Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 6, No. 3, 2018, pp. 380-395

Numerical solution of Convection-Diffusion equations with memory term based on sinc method

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Abstract In this essay, we study the numerical solution of Convection-Diffusion equation with a memory term subject to initial boundary value conditions. Finite difference method in combination with product trapezoidal integration rule is used to discretize the equation in time and sinc collocation method is employed in space. The accuracy and error analysis of the method are discussed. Numerical examples and illustrations are presented to prove the validity of the suggested method.

Keywords. Partial integro-differential equation, Sinc collocation method, Finite difference method, Product trapezoidal integration rule, Convection-diffusion equation.

2010 Mathematics Subject Classification. 65R99, 45A99, 45K05.

1. Introduction

We consider convection-diffusion equation with memory term of the form [10]

$$u_t(x,t) + mu_x(x,t) - bu_{xx}(x,t) = \int_0^t k_0\,(t-s)\,u\,(x,s)\,ds + f\,(x,t),$$

$$x \in \Omega = [0,1], t \in J = [0,T]. \tag{1.1}$$

Subjected to initial and boundary conditions,

$$\begin{aligned} u(0,t) &= u(1,t) = 0, \quad t \in J, \\ u(x,0) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{1.2}$$

where $m \geqslant 0$ and $b > 0$ are constants quantity that are denoted the advection(convection) and diffusion processes, respectively. Here $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, k_0 is assumed to be weakly singular kernel, i.e. satisfies $k_0(t) = t^{-\beta}$, $0 < \beta < 1$ and f is a given smooth function.

Received: 19 June 2017 ; Accepted: 12 May 2018.
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When the effects of the memory in the system are considered, the model involves the integral or partial differential operator containing the unknown function. Thus, the arising equation appears as partial integro-differential equation (PIDE), consists of partial differentiations and integral terms that is well known as memory term. Modeling phenomena in viscoelasticity, biological models, chemical kinetics, heat conduction in materials with memory, population dynamics, fluid dynamics and nuclear reactor dynamics, mathematical biology, financial mathematics, compression of visco-elastic media, and other similar areas are all done by partial integro-differential equations. See for example [4, 8] and the references cited within.

The convection-diffusion equation is a parabolic partial differential equation, which describes physical phenomena where the energy is transformed inside a physical system due to two processes, convection and diffusion. This takes place when the integral term in Eq. (1.1) is zero. When, it is necessary analyzed a heat transfer system where high-temperature gas passes through a porous medium placed in a duct, and showed that the porous medium can effectively convert the gas enthalpy into thermal radiation, convection-diffusion with memory term of the form Eq. (1.1) is performed [2, 15].

Partial integro-differential equations such as Eq. (1.1) are usually difficult to solve analytically. So, it is required to obtain an efficient approximate solution with use of numerical methods. One idea to solve Eq. (1.1) is discretization of the equation in time and spatial directions numerically. Sidiqqi and Arshed [10], used backward Euler method in combination with Euler's product integration rule in time and cubic B-spline collocation method in space. In Euler's product integration rule, a weakly singular integral, replaced with a quadrature formula that approximated the integrand by a piecewise constant function agreeing with the integrand at the left endpoint of each subinterval and an error of magnitude is $\mathcal{O}(\Delta t)$ [5].

In this paper, we apply the backward Euler method in time direction and in addition using product trapezoidal integration rule [5], for integral term. Consequently, Eq.(1.1) reduced to a system of ordinary differential equations (ODEs) that is discretized with sinc collocation method. In addition, the accuracy and efficiency of the suggested method will be tested with some examples and illustrations and compared with the method of Sidiqqi and Arshed [10].

The organization of this paper is as follows. Section 2, contains some notations, definitions, assumptions and preliminaries of sinc approximation. The description of the method in order to discrete Convection-Diffusion equation with memory term is devoted in two subsections in section 3. In section 4, the error analysis of the method is described in detail. Finally, in section 5, numerical examples are solved to verify the accuracy and efficiency of the proposed approach.

2. Preliminaries

The goal of this section is to recall notations and definitions of the sinc function and state some known theorems that are important for the rest of this paper which were discussed thoroughly in [6, 12, 13].

Sinc method is defined on the real line basically. So, for any $h > 0$, the translated sinc functions with evenly spaced nodes are given as

$$S(j,h)(z) = \operatorname{sinc}\left(\frac{z - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \ldots, \tag{2.1}$$

where, the sinc function is defined by

$$\operatorname{sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

The sinc function for the interpolating points $x_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

They are based on the infinite strip D_d in the complex plane

$$D_d = \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\}.$$

Let f be a function defined on $\mathbb{R}$, and $h > 0$ is the mesh size, then the Whittaker Cardinal function [11] is defined by the infinite series as follows:

$$C(f,h,x) = \sum_{j=-\infty}^{\infty} f(jh) S(j,h)(x). \tag{2.2}$$

But in practical, in order to coding the finite number of terms are used in the above series such as $j = -N, ..., N$, where $2N + 1$ is the number of sinc grid points. So, the approximation of Eq. (2.2) is rewritten as

$$C(f,h,x) \approx \sum_{j=-N}^{N} f(jh) S(j,h)(x),$$

where h suitably selected depending on properties of the function f and a given positive integer N.

To construct an approximation on the interval $\Gamma = (a,b)$, we consider the conformal map

$$\phi(z) = \log\left(\frac{z-a}{b-z}\right). \tag{2.3}$$

The map ϕ carries the eye-shaped region

$$D_E = \left\{ z = x + iy : \left| \arg\left(\frac{z-a}{b-z}\right) \right| < d \leqslant \frac{\pi}{2}, i^2 = -1 \right\},$$

onto D_d such that $\phi(a) = -\infty, \phi(b) = \infty$, where a, b are boundary points of D_E with $a, b \in \partial D_E$. For the sinc method on the interval $\Gamma = (a,b)$, basis functions are from the composite translated sinc functions,

$$S_j(z) = S(j,h) \circ (\phi(z)) = \operatorname{sinc}\left(\frac{\phi(z) - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \ldots.$$

The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \frac{a + be^{w}}{1 + e^{w}}.$$

Let ψ denote the inverse map of ϕ, so we define the range of ϕ^{-1} on the real line as

$$\Gamma = \left\{\psi(u) = \phi^{-1}(u) \in D_E : -\infty < u < \infty\right\} = (a, b).$$

For $h > 0$, let the points x_k on Γ given by

$$x_k = \psi(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \ k \in \mathbb{Z}. \tag{2.4}$$

Definition 2.1. ([13]) Let $L_\alpha(D_E)$ be the set of all analytic functions f, for which there exists a constant, C, such that

$$|f(z)| \leq C\frac{|\rho(z)|^{\alpha}}{(1 + |\rho(z)|)^{2\alpha}}, \quad z \in D_E, \quad 0 < \alpha \leq 1, \tag{2.5}$$

where $\rho(z) = e^{\phi(z)}$.

Definition 2.2. ([11]) Let $B(D_E)$ is the class of functions f which are analytic in D_E such that

$$\int_{\psi(u+\sum)} |f(z)|\,|dz| \to 0, \quad as \quad u \to \pm\infty,$$

where $\sum = \left\{\cancel{i}\eta : |\eta| < d \leqslant \frac{\pi}{2}, \cancel{i}^2 = -1\right\}$ and satisfy

$$\aleph(f) \equiv \int_{\partial D_E} |f(z)|\,|dz| < \infty,$$

where ∂D_E represents the boundary of D_E.

Theorem 2.3. *([13]) Let $F \in L_\alpha(D_E)$, let N be a positive integer, and let h be selected by the formula*

$$h = \frac{c}{\sqrt{N}},$$

with c a positive constant. Then there exist positive constants C and κ, independent of N, such that

$$\sup_{z\in\Gamma}\left|F(z) - \sum_{k=-N}^{N} F(z_k)\, S(k,h) \circ \phi(z)\right| \leqslant Ce^{-\kappa\sqrt{N}}.$$

The sinc collocation method requires that the derivatives of composite sinc function be evaluated at the nodes. So, we need to recall the following lemma.

Lemma 2.4. *([6], p. 106) Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d , given by (2.3). Then*

$$\delta_{jk}^{(0)} = \left.[S(j,h) \circ \phi(x)]\right|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{2.6}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}\left[S\left(j,h\right)\circ\phi\left(x\right)\right]|_{x=x_k} = \begin{cases} 0, & j=k, \\ \dfrac{(-1)^{k-j}}{k-j}, & j\neq k, \end{cases} \tag{2.7}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}\left[S\left(j,h\right)\circ\phi\left(x\right)\right]|_{x=x_k} = \begin{cases} \dfrac{-\pi^2}{3}, & j=k, \\ \dfrac{-2(-1)^{k-j}}{(k-j)^2}, & j\neq k. \end{cases} \tag{2.8}$$

In Eqs. (2.6)-(2.8), h is step size and x_k is sinc grid given by (2.4).

3. Description of the method

In this section, a description of the spatial-temporal discretization about this type of equations is provided in detail.

3.1. **Discretization in time:** The backward Euler method is applied for time-derivatives in Eq. (1.1). Let $t_n = n\Delta t$ with $\Delta t = \frac{T}{K}, K\in\mathbb{N}$ being the time step and $u^n = u(x,t_n)$ and $f^n = f(x,t_n)$ for $n = 0,1,...,M, 1\leqslant M\leqslant K$. By replacing $t = t_{n+1}$ into the left hand side of Eq. (1.1) for the first term we get

$$u_t\left(x,t_{n+1}\right) \approx \frac{u^{n+1}\left(x\right)-u^n\left(x\right)}{\Delta t}, \quad 0<x<1, \quad n\geqslant 0, \tag{3.1}$$

and the integral term of (1.1) can be approximated by the product trapezoidal integration rule [5] as follows:

$$\begin{aligned} \int_0^{t_{n+1}}\left(t_{n+1}-s\right)^{-\beta}u\left(x,s\right)ds &= \sum_{l=0}^{n}\int_{t_l}^{t_{l+1}}\left(t_{n+1}-s\right)^{-\beta}u\left(x,s\right)ds \\ \approx \sum_{l=0}^{n}\int_{t_l}^{t_{l+1}}\left(t_{n+1}-s\right)^{-\beta}&\left\{\frac{t_{l+1}-s}{\Delta t}u^l(x)+\frac{s-t_l}{\Delta t}u^{l+1}(x)\right\}ds \\ &= \frac{1}{\Delta t}\sum_{l=0}^{n}\left(\lambda_{n,l}u^l(x)+\eta_{n,l}u^{l+1}(x)\right), \end{aligned} \tag{3.2}$$

where,

$$\begin{aligned} \lambda_{n,l} &= \int_{t_l}^{t_{l+1}}\left(t_{n+1}-s\right)^{-\beta}\left(t_{l+1}-s\right)ds, \\ \eta_{n,l} &= \int_{t_l}^{t_{l+1}}\left(t_{n+1}-s\right)^{-\beta}\left(s-t_l\right)ds. \end{aligned} \tag{3.3}$$

Substituting Eqs. (3.1) and Eq. (3.2) into Eq. (1.1), we can get the temporal semi-discrete form of Eq. (1.1), as follows:

$$\begin{aligned}&(1-\eta_{n,n})\, u^{n+1}(x)+\Delta t\left(m u_x^{n+1}(x)-b u_{xx}^{n+1}(x)\right)\\&\quad= u^n(x)+\Delta t f^{n+1}(x)+\sum_{l=0}^{n}\rho_{n,l} u^l(x), 0<x<1, n\geqslant 0,\end{aligned} \tag{3.4}$$

$$u^{n+1}(0)=0, \quad u^{n+1}(1)=0, \tag{3.5}$$

where,

$$\begin{aligned}\rho_{n,0} &= \lambda_{n,0},\\ \rho_{n,l} &= \lambda_{n,l}+\eta_{n,l-1}, \quad l=1,2,\ldots,n,\end{aligned} \tag{3.6}$$

and with additional initial condition

$$u^0(x)=u_0(x).$$

As a consequence, a linear ordinary differential equation is obtained in the form of (3.4) with boundary conditions (3.5) in each time level. Now, we can use the sinc collocation method to estimate the solution of this linear boundary value problem.

3.2. **Discretization in space:** We discretize the spatial direction by the described sinc collocation method. Assume that the approximate solution of (3.4) defined by

$$u_m^n(x)=\sum_{j=-N}^{N} c_j^n S(j,h)\circ\phi(x), \quad m=2N+1, \tag{3.7}$$

and

$$\phi(x)=\log\left(\frac{x}{1-x}\right),$$

and the unknown coefficient c_j^n in Eq. (3.7) are determined by sinc collocation method. The collocation points are as

$$x_i=\frac{e^{ih}}{1+e^{ih}}, \quad i=-N,\ldots,N, \quad h=\frac{c}{\sqrt{N}}, \tag{3.8}$$

so,

$$\begin{aligned}\frac{d}{dx}u_m^n(x) &= \sum_{j=-N}^{N} c_j^n \frac{d}{dx}\left[S(j,h)\circ\phi(x)\right]\\ &= \sum_{j=-N}^{N} c_j^n\left[\phi'(x)\, S_j^{(1)}(x)\right],\end{aligned} \tag{3.9}$$

and,

$$\begin{aligned}\frac{d^2}{dx^2}u_m^n(x) &= \sum_{j=-N}^{N} c_j^n \frac{d^2}{dx^2}\left[S(j,h)\circ\phi(x)\right]\\ &= \sum_{j=-N}^{N} c_j^n\left[\phi''(x)\, S_j^{(1)}(x)+(\phi'(x))^2 S_j^{(2)}(x)\right],\end{aligned} \tag{3.10}$$

where,

$$S_j^{(l)}(x) = \frac{d^l}{d\phi^l}\left[S(j,h)\circ\phi(x)\right], \quad l = 1, 2,$$

thus, by applying notations in Lemma (2.4),

$$\frac{d}{dx}u_m^n(x_i) = \sum_{j=-N}^{N} c_j^n \left[\phi'(x_i)\frac{\delta_{ji}^{(1)}}{h}\right], \tag{3.11}$$

and,

$$\frac{d^2}{dx^2}u_m^n(x_i) = \sum_{j=-N}^{N} c_j^n \left[\phi''(x_i)\frac{\delta_{ji}^{(1)}}{h} + (\phi'(x_i))^2\frac{\delta_{ji}^{(2)}}{h^2}\right]. \tag{3.12}$$

Substituting (3.7), (3.11) and (3.12) into (3.4) we obtain,

$$\begin{aligned}(1-\eta_{n,n})\sum_{j=-N}^{N} c_j^{n+1}\delta_{ji}^{(0)} &+ m\Delta t\sum_{j=-N}^{N} c_j^{n+1}\left[\phi'(x_i)\frac{\delta_{ji}^{(1)}}{h}\right]\\ &-b\Delta t\sum_{j=-N}^{N} c_j^{n+1}\left[\phi''(x_i)\frac{\delta_{ji}^{(1)}}{h} + (\phi'(x_i))^2\frac{\delta_{ji}^{(2)}}{h^2}\right]\\ &= c_i^n + \Delta t f_i^{n+1} + \sum_{l=0}^{n}\sum_{j=-N}^{N}\rho_{n,l}c_j^l\delta_{ji}^{(0)},\end{aligned} \tag{3.13}$$

with additional initial condition,

$$c_i^0 = u_0(x_i), \quad i = -N, \dots, N. \tag{3.14}$$

We note that $\delta_{ji}^{(0)} = \delta_{ij}^{(0)}$, $\delta_{ji}^{(1)} = -\delta_{ij}^{(1)}$ and $\delta_{ji}^{(2)} = \delta_{ij}^{(2)}$. We denote $I^{(r)} = [\delta_{ij}^{(r)}], r = 0, 1, 2$ where $I^{(0)}$ is identity matrix and $I^{(1)}$, $I^{(2)}$ are symmetric and skew-symmetric Toplitz matrix of order $2N+1$ respectively. We define the $(2N+1)\times(2N+1)$ diagonal matrix as follows:

$$D(g(x))_{ij} = \begin{cases} g(x_i), & i = j,\\ 0, & i \neq j.\end{cases} \tag{3.15}$$

By multiplying both sides of (3.13) in $\frac{1}{(\phi'(x_i))^2}$ we conclude

$$\begin{aligned}(1-\eta_{n,n})\left(\frac{1}{(\phi'(x_i))^2}\right)c_i^{n+1} &- m\Delta t\sum_{j=-N}^{N}\left[\left(\frac{1}{\phi'(x_i)}\right)\frac{\delta_{ij}^{(1)}}{h}\right]c_j^{n+1}\\ &-b\Delta t\sum_{j=-N}^{N}\left[\left(\frac{-\phi''(x_i)}{(\phi'(x_i))^2}\right)\frac{\delta_{ij}^{(1)}}{h} + \frac{\delta_{ij}^{(2)}}{h^2}\right]c_j^{n+1}\\ = \left(\frac{1}{(\phi'(x_i))^2}\right)c_i^n &+ \Delta t\left(\frac{1}{(\phi'(x_i))^2}\right)f_i^{n+1} + \sum_{l=0}^{n}\rho_{n,l}\left(\frac{1}{(\phi'(x_i))^2}\right)c_i^l.\end{aligned} \tag{3.16}$$

Therefore, the system (3.16) can be denoted by the following matrix form:

$$PC^{n+1} = Q, \tag{3.17}$$

where,

$$P = (1-\eta_{n,n})\, D\left(\left(\frac{1}{\phi'}\right)^2\right) I^{(0)} - \Delta t\left(\frac{m}{h} D\left(\frac{1}{\phi'}\right) + \frac{b}{h} D\left(\left(\frac{1}{\phi'}\right)'\right) I^{(1)} + \frac{b}{h^2} I^{(2)}\right),$$

$$Q = D\left(\left(\frac{1}{\phi'}\right)^2\right)\left(C^n + \Delta t F^{n+1}\right) + \sum_{l=0}^{n} \rho_{n,l} D\left(\left(\frac{1}{\phi'}\right)^2\right) C^l, \tag{3.18}$$

and,

$$\begin{aligned} C^{n+1} &= \left(c_{-N}^{n+1}, c_{-N+1}^{n+1}, \ldots, c_N^{n+1}\right)^t, \\ F^{n+1} &= \left(f_{-N}^{n+1}, f_{-N+1}^{n+1}, \ldots, f_N^{n+1}\right)^t, \end{aligned} \tag{3.19}$$

with additional initial condition,

$$C^0 = \left(u_0\left(x_{-N}\right), u_0\left(x_{-N+1}\right), \ldots, u_0\left(x_N\right)\right)^t. \tag{3.20}$$

For each n, system (3.17) is a linear system of equations which consists of $2N+1$ equations and $2N+1$ unknowns. The coefficients c_j^n, in the approximate solution (3.7), can be determined by solving (3.17).

4. Error Analysis

When we apply backward Euler method in Eq. (1.1), u_t and integral term are approximated as follow:

$$u_t\left(x, t_{n+1}\right) \approx \frac{u^{n+1}(x) - u^n(x)}{\Delta t} + \mathcal{O}(\Delta t), \tag{4.1}$$

and,

$$\int_0^{t_{n+1}} \left(t_{n+1}-s\right)^{-\beta} u(x,s)\, ds \approx \frac{1}{\Delta t}\sum_{l=0}^{n}\left(\lambda_{n,l} u^l + \eta_{n,l} u^{l+1}\right) + \mathcal{O}(\Delta t^{2-\beta}), \tag{4.2}$$

the order of product trapezoidal integration rule is $\mathcal{O}(\Delta t^{2-\beta})$ that is proved by Dixon in [1].

Lemma 4.1. [3] *The following estimates hold*

$$\begin{aligned} u(x,t) &\approx u(x_i,t) + C_1 e^{-\kappa\sqrt{N}}, \\ u_x(x,t) &\approx u_x(x_i,t) + C_2 e^{-\kappa\sqrt{N}}, \\ u_{xx}(x,t) &\approx u_{xx}(x_i,t) + C_3 e^{-\kappa\sqrt{N}}, \end{aligned}$$

where $\kappa > 0$, $-N \leqslant i \leqslant N$, C_1, C_2, and C_3 are constants independent of N.

The following theorem gives the truncation error for proposed approach.

Theorem 4.2. *By using the finite difference method and product integration rule in combination with* sinc *collocation method the truncation error of the proposed approach is* $\mathcal{O}((\Delta t)+e^{-\kappa\sqrt{N}})$.

Proof. Replacing Eqs. (4.1) and (4.2) into Eq. (1.1) and discretization in time, we obtain,

$$u_t\left(x,t_{n+1}\right)+\mathcal{O}\left(\Delta t\right)-\int_0^{t_{n+1}}\left(t_{n+1}-s\right)^{-\beta}u\left(x,s\right)ds+\mathcal{O}\left(\Delta t^{2-\beta}\right)=\chi(x,t_{n+1}), \tag{4.3}$$

where,

$$\chi(x,t)=f\left(x,t\right)-mu_x\left(x,t\right)+bu_{xx}\left(x,t\right). \tag{4.4}$$

From lemma 4.1 and discretization in spatial direction,

$$\chi(x_i,t_{n+1})=f\left(x_i,t_{n+1}\right)-m\left(u_x\left(x_i,t_{n+1}\right)+c_1e^{-\kappa\sqrt{N}}\right)+b\left(u_{xx}\left(x_i,t_{n+1}\right)+c_2e^{-\kappa\sqrt{N}}\right), \tag{4.5}$$

if we suppose that the truncation error estimated as

$$TE(x,t)=u_t-\int_0^{t}\left(t-s\right)^{-\beta}u\left(x,s\right)ds-\chi(x,t),$$

then,

$$TE(x_i,t_{n+1})=\mathcal{O}\left(\Delta t\right)+\mathcal{O}\left(\Delta t^{2-\beta}\right)+(mc_1-bc_2)e^{-\kappa\sqrt{N}},$$

so, the order of error is $\mathcal{O}((\Delta t)+e^{-\kappa\sqrt{N}})$. □

Algorithm 1

1: Input $T, K, M, N, u_0(x), f(x,t), u_{ex}(x,t)$,

2: Set $x_i := \dfrac{e^{ih}}{1+e^{ih}}, i = -N, ..., N, z_k := kp, k = 0, \ldots, N$,

3: Set $t_j := j\Delta t, j = 0, 1, ..., M$,

4: Compute $u_{ex}(x_i, t_j), u_{ex}(z_k, t_j)$,

5: Compute $u_{app}(x_i, t_j)$, as follows:

6: Set $u_{app}(x_i, t_0) := c_i^0, i = -N, ..., N$, based on Eq. (3.20)

  for $j = 0 : M - 1$ do

    for $i = -N : N$ do

      $u_{app}(x_i, t_{j+1}) := c_i^{j+1}$, by applying Eqs. (3.17) and (3.18)

    end do

  end do

7: Set $u_{app}(z_k, t_M) := \sum_{j=-N}^{N} c_j^M S(j,h) \circ \phi(z_k)$, by applying (3.7)

8: $error1(i) := |u_{ex}(x_i, t_M) - u_{app}(x_i, t_M)|, i = -N, ..., N$,

  $error2(k) := |u_{ex}(z_k, t_M) - u_{app}(z_k, t_M)|, k = 0, ..., N$,

  $error3 := \frac{1}{2N+1}(\sum_{i=-N}^{N} |u_{app}(x_i, t_M) - u_{ex}(x_i, t_M)|^2)^{\frac{1}{2}}$,

  $error4 := \frac{1}{N+1}(\sum_{k=0}^{N} |u_{app}(z_k, t_M) - u_{ex}(z_k, t_M)|^2)^{\frac{1}{2}}$,

9: Print $\|E_M\|_\infty := max(error1(i)), i = -N, ..., N$,

  Print $\|e_M\|_\infty := max(error2(k)), k = 0, ..., N$,

  Print $\|E_M\|_2 := error3$,

  Print $\|e_M\|_2 := error4$.

5. Numerical Results

In this section, the numerical experiments of the proposed approach are provided. In all examples, we set parameters $c = \sqrt{\pi d/\alpha}$, $d = \frac{\pi}{2}$ and $\alpha = 1$ and denote computed solution and exact solution by u_{app} and u_{ex}, respectively. Let $t_n = n\Delta t, n = 0, 1, \ldots, M$, which M denotes the last time t_M and Δt is the small time step. The maximum error norm and Euclidian error norm in points $z_k = kp, k = 0, 1, ..., N, p = \frac{1}{N}$ between the computed and exact solutions are given as $\|e_M\|_\infty$ and $\|e_M\|_2$, respectively. Also, the maximum error norm and Euclidian error norm in points $x_i = \frac{e^{ih}}{1+e^{ih}}, i = -N, ..., N, h = \frac{c}{\sqrt{N}}$ are given as $\|E_M\|_\infty$ and $\|E_M\|_2$, respectively as follows:

$$\begin{aligned}
\|e_M\|_\infty &= \underset{k}{Max}\, |u_{app}(z_k, t_M) - u_{ex}(z_k, t_M)|, \quad k = 0, \ldots, N,\\
\|E_M\|_\infty &= \underset{i}{Max}\, |u_{app}(x_i, t_M) - u_{ex}(x_i, t_M)|, \quad i = -N, \ldots, N,\\
\|e_M\|_2 &= \frac{1}{N+1}(\sum_{k=0}^{N} |u_{app}(z_k, t_M) - u_{ex}(z_k, t_M)|^2)^{\frac{1}{2}},\\
\|E_M\|_2 &= \frac{1}{2N+1}(\sum_{i=-N}^{N} |u_{app}(x_i, t_M) - u_{ex}(x_i, t_M)|^2)^{\frac{1}{2}}.
\end{aligned}$$

TABLE 1. Results of example 1.

		$\Delta t = 10^{-3}$	$\Delta t = 10^{-4}$		$\Delta t = 10^{-5}$	
$\|.\|$	M	proposed approach	proposed approach	method in [10]	proposed approach	method in [10]
$\|e_M\|_\infty$	10	$1.9089E-04$	$1.5230E-05$	$2.6417E-05$	$8.8082E-07$	$6.7929E-06$
	50	$1.2601E-03$	$4.2241E-05$	$3.9890E-05$	$1.0096E-05$	$2.0040E-05$
	100	$2.6432E-03$	$9.0542E-05$	$4.4030E-05$	$1.5412E-05$	$2.6950E-05$
	500	$1.3863E-02$	$4.8595E-04$	$5.3632E-05$	$2.5124E-05$	$4.0214E-05$
$\|e_M\|_2$	10	$1.8904E-05$	$7.6715E-07$	$5.6276E-07$	$3.1893E-08$	$1.3756E-07$
	50	$1.2422E-04$	$4.3127E-06$	$1.0863E-06$	$3.1303E-07$	$4.1125E-07$
	100	$2.6009E-04$	$9.1029E-06$	$1.3976E-06$	$5.3657E-07$	$5.8715E-07$
	500	$1.3611E-03$	$4.8350E-05$	$2.8654E-06$	$1.8721E-06$	$1.2581E-06$

In order to implement the proposed approach, algorithm 1 is given.

The linear algebraic system in step 6 of algorithm 1 is solved directly by using "linsolve" command from "LinearAlgebra" package in Matlab R2014a software and all the calculations were supported by intel CORE Dual-Core at 2.20 GHz CPU with 4 GB RAM.

Example 1. Consider Eq. (1.1), when $m = 0.05$, $b = 0.4$, $\beta = \frac{1}{2}$, $u_0(x) = \sin(\pi x)$ and

$$f(x,t) = \left(2(1+t) + b\pi^2(1+t)^2 - 2\sqrt{t}\left(1 + \frac{4}{3}t + \frac{8}{15}t^2\right)\right)\sin(\pi x) + m\pi(1+t)^2\cos(\pi x).$$

The analytic solution is given by $u(x,t) = (1+t)^2\sin(\pi x)$ [10]. In Table 1, maximum and Euclidian error norms of the presented approach with regarding $N = 25$ are compared with cubic B-spline collocation method in [10] when $N = 50$, for $T = 1$ and $\Delta t = 10^{-3}$, $\Delta t = 10^{-4}$ and $\Delta t = 10^{-5}$ in the different time levels t_M. This table shows that, the presented approach is in good agreement with the method in [10]. In Table 2, we set $\Delta t = 10^{-5}$ for different values of N. Results of this table demonstrate the exponential convergence rate of the proposed approach by increasing values of N. Furthermore, in this table condition number with use of Euclidian norm and CPU time based on second are reported. Convergence curves of Table 2 are plotted in Figure 1. Based on the idea mentioned in [9], Figure 2 reports that the sinc collocation method achieves an approximation convergence rate because the $|c_j^{100}|$ falls off when $|j|$ tends to N and also, shows that if N is increased, the approximate solution converges to the analytic solution. In addition, the analytic and computed solutions in the different values of x is ploted in Figure 3 (a), and the maximum error norm in the sinc collocation points is ploted in Figure 3 (b) for $N = 35$ and $\Delta t = 10^{-5}$.

Example 2. Consider Eq. (1.1), when $m = 0.005$, $b = 0.5$, $\beta = \frac{1}{3}$, $u_0(x) = 1 - \cos(2\pi x) + 2\pi^2 x(1-x)$ and

$$f(x,t) = \left(1 - \cos(2\pi x) + 2\pi^2 x(1-x)\right)\left(2(t+1) - \frac{(t+1)^2 t^{1-\beta}}{1-\beta} + \frac{2(t+1)t^{2-\beta}}{2-\beta} - \frac{t^{3-\beta}}{3-\beta}\right)$$
$$+ \left(2(m+2b)\pi^2 - 4m\pi^2 x + 2m\pi\sin(2\pi x) - 4b\pi^2\cos(2\pi x)\right)(t+1)^2.$$

The exact solution is given by [10]

$$u(x,t) = \left(1 - \cos(2\pi x) + 2\pi^2 x(1-x)\right)(t+1)^2.$$

TABLE 2. Results of example 1 at $t = 0.001$.

$\Delta t = 10^{-5}$, $M = 100$				
N	$\|E_M\|_\infty$	$\|E_M\|_2$	$Cond(P)$	$CPUtime(s)$
5	8.8737×10^{-3}	1.3276×10^{-3}	1.04×10^{3}	0.093
10	1.4835×10^{-3}	1.9793×10^{-4}	6.26×10^{3}	0.099
15	3.5188×10^{-4}	4.2850×10^{-5}	1.49×10^{4}	0.104
20	9.9114×10^{-5}	1.1222×10^{-5}	2.49×10^{4}	17.280
25	3.2173×10^{-5}	3.4096×10^{-6}	3.50×10^{4}	20.363
30	1.1519×10^{-5}	1.1572×10^{-6}	4.48×10^{4}	20.912
35	4.4614×10^{-6}	4.3661×10^{-7}	5.41×10^{4}	34.437
40	2.9079×10^{-6}	1.9241×10^{-7}	6.28×10^{4}	38.519

FIGURE 1. Convergence curve for example 1 at $t = 0.001$.

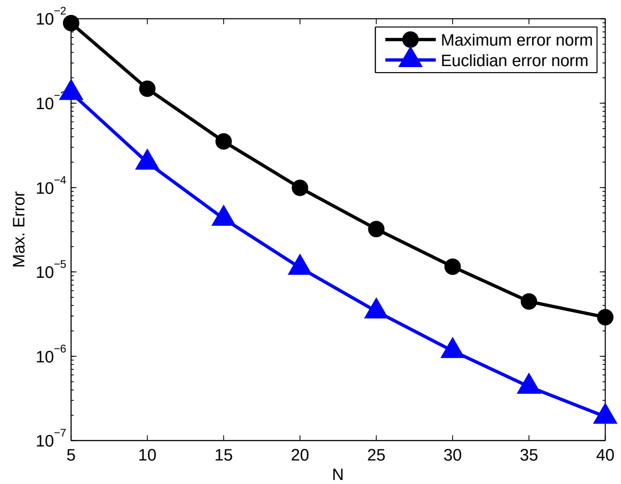


TABLE 3. Results of example 2.

		$\Delta t = 10^{-3}$	$\Delta t = 10^{-4}$		$\Delta t = 10^{-5}$	
$\|.\|$	M	proposed approach	proposed approach	method in [10]	proposed approach	method in [10]
$\|e_M\|_\infty$	10	$2.4967E-04$	$6.2488E-06$	$6.6290E-06$	$1.7842E-07$	$1.9020E-06$
	50	$1.8686E-03$	$4.6559E-05$	$3.1392E-05$	$1.0502E-06$	$3.0941E-06$
	100	$3.9969E-03$	$1.0301E-04$	$5.9245E-05$	$2.3308E-06$	$5.9627E-06$
	500	$2.0120E-02$	$5.7307E-04$	$2.2874E-04$	$1.3555E-05$	$2.6198E-05$
$\|e_M\|_2$	10	$2.4059E-05$	$6.0246E-07$	$5.0793E-07$	$1.4072E-08$	$4.5422E-08$
	50	$1.8132E-04$	$4.4850E-06$	$2.4155E-06$	$1.0410E-07$	$2.0137E-07$
	100	$3.8922E-04$	$9.9352E-06$	$4.6368E-06$	$2.2789E-07$	$3.9130E-07$
	500	$1.9666E-03$	$5.5665E-05$	$1.9337E-05$	$1.3121E-06$	$1.7531E-06$

The numerical solutions for $N = 50$, $T = 1$ with $\Delta t = 10^{-3}$, $\Delta t = 10^{-4}$ and $\Delta t = 10^{-5}$ at different time levels t_M of the proposed approach are listed and compared with cubic B-spline collocation method in [10] with $N = 100$ in Table 3. This table shows the computed solution is in good agreement with the numerical results reported in [10], when Δt is selected small enough. In Table 4, we set $\Delta t = 10^{-5}$ for different values of N, that illustrates maximum and Euclidian errors decrease when N

FIGURE 2. Plot of $|c_j^{100}|$ coefficients for example 1 defined by Eq. (3.7) for $N = 5, 10, 20$ and $\Delta t = 10^{-5}$.

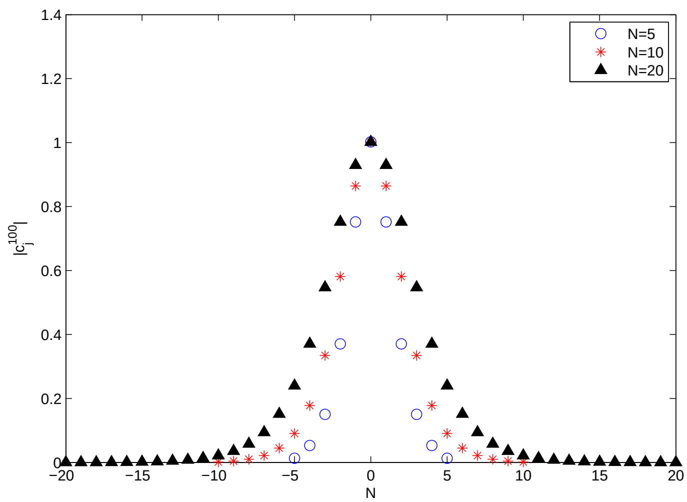
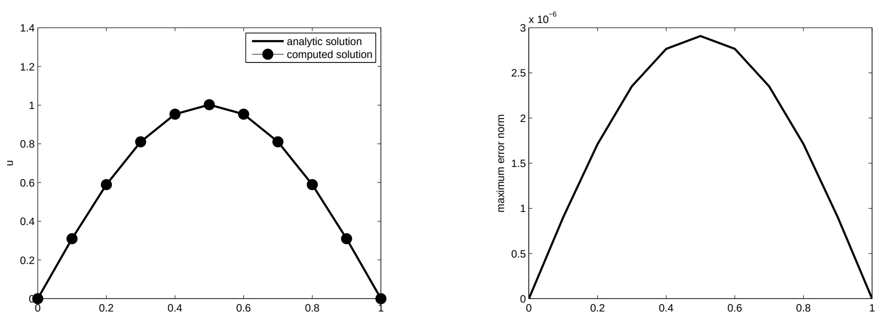


FIGURE 3. (a) The analytic and computed solutions and (b) the maximum error norm at $t = 0.001$ for example 1.



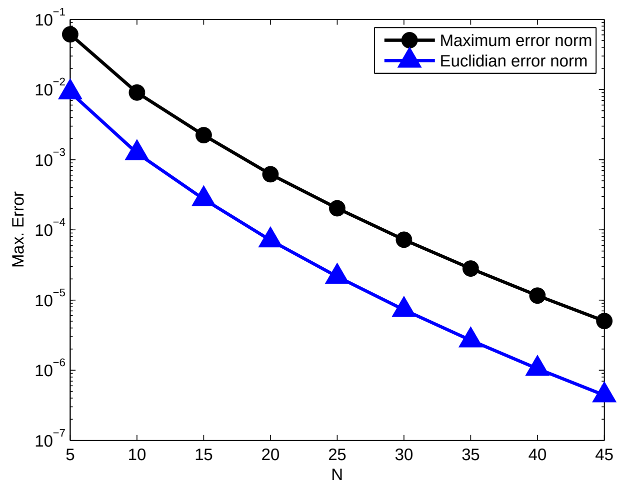
(a) (b)

increases. This table demonstrates the convergence rate in sinc method is exponential. Furthermore, condition number with use of Euclidian norm and CPU time based on second is reported in this table. in Figure 1, convergence curves of Table 2 are plotted. Figure 5 shows that the sinc collocation method achives an approximation convergence rate because the $|c_j^{100}|$ falls off when $|j|$ tends to N. In Figure 6 (a) the analytic and computed solutions for different values of x are ploted and in Figure

Table 4. Results of example 2 at $t = 0.001$.

	$\Delta t = 10^{-5}$, $M = 100$			
N	$\|E_M\|_\infty$	$\|E_M\|_2$	$Cond(P)$	$CPUtime(s)$
5	6.1135×10^{-2}	9.1320×10^{-3}	9.90×10^{2}	0.093
10	9.0552×10^{-3}	1.2442×10^{-3}	5.43×10^{3}	0.101
15	2.2376×10^{-3}	2.7474×10^{-4}	1.26×10^{4}	0.110
20	6.1945×10^{-4}	7.0968×10^{-5}	2.07×10^{4}	0.122
25	2.0268×10^{-4}	2.1606×10^{-5}	2.89×10^{4}	0.126
30	7.2279×10^{-5}	7.2665×10^{-6}	3.67×10^{4}	0.127
35	2.8052×10^{-5}	2.6709×10^{-6}	4.42×10^{4}	0.157
40	1.1546×10^{-5}	1.0505×10^{-6}	5.12×10^{4}	0.162
45	5.0140×10^{-6}	4.4109×10^{-7}	5.78×10^{4}	0.210

Figure 4. Convergence curve for example 2 at $t = 0.001$.



6 (b) the maximum error norm in sinc collocation points is ploted for $\Delta t = 10^{-5}$, $N = 50$.

6. Conclusions

In this paper, finite difference method in combination with product trapezoidal integration rule was used to discretize the Convection-Diffusion equation in time. Then, the sinc collocation method was applied to solve outcoming equation in spatial direction. To illustrate effectiveness of the given approach, some examples were solved based on the proposed algorithm. Also, the error of the scheme was given. The results show that the proposed approach and the cubic B-spline collocation method for Convection-Diffusion equation are the efficient methods and have a same rather numerical results. Furthermore, the proposed approach is efficient for different values of β , $(0 < \beta < 1)$.

FIGURE 5. Plot of $|c_j^{100}|$ coefficients for example 2 defined by Eq. (3.7) for $N = 5, 10, 20, 30$ and $\Delta t = 10^{-5}$.

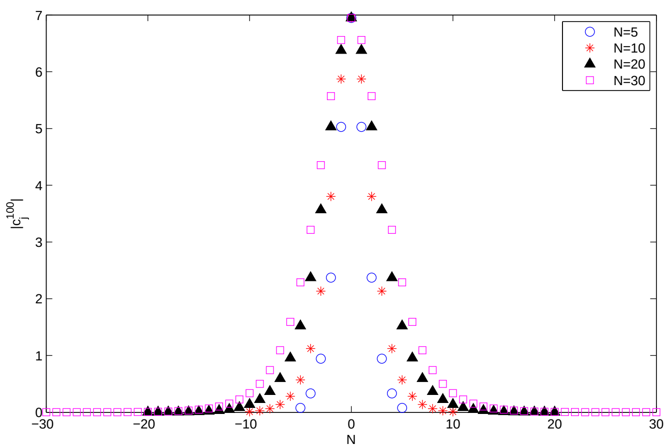
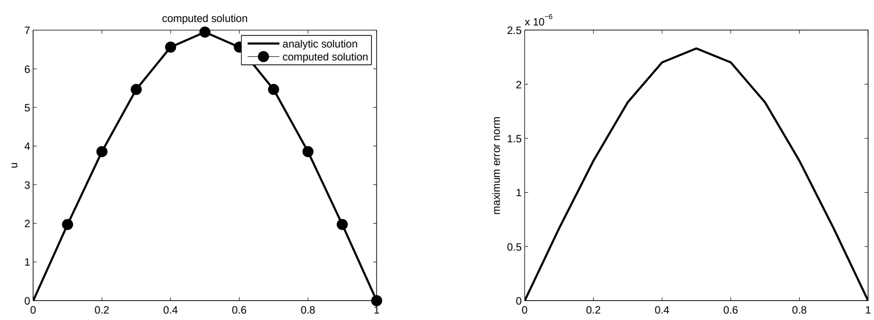


FIGURE 6. (a) The analytic and computed solutions and (b) the maximum error norm for example 2 at $t = 0.001$.



(a) (b)

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their constructive comments to improve the quality of this work.

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