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Chebyshev finite difference method for solving a mathematical model arising in wastewater treatment plants

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Abstract The Chebyshev finite difference method is applied to solve a system of two coupled nonlinear Lane-Emden differential equations arising in mathematical modelling of the excess sludge production from wastewater treatment plants. This method is based on a combination of the useful properties of Chebyshev polynomials approximation and finite difference method. The approach consists of reducing the problem to a set of algebraic equations. Numerical results are included to demonstrate the validity and applicability of the technique and a comparison is made with the existing results.

Keywords. Chebyshev finite difference method, Gauss-Lobatto nodes, Excess sludge production, Activated sludge, Carbon substrate.

2010 Mathematics Subject Classification. 65L10, 65M70.

1. Introduction

Domestic and industrial wastewater have a rich concentration of biodegradable carbonaceous organics (carbon substrate). Activated sludge is the primary method for treatment of organic wastes [1]. This biological method can efficiently oxidize carbon substrate and transform it into new cells (sludge), CO_2 and H_2O. Excess sludge is the main by-product that is costly to treat and dispose. About half of the total cost of wastewater treatment is accounted for the produced excess sludge treatment and disposal [17]. Thus, it is crucial to investigate a new method for minimization of sludge production. The amount of sludge is mainly influenced by the concentrations of carbon substrate and oxygen [1].
In this paper, following [1, 5, 10, 14, 16], we consider a system of two coupled nonlinear Lane-Emden differential equations, that governs the concentrations of oxygen and the

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carbon substrate. This system models the excess sludge production from wastewater treatment plants and is given by

$$x\frac{d^2u}{dx^2}+2\frac{du}{dx}=-\alpha_2 x+xF_1(u(x),v(x)), \tag{1.1}$$

$$x\frac{d^2v}{dx^2}+2\frac{dv}{dx}=xF_2(u(x),v(x)), \tag{1.2}$$

subject to the two mixed sets of Neumann and Dirichlet boundary conditions:

$$u'(0)=0,\ u(1)=1, \qquad v'(0)=0,\ v(1)=1, \tag{1.3}$$

where x denotes the radius of a spherical floc particle. Also, the functions $u(x)$ and $v(x)$ are the concentrations of carbon substrate and oxygen, respectively. Moreover, F_1, F_2 are given by

$$F_1(u(x),v(x))=\alpha_1\frac{u(x)v(x)}{(\ell_1+u(x))(m_1+v(x))}+\alpha_3\frac{u(x)v(x)}{(\ell_2+u(x))(m_2+v(x))}, \tag{1.4}$$

$$F_2(u(x),v(x))=\alpha_4\frac{u(x)v(x)}{(\ell_1+u(x))(m_1+v(x))}+\alpha_5\frac{u(x)v(x)}{(\ell_2+u(x))(m_2+v(x))}. \tag{1.5}$$

Here ℓ_1,ℓ_2,m_1,m_2 and $\alpha_i,\ i=1,2,..,5$ are some known constants.

The literature of numerical analysis contains little on the solutions of this problem. Muthukumar et al. [10] used the Adomian decomposition method and Duan et al. [5] employed the Adomian decomposition method combined with the Duan-Rach modified recursion scheme. Also, variational iteration method [5] is used to solve this problem. Recently, Saadatmandi and Fayyaz [14] solved this problem by sinc-collocation method.

In this work, a different approach is used. Our idea is to apply the Chebyshev finite difference (ChFD) method to discretize the problem (1.1)-(1.3) to get a nonlinear system of algebraic equations, thus greatly simplifying the problem. The main advantage of this method lies in its accuracy for a given number of unknowns. Chebyshev polynomials are well-known family of orthogonal polynomials on the interval $[-1,1]$. These polynomials present very good properties in the approximation of functions. Therefore, Chebyshev polynomials appear frequently in numerical computation (e.g., see [2, 11]). ChFD method can be regarded as a nonuniform finite difference scheme [6]. In this method the derivative of the function $u(t)$ at a point t_j is a linear combination of the values of the function u at the Chebyshev-Gauss-Lobatto points $t_k=\cos(k\pi/N)$, where $k,j=0,1,2,...,N$. This method can obtain a better approximation and satisfactory simulation results than finite difference and finite elements methods because the approximation of the derivatives is defined over the whole domain [6, 7]. As pointed by [8], in this method, the error decreases exponentially. ChFD method has been recognized as a powerful tool for problems in applied physics and engineering. This method has been extended to handle boundary value problems [6], boundary layer equations [7], heat transfer problem [9], a problem arising from chemical reactor theory [12], problems in calculus of variation [13], Fredholm integro-differential equation [4] and the flow of a third-grade fluid in a porous half space [15].

The outline of the paper is as follows. In section 2, we review the basic formulation of ChFD method. In sections 3, we present a computational method for solving the problem (1.1)-(1.3) by ChFD method. In section 4 some numerical results are given to clarify the method and a comparison is made with existing results. Finally, we conclude the paper in section 5.

2. Chebyshev finite difference method

Consider the Chebyshev polynomial of the first kind of degree n given by the formula:

$$T_n(t) = \cos(n \cos^{-1} t), \qquad t \in [-1, 1].$$

The well known Chebyshev-Gauss-Lobatto points are given by [3]

$$t_k = \cos\left(\frac{k\pi}{N}\right), \qquad k = 0, 1, 2, ..., N. \tag{2.1}$$

These grid (interpolation) points, $t_N = -1 < t_{N-1} < ... < t_1 < t_0 = 1$ are also viewed as the zeros of $(1 - t^2)\dot{T}(t)$, where $\dot{T}(t)$ is the first derivative of $T_n(t)$.

Authors of [3] introduced the following approximation of the function $u(t)$,

$$u_N(t) = \sum_{n=0}^{N}{}'' a_n T_n(t), \qquad a_n = \frac{2}{N} \sum_{j=0}^{N}{}'' u(t_j) T_n(t_j). \tag{2.2}$$

Here, the summation symbol with double primes denotes a sum with both the first and last terms halved. Also, the derivatives of the function $u(t)$ at the point t_k are given by [6, 7]

$$u_N^{(n)}(t_k) \approx \sum_{j=0}^{N} d_{k,j}^{(n)} u(t_j), \qquad n = 1, 2. \tag{2.3}$$

where $d_{k,j}^{(1)}$ and $d_{k,j}^{(2)}$ for $k, j = 0, 1, ..., N$, are given by

$$d_{k,j}^{(1)} = \frac{4\theta_j}{N} \sum_{n=0}^{N} \sum_{\substack{\ell=0 \\ (n+\ell) odd}}^{n-1} \frac{n\theta_n}{c_\ell} T_n(t_j) T_\ell(t_k), \tag{2.4}$$

$$d_{k,j}^{(2)} = \frac{2\theta_j}{N} \sum_{n=0}^{N} \sum_{\substack{\ell=0 \\ (n+\ell) even}}^{n-2} \frac{n(n^2 - \ell^2)\theta_n}{c_\ell} T_n(t_j) T_\ell(t_k), \tag{2.5}$$

with $\theta_0 = \theta_N = 1/2$, $\theta_j = 1$ for $j = 1, 2, ..., N-1$, and $c_0 = 2$, $c_i = 1$, for $i \geq 1$.

3. Applying the ChFD method to the problem (1.1)-(1.3)

In this section, we solve problem (1.1)-(1.3) by using ChFD method. The domain is $0 \leq x \leq 1$. Using the algebraic mapping $t = 2x - 1$, the domain $[0, 1]$ is mapped into the computational domain $[-1, 1]$ and the Eqs. (1.1)-(1.3) are transformed into the following equations:

$$4(t+1))u'' + 8u' = -\alpha_2(t+1) + (t+1)F_1(u(t), v(t)), \tag{3.1}$$

$$4(t+1))v'' + 8v' = (t+1)F_2(u(t), v(t)), \tag{3.2}$$

$$u'(-1) = 0,\ u(1) = 1, \qquad v'(-1) = 0,\ v(1) = 1. \tag{3.3}$$

Now using Eq. (2.3) to approximate $u(t)$ and $v(t)$ as

$$u_N(t) = \sum_{n=0}^{N}{}'' \lambda_n T_n(t), \qquad v_N(t) = \sum_{n=0}^{N}{}'' \mu_n T_n(t), \tag{3.4}$$

where

$$\lambda_n = \frac{2}{N}\sum_{j=0}^{N}{}'' u(t_j)T_n(t_j), \qquad \mu_n = \frac{2}{N}\sum_{j=0}^{N}{}'' v(t_j)T_n(t_j). \tag{3.5}$$

Substituting Eq. (3.4) into Eqs. (3.1)-(3.2) and evaluating the result at the Gauss-Lobatto nodes t_k for $k = 1, ..., N-1$, we obtain

$$\sum_{j=0}^{N}\left\{4(t_k+1)d^{(2)}_{k,j} + 8d^{(1)}_{k,j}\right\}u(t_j) = -\alpha_2(t_k+1)+(t_k+1)F_1(u(t_k), v(t_k)), \tag{3.6}$$

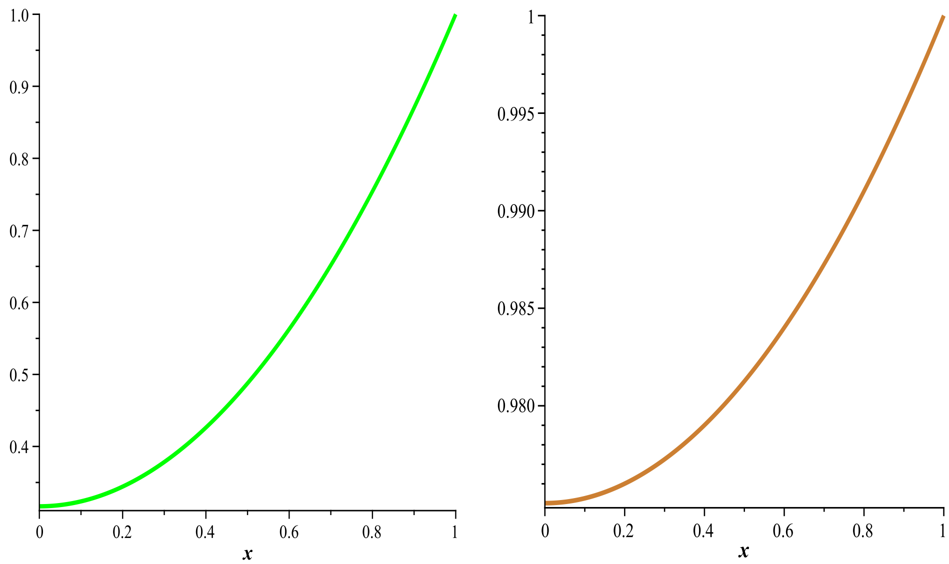
$$\sum_{j=0}^{N}\left\{4(t_k+1)d^{(2)}_{k,j} + 8d^{(1)}_{k,j}\right\}v(t_j) = (t_k+1)F_2(u(t_k), v(t_k)), \tag{3.7}$$

where $d^{(1)}_{k,j}$ and $d^{(2)}_{k,j}$ are given in Eqs. (2.4) and (2.5) respectively. For $k = 0$ and $k = N$ by using the boundary conditions (3.3) we get

$$\sum_{j=0}^{N} d^{(1)}_{N,j}u(t_j) = 0, \quad u(t_0) = 1, \quad \sum_{j=0}^{N} d^{(1)}_{N,j}v(t_j) = 0, \quad v(t_0) = 1. \tag{3.8}$$

Eqs. (3.6) and (3.7) together with Eq. (3.8) generate $2N + 2$ non-linear algebraic equations which can be solved for the unknown coefficients $u(t_k)$ and $v(t_k)$ for $k = 0, ..., N$. Throughout this paper, we use the Maple's fsolve command for solving this non-linear system. Consequently $u_N(t)$ and $v_N(t)$ given in Eq. (3.4) can be calculated.

FIGURE 1. Plot of the approximate solutions $u_4(x)$ (left) and $v_4(x)$ (right).



4. Results and discussion

This section is devoted to the presentation of some numerical simulations obtained by applying the ChFD method. Following [5], in order to evaluate the accuracy of our approximate solutions, we construct the error remainder functions

$$\mathrm{ER}_{1,N}(x) = x\frac{d^2u_N}{dx^2} + 2\frac{du_N}{dx} + \alpha_2 x - xF_1(u_N(x), v_N(x)), \tag{4.1}$$

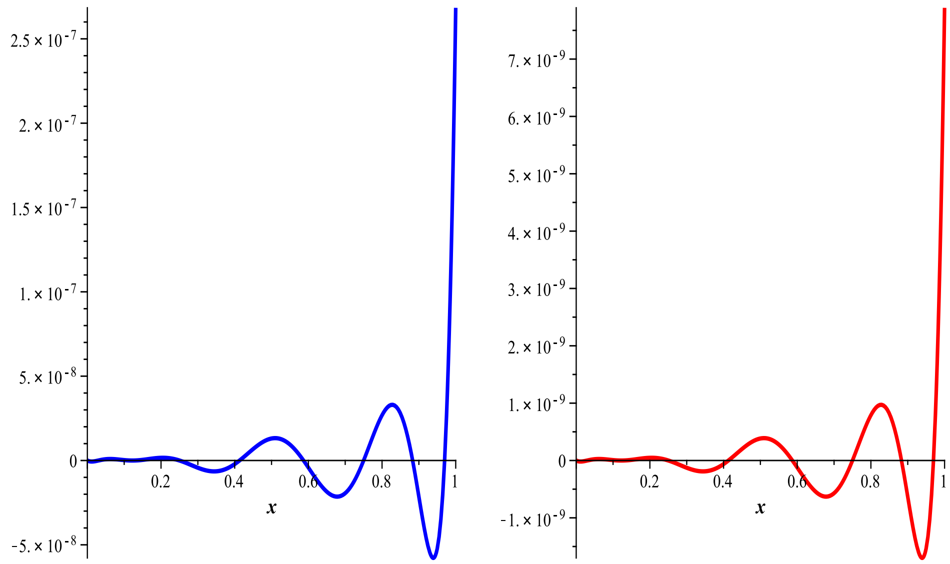
$$\mathrm{ER}_{2,N}(x) = x\frac{d^2v_N}{dx^2} + 2\frac{dv_N}{dx} - xF_2(u_N(x), v_N(x)), \tag{4.2}$$

and the maximal error remainder parameters

$$\mathrm{MER}_{1,N} = \max_{0\le x\le 1} |\mathrm{ER}_{1,N}(x)|, \quad \mathrm{MER}_{2,N} = \max_{0\le x\le 1} |\mathrm{ER}_{2,N}(x)|, \tag{4.3}$$

whenever the solutions are unknown in advance. Here, we have shown the approximate solutions of problem (1.1)-(1.3) for some typical values of parameters $m_1 = \ell_1 = m_2 = \ell_2 = 0.0001, \alpha_1 = 5, \alpha_2 = 1, \alpha_3 = 0.1, \alpha_4 = 0.1$ and $\alpha_5 = 0.05$ as in [5, 10, 16]. In Figure 1 the approximate solutions $u_N(x)$ and $v_N(x)$ are plotted for $N = 4$. Also, in Figure 2, the curves of the error remainder functions $\mathrm{ER}_{1,N}(x)$ and $\mathrm{ER}_{2,N}(x)$ are plotted for $N = 9$. To make a comparison, in Table 1, we compare the maximal error remainder parameters $\mathrm{MER}_{1,N}$ and $\mathrm{MER}_{2,N}$, for different values of N, together with the result obtained by using the Adomian decomposition method combined with the Duan-Rach modified recursion scheme given in [5]. From Table 1, we see that the present method is clearly reliable if compared with the Adomian decomposition

Figure 2. Plot of error remainder functions $\mathrm{ER}_{1,9}(x)$ (left) and $\mathrm{ER}_{2,9}(x)$ (right).



method. Furthermore, to show the efficiency of our method we define the norm of remainder functions as follows:

$$\|\mathrm{MER}_{1,N}\|_2 = \left(\int_0^1 (\mathrm{ER}_{1,N}(x))^2 dx \right)^{1/2},$$

$$\|\mathrm{MER}_{2,N}\|_2 = \left(\int_0^1 (\mathrm{ER}_{2,N}(x))^2 dx \right)^{1/2}.$$

The logarithmic graphs of $\|\mathrm{MER}_{1,N}\|_2$ and $\|\mathrm{MER}_{2,N}\|_2$ of the present method for different values of N are shown in Figure 3. From this figure, it is found that by increasing N the norm of remainder functions decrease. Also, Figure 3 illustrates the accuracy of the present method.

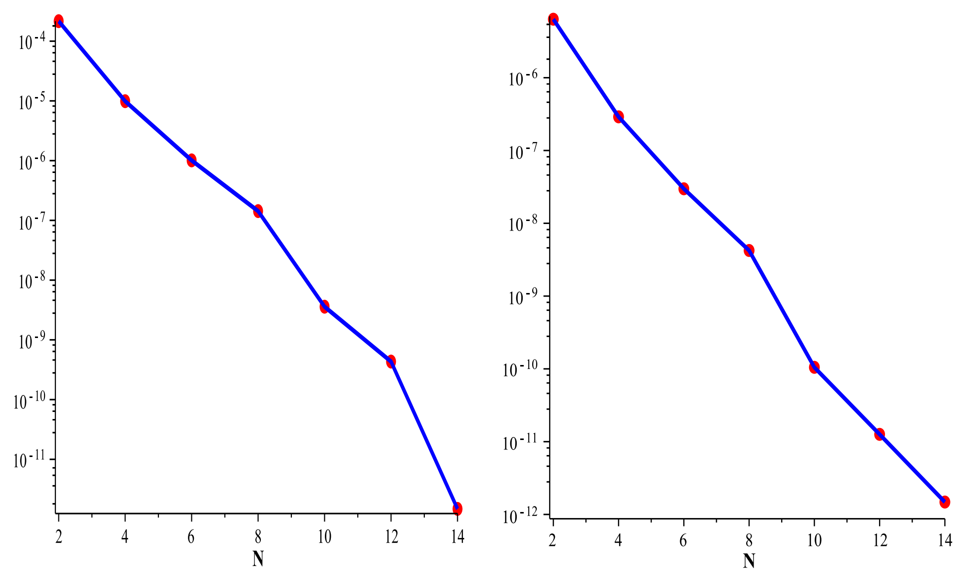
5. Conclusion

An alternative method for solving the system models the excess sludge production from wastewater treatment plants is proposed in this article. Our approach was based on the Chebyshev finite difference method. The new described computational technique produces very accurate results even with a small number of collocation points. The results presented indicate that the ChFD method provides powerful tool to solve the system of two coupled nonlinear Lane-Emden differential equations, that governs the concentrations of oxygen and the carbon substrate.

Table 1. Comparison of the maximal error remainder parameters $\mathrm{MER}_{1,N}$ and $\mathrm{MER}_{2,N}$.

	$\mathrm{MER}_{1,N}$		$\mathrm{MER}_{2,N}$	
N	Method in Ref. [5]	ChFD method	Method in Ref. [5]	ChFD method
2	1.18506×10^{-3}	5.45255×10^{-4}	3.48547×10^{-5}	1.60369×10^{-5}
3	8.53760×10^{-4}	1.25562×10^{-4}	2.51106×10^{-5}	3.69301×10^{-6}
4	6.21465×10^{-4}	3.64959×10^{-5}	1.82784×10^{-5}	1.07341×10^{-6}
5	4.52526×10^{-4}	3.11908×10^{-5}	1.33096×10^{-5}	9.17378×10^{-7}
6	3.29508×10^{-4}	6.84916×10^{-6}	9.69142×10^{-6}	2.01446×10^{-7}
7	2.39928×10^{-4}	1.15556×10^{-6}	7.05670×10^{-6}	3.39872×10^{-8}

Figure 3. Graphs of $\|\mathrm{MER}_{1,N}\|_2$ (left) and $\|\mathrm{MER}_{2,N}\|_2$ (right) for different values of N.



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References

[1] B. Abbassi, S. Dullstein, and N. Rabiger, *Minimization of excess sludge production by increase of oxygen concentration in activated sludge ocs: experimental and theoretical approach*, Water Res., *34* (2000), 139–146.

[2] M. Ahmadi Darani and M. Nasiri, *A fractional type of the Chebyshev polynomials for approximation of solution of linear fractional differential equations*, Comput. Methods Differ. Equ., *1* (2013), 96-107.

[3] C. W. Clenshaw and A. R. Curtis, *A method for numerical integration on an automatic computer*, Numer. Math., *2* (1960), 197–205.

[4] M. Dehghan and A. Saadatmandi, *Chebyshev finite difference method for Fredholm integro-differential equation*, Int. J. Comput. Math., *85* (2008), 123–130.

[5] J. S. Duan, R. Rach, and A. M. Wazwaz, *Oxygen and carbon substrate concentrations in microbial floc particles by the Adomian decomposition method*, MATCH Commun. Math. Comput. Chem., *73* (2015), 785–796.

[6] E. M. E. Elbarbary, *Chebyshev finite difference approximation for the boundary value problems*, Appl. Math. Comput., *139* (2003), 513–523.

[7] E. M. E. Elbarbary, *Chebyshev finite difference method for the solution of boundary- layere equations*, Appl. Math. Comput., *160* (2005), 487–498.

[8] N. T. Eldabe and M. E. M. Ouaf, *Chebyshev finite difference method for heat and mass transfer in a hydromagnetic flow of a micropolar fluid past a stretching surface with Ohmic heating and viscous dissipation*, Appl. Math. Comput., *177* (2006), 561–571.

[9] A. Y. Ghaly and M. A. Seddeek, *Chebyshev finite difference method for the effects of chemical reaction, heat and mass transfer on laminar flow along a semi infinite horizontal plate with temperature dependent viscosity*, Chaos Solitons Fractals, *19* (2004), 61–70.

[10] S. Muthukumar, M. Veeramuni, and R. Lakshmanan, *Analytical expression of concentration of substrate and oxygen in excess sludge production using Adomian decomposition method*, Indian J. Appl. Res., *4* (2014), 387–391.

[11] M. Ramezani, *Numerical solution of optimal control problems by using a new second kind Chebyshev wavelet*, Comput. Methods Differ. Equ., *4* (2016), 162–169.

[12] A. Saadatmandi and M. R. Azizi, *Chebyshev finite difference method for a two-point boundary value problems with applications to chemical reactor theory*, Iran. J. Math. Chem., *3* (2012), 1–7.

[13] A. Saadatmandi and M. Dehghan, *The numerical solution of problems in calculus of variation using Chebyshev finite difference method*, Phys. Lett. A, *372* (2008), 4037–4040.

[14] A. Saadatmandi, S. Fayyaz, *Numerical study of oxygen and carbon substrate concentrations in excess sludge production using sinc-collocation method*, MATCH Commun. Math. Comput. Chem., *80* (2018), 355–368.

[15] A. Saadatmandi, Z. Sanatkar, and S. P. Toufighi, *Computational methods for solving the steady flow of a third grade fluid in a porous half space*, Appl. Math. Comput., *298* (2017), 133–140.

[16] A. M. Wazwaz, R. Rach, and J. S. Duan, *Variational iteration method for solving oxygen and carbon substrate concentrations in microbial floc particles*, MATCH Commun. Math. Comput. Chem., *76* (2016), 511–523.

[17] Y. Wei, R. T. V. Houten, A. R. Borger, D. H. Eikelboom, and Y. Fan, *Minimization of excess sludge production for biological wastewater treatment*, Water Res., *37* (2003), 4453–4467.