principle-satisfying.*

Proof. The scheme (2.9) can be written as

$$u_j^{n+1} = u_j^n + \nu\Big(1 - \phi_{j-1}^n + \frac{1}{\theta_j^n}\phi_j^n\Big)(u_{j-1}^n - u_j^n),$$

we assume

$$D_{j-\frac{1}{2}} = \nu\Big(1 - \phi_{j-1}^n + \frac{1}{\theta_j^n}\phi_j^n\Big),$$

for positivity preserving we require $D_{j-\frac{1}{2}} \geq 0$. A sufficient condition on the limiter function is

$$0 \leq \phi \leq 1,\ 0 \leq \phi \leq \theta,$$

therefore $0 \leq 1 - \phi$, $0 \leq \frac{\phi}{\theta} \leq 1$, this completes the proof. Hence scheme (2.9) is positivity preserving. On the other hand, let $\{u_j^n\}$ satisfies (3.2) for some constants m and M. Then we have

$$u_j^{n+1} = (1 - D)u_j^n + Du_{j-1}^n,$$

and since $0 \leq D_{j-\frac{1}{2}} \leq 1$, therefore, we readily find that (3.2) with the same bounds m and M. □

Theorem 3.5. *The scheme (2.12) is TVD under the constraints* $0 \leq \phi \leq 1, 0 \leq \phi \leq \mu\theta, \mu > 0$.

Proof. The scheme (2.12) can be written as

$$u_j^{n+1} = u_j^n - \nu\Big(1 - \phi_{j-1}^n + \frac{1}{\theta_j^n}\phi_j^n\Big)(u_j^n - u_{j-1}^n),$$

for TVD property we require $0 \leq 1 - \phi(\theta_{j-1}^n) \leq 1$, $0 \leq \frac{\phi(\theta_j^n)}{\theta_j^n} \leq \mu$, therefore, under the CFL condition $\nu \leq \frac{1}{1+\mu}$, $\mu > 0$ we have $0 \leq \phi \leq 1, 0 \leq \phi \leq \mu\theta$. □

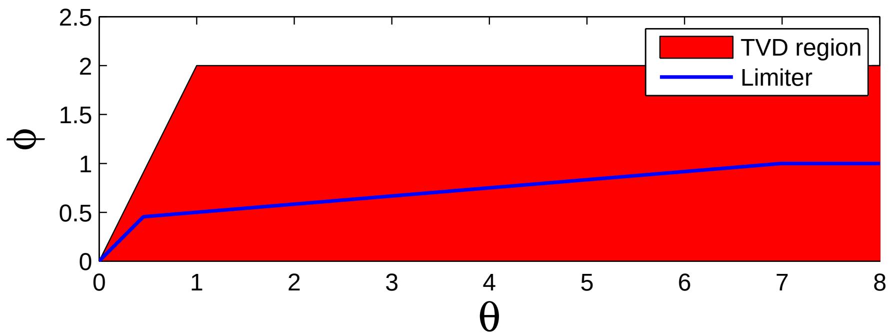
In this step, we can identify the appropriate TVD limiters according to the restrictions derived in Theorem 3.2 and Theorem 3.3, Theorem 3.5. To gain high order accuracy of the methods (2.6) and (2.9), (2.12) we need to design the limiters such that in smooth regions we maintain the original high order schemes. Getting together these considerations we find the following limiter functions for schemes (2.6) and (2.9), (2.12) respectively,

$$0 \leq \phi(\theta) \leq \text{minmod}(2, 2\theta), \tag{3.4}$$

and

$$\phi(\theta) = \max\Big(0, \min(1, \frac{5}{12} + \frac{1}{12}\theta, \theta)\Big), \tag{3.5}$$

FIGURE 1. The shaded region shows where function values must lie for the method to be TVD. The proposed limiter (2.9) is have function $\phi(\theta)$ that in this region.



and

$$\phi(\theta_j) = \max\Big(0, \min(1, \frac{1}{24}(9 - \theta_j\theta_{j-1} + 4\theta_j), \mu\theta_j)\Big). \tag{3.6}$$

Figure 1, illustrates the TVD region of the corresponding limiter functions (3.4) and (3.5), (3.6). The curve $\phi(\theta)$ must lie in the TVD region, which is shown as the shaded region in θ-ϕ plane.

4. Numerical results

In this section we present the numerical results for solving advection and Burgers equations with the TVD schemes (2.9). We give a comparison of the new limiter by the Sweby scheme [18]

$$u_j^{n+1} = u_j^n - \nu\Big((u_j^n - u_{j-1}^n) + \frac{1}{2}(1-\nu)(\phi_j^n(u_j^n - u_{j-1}^n) - \phi_{j-1}^n(u_{j-1}^n - u_{j-2}^n))\Big),$$

TABLE 1. Estimated orders and errors for advection equation with limiter (3.5) in Test 5.

Δx	l_1-norm	order	l_2-norm	order	l_∞-norm	order
$\frac{1}{10}$	$7.81\,10^{-2}$		$1.03\,10^{-1}$		$1.58\,10^{-1}$	
$\frac{1}{20}$	$1.90\,10^{-2}$	2.03	$2.27\,10^{-2}$	1.91	$5.51\,10^{-2}$	1.52
$\frac{1}{40}$	$4.20\,10^{-3}$	2.17	$7.10\,10^{-3}$	1.92	$1.90\,10^{-2}$	1.53

with Superbee and minmod limiters, respectively are given by

$$\phi(\theta) = \max\Big(0, \min(1, 2\theta), \min(2, \theta)\Big), \tag{4.1}$$

and

$$\phi(\theta) = \max\Big(0, \min(1, \theta)\Big). \tag{4.2}$$

Test 1: In this test we consider the advection equation (2.2) with $a = 0.01$, $x \in [-1, 2]$ and the following initial condition

$$u(x,0) = \begin{cases} 1, & \text{if } x \le 0.1, \\ 0, & \text{if } x > 0.1. \end{cases} \tag{4.3}$$

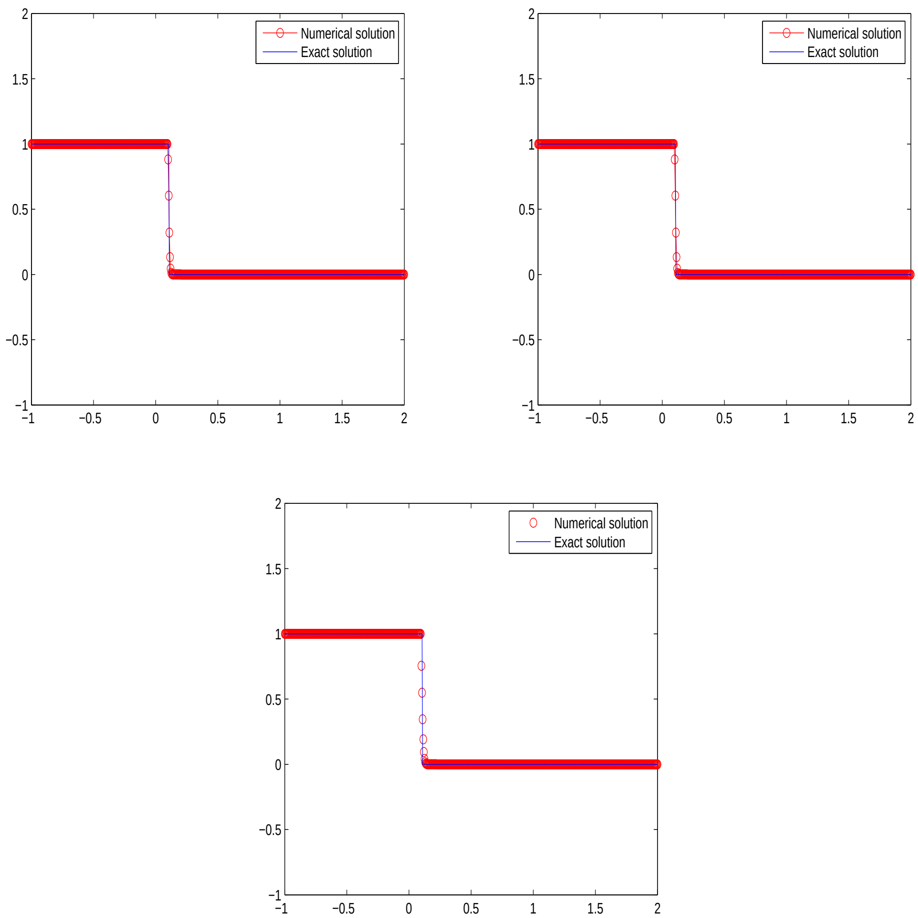
Figure 2 demonstrates the numerical solution obtained by the TVD scheme (2.9) along with the true solution at the final time $T = 1$. We put $\Delta x = 0.005$, $\Delta t = 0.005$ and $\nu = 0.01$. The nonoscillatory behavior of the numerical solution is evident in the graphs. As we expect from TVD schemes we find a sharp and oscillations free transition through the shock.

Test 2: In Figure 2 we observe the numerical results for the following compact support initial condition

$$u(x,0) = \begin{cases} 1, & \text{if } 0.3 \le x \le 0.7, \\ 0, & \text{otherwise}. \end{cases} \tag{4.4}$$

In this test we set $\nu = 0.5$ and the other parameters are the same as previous test problem. In this test we have two discontinuities propagating with same speed to the right. Again, it is clear that the high resolution solution scheme (2.9) produces sharp results in the vicinity of shock. Figure 6 demonstrate the numerical solution obtained by the TVD scheme (2.12) at the final time $T = 1$. We put $\Delta x = 0.01$, $\Delta t = 0.002$ and $\nu = 0.2$, . The nonoscillatory behavior of the numerical solution is evident in the graph.

FIGURE 2. Numerical solution of the advection equation in Test 1: Sweby's scheme by Superbee limiter (top left), minmod limiter (2.9) (top right) and the TVD scheme (bottom).

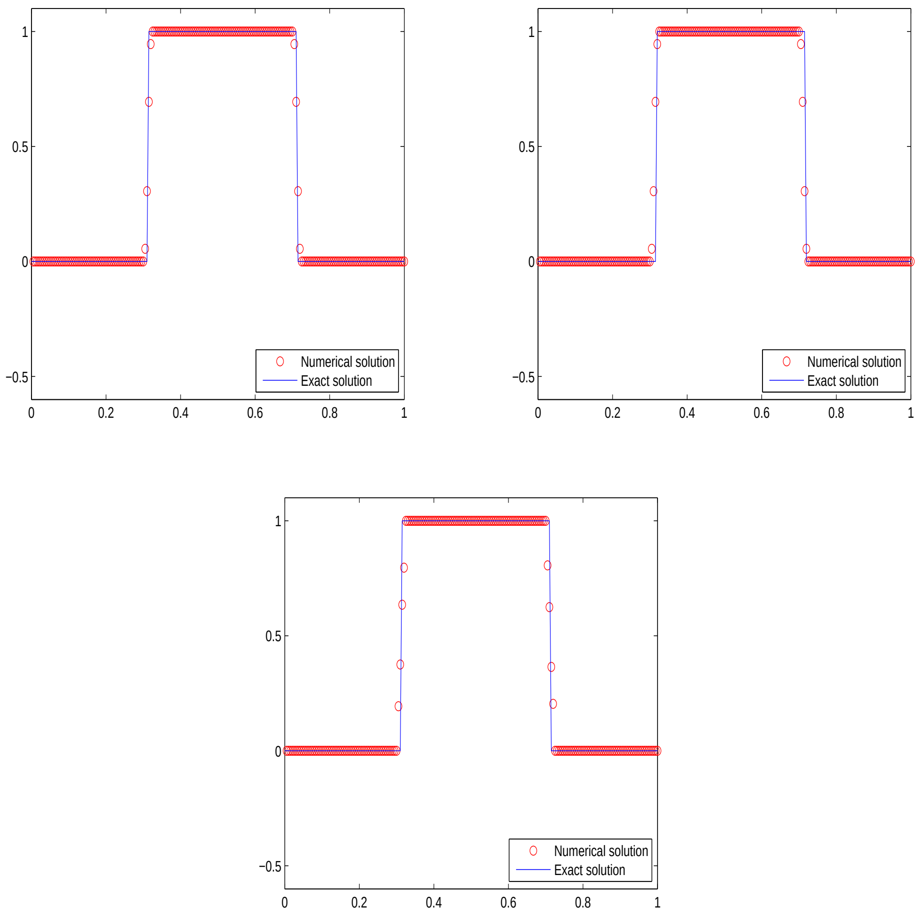


Test 3: In this test we verify the dissipation behavior of the new TVD scheme (2.9) with the following smooth initial profile

$$u(x,0) = (\sin(\pi x))^{100}. \tag{4.5}$$

Figure 4 illustrates the results of (2.9) and a comparison with Sweby scheme at final time $T = 1$ and $x \in [0,1]$. In this test we let $\Delta x = 0.005$, $\Delta t = 0.0025$ and $\nu = 0.5$.

Figure 3. Numerical solution of the advection equation in Test 2: Sweby's scheme by Superbee limiter (top left), minmod limiter (top right) and the TVD scheme (2.9) (bottom).



It is well known that the further amount of dissipation flatten the smooth extrema. However, the comparison results in Figure 4 demonstrate that the dissipation of the new TVD scheme (2.9) is approximately in the same order as the Superbee and minmod limiters.

Test 4: In this test We apply the proposed scheme (2.9) to the Riemann problem for Burger's equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0. \tag{4.6}$$

Table 2. Estimated orders and errors for advection equation with minmod limiter in Test 5.

Δx	l_1-norm	order	l_2-norm	order	l_∞-norm	order
$\frac{1}{10}$	$1.78\,10^{-1}$		$1.81\,10^{-1}$		$1.93\,10^{-1}$	
$\frac{1}{20}$	$7.91\,10^{-2}$	1.17	$8.20\,10^{-2}$	1.14	$9.10\,10^{-2}$	1.09
$\frac{1}{40}$	$3.01\,10^{-2}$	1.39	$3.14\,10^{-2}$	1.38	$4.09\,10^{-2}$	1.15

where $f(u) = \frac{1}{2}u^2$, with the following initial data

$$u(x,0) = \begin{cases} 1.2, & \text{if } x \leq 0, \\ 0, & \text{if } x > 0. \end{cases} \tag{4.7}$$

The non-TVD methods typically produce oscillations around the points of discontinuity [9, 14].

The similar variant of the scheme (2.9) for Burgers equation (4.6) is given as follow

$$u_j^{n+1} = u_j^n + \frac{1}{2}\frac{\Delta t}{\Delta x}[(1 - \phi_{j-1}^n + \frac{1}{\theta_j^n}\phi_j^n)]((u_{j-1}^n)^2 - (u_j^n)^2),$$

where,

$$\theta_j^n = \frac{(u_j^n)^2 - (u_{j-1}^n)^2}{(u_{j+1}^n)^2 - (u_j^n)^2}.$$

The limiter function ϕ is , also, given as follow

$$\phi(\theta) = \max(0, \min(1, \frac{5}{12} + \frac{1}{12}\theta, \theta)).$$

The true solution of (4.6) is a shock propagating with shock speed derived from the following Rankine-Hugoniot condition

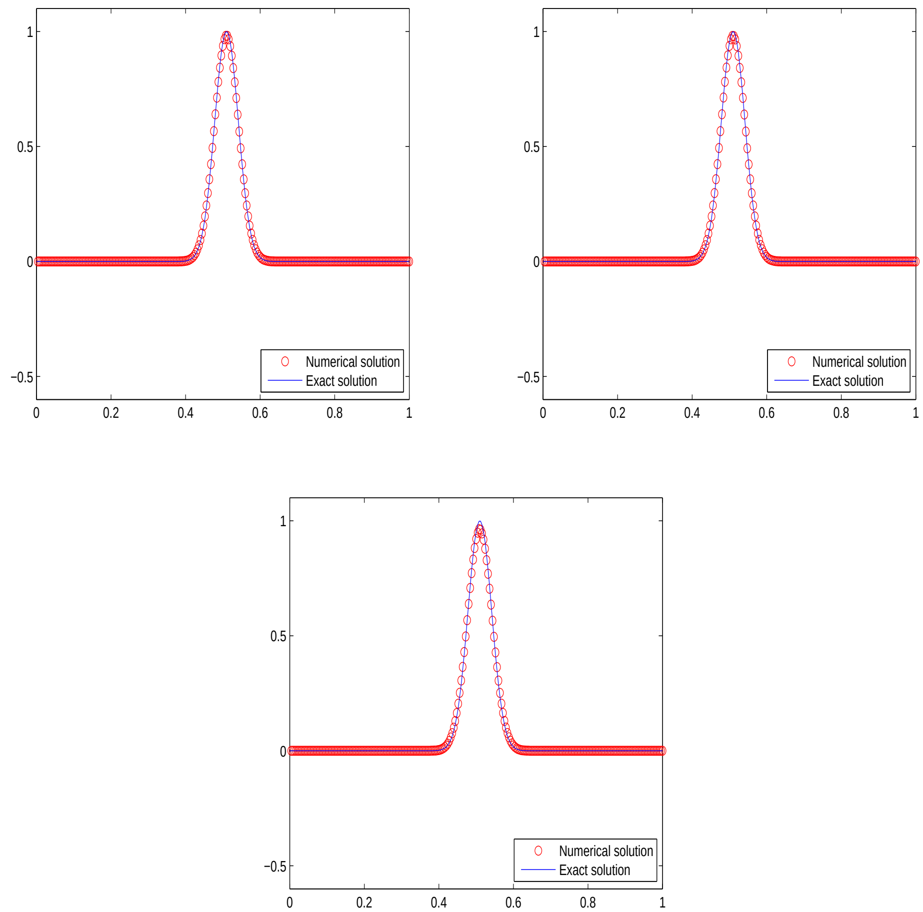
$$s = \frac{[f]}{[u]} = \frac{1}{2}(u_l + u_r) = \frac{(1.2+0)}{2} = 0.6.$$

In this test we set $\Delta x = 0.08$, $\Delta t = 0.01$ and $T = 1.2$.

Once again, in Figure 5, we find high resolution simulation of the shock problem. The numerical solution is oscillation free, with sharp results in the shock position.

Test 5: The adverse effect of limiting is the accuracy drop near the smooth extrema. In this test, we give a comparision between minmod limiter function

FIGURE 4. Numerical solution of the advection equation in Test 3: Sweby's scheme by Superbee limiter (top left), the TVD scheme (2.9) (top right) and minmod limiter (bottom).



and the new developed limiter function (3.5). We consider advection equation [8]

$$u_t + au_x = 0, \quad a = 1, \quad 0 < x < 1, \tag{4.8}$$

with the following initial data

$$u(x, 0) = \sin^2(\pi x). \tag{4.9}$$

FIGURE 5. Numerical solution of the Burger's equation in Test 4.

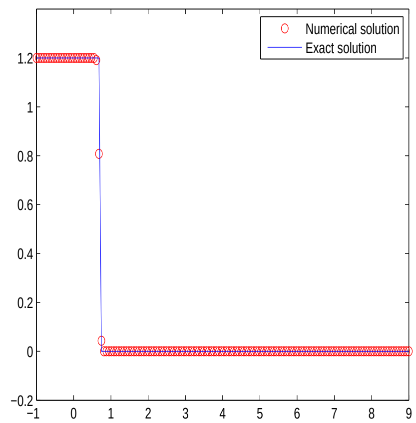
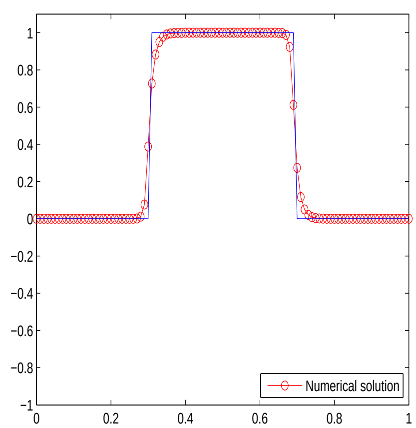


FIGURE 6. Numerical solution with scheme (2.12) for advection equation in Test 2.



In Table 1 the numerical results are given for limiter (3.5) and the simillar results for minmod limiter are given in Table 2. The orders comparision, in this worst case (presence of smooth extrema and order drop), illustrates the superior behavior of the new limiter function.

5. Conclusions

In this paper, we developed a novel TVD high resolution scheme for solving linear advection and nonlinear Burgers equations. A new limiter function introduced to preserve the total variation diminishing and positivity properties of the numerical scheme in conservative form.

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