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Approximate solution of the fuzzy fractional Bagley–Torvik equation by the RBF collocation method

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Abstract In this paper, we propose the spectral collocation method based on radial basis functions to solve the fractional Bagley-Torvik equation under uncertainty, in the fuzzy Caputo's H-differentiability sense with order $(1 < \nu < 2)$. We define the fuzzy Caputo's H-differentiability sense with order ν $(1 < \nu < 2)$, and employ the collocation RBF method for upper and lower approximate solutions. The main advantage of this approach is that the fuzzy fractional Bagley-Torvik equation is reduced to the problem of solving two systems of linear equations. Determining a good shape parameter is still an outstanding research topic. To eliminate the effects of the radial basis function shape parameter, we use thin plate spline radial basis functions which have no shape parameter, and also we use the variable shape parameter for Matérn radial basis function which give almost optimal shape parameter. The numerical investigation is presented in this paper shows that excellent accuracy can be obtained even when few nodes are used in analysis. Efficiency and effectiveness of the proposed procedure is examined by solving two benchmark problems.

Keywords. The fractional Bagley-Torvik equation, Meshless method, RBF collocation, Thin plate splines, Fuzzy theory, Fuzzy Caputo's H-differentiability.

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1. Introduction

Fractional-order differential equations are encountered in many fields in science and engineering such as viscoelasticity, heat conduction, electrode-electrolyte polarization, electromagnetic waves, diffusion wave, and control theory. Since, it is too difficult to obtain the exact solution of fractional differential equation so, one may need a reliable and efficient numerical scheme for the solution of fractional differential equations.

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Many important works have been reported regarding fractional calculus in the last few decades. Relating to this field several excellent books have also been written by different authors representing the scope and various aspects of fractional calculus such as in [37, 46, 49, 53, 56]. Fractional-order derivatives have been successfully used to model damping forces with memory effect or to describe state feedback controllers [9, 10, 49, 59]. The Bagley-Torvik equation with 1/2-order derivative or 3/2-order derivative describes motion of real physical systems, an immersed plate in a Newtonian fluid and a gas in a fluid, respectively [9, 10, 49, 52]. The motion of a rigid plate of mass m and area A connected by a mass less spring of stiffness k, immersed in a Newtonian fluid, was originally proposed by Bagley and Torvik.

Let μ be the viscosity and ρ be the fluid density. The displacement of the plate u is described by Bagley-Torvik equation

$$Au''(t) + BD^{3/2}u(t) + Cu(t) = f(t), \qquad 0 \leq t \leq T, \tag{1.1}$$

with the initial conditions

$$u(0) = b_0, \qquad u'(0) = b_1, \tag{1.2}$$

where $A = m$, $B = 2A\sqrt{\mu\rho}$, $C = k$, and $D^{3/2}$ is Caputo-type fractional derivative. Podlubny [49] gave the analytical solution of the Bagley-Torvik equation with homogeneous initial conditions by using Green's function. But, in practice, these equations can not be evaluated easily for different functions $f(t)$. Ray and Bera [52] have introduced Adomian decomposition method for the analytical solution of the Bagley-Torvik equation. Recently, the Eq. (1.1) has been numerically solved by the generalized Taylor collocation method [17], the Bessel collocation method [58], and the Haar wavelet operational matrix [51]. J. Cermk and T. Kisela et al. [18] have proved exact and discretized stability of the Bagley-Torvik equation. Also, there have been numerous investigations to numerically solve some fractional differential equations, based on the multi-domain spectral method [19], the modified generalized Laguerre operational matrix [14], the spline [47], the predictor-corrector approach [22], and the radial basis function [36], etc.

In recent years, meshless methods have been used in many different areas ranging from geology, biology, physical and engineering sciences, applied mathematics, computer science, and business studies [27, 29, 30, 61]. Meshless methods use a set of uniform or random points which are not necessarily interconnected in the form of a mesh. One class of meshless methods are radial basis functions (RBFs) collocation methods which use radial functions as the basis functions for the collocation.

The use of meshless methods based on radial basis functions has gained popularity in science community for a number of reasons. The most prominent characteristics of these methods are: meshfree character and flexibility in dealing with complex geometries and easy extension to multi-dimensional problems. A radial basis function depends upon the separation distances of a subset of trial centres. Sums of RBFs are typically used to approximate given functions. This approximation process can also be interpreted as a simple kind of neural network. RBFs are also used as a kernel in support vector classification.

The theory of radial basis functions has under gone intensive research and enjoyed considerable success as a technique for interpolating multivariable data and functions. RBF approximation method is a generalization of multiquadric (MQ) method developed by a geologist, Hardy in 1968 and was successfully applied in interpolation of cartographic scattered data [31]. It was not recognized by most of the academic researchers until Franke [26] published a review paper in the evaluation of two-dimensional interpolation methods. This was a motivation for Kansa [34, 35] to develop Hardy's method to approximate the solutions of elliptic, parabolic and hyperbolic PDEs. Kansa's method was recently extended to solve various ordinary and partial differential equations [24, 25]. Some examples of popular choices of RBFs are given in Table 1.

Structural design and analysis plays a vital role for the structural safety. Most of the structures fail due to the poor design. In the design process the system parameters involved such as mass, geometry, material properties, external loads, or initial conditions are considered as crisp or defined exactly. But, rather than the particular value we may have only the vague, imprecise and incomplete information about the variables and parameters being a result of errors in measurement, observations, experiment, applying different operating conditions or it may be maintenance induced error, etc. which are uncertain in nature. Basically, these uncertainties can be modelled through probabilistic, interval and fuzzy theory.

In probabilistic practice, the variables of uncertain nature are assumed as random variables with joint probability density functions. If the structural parameters and the external load are modeled as random variables with known probability density functions, the response of the structure can be predicted using the theory of probability and stochastic processes which have been studied by Elishakoff [23]. Unfortunately, probabilistic methods may not able to deliver reliable results at the required precision without sufficient experimental data. It may be due to the probability density functions involved in it. As such in the recent decades, interval analysis and fuzzy theory are becoming powerful tools for many real life applications. In these approaches, the uncertain variables and parameters are represented by interval and fuzzy numbers, vectors, or matrices.

Interval computations introduced by Moore [42] and various aspects of interval analysis along with applications are explained by moore [43]. If only incomplete information is available, it is possible to establish the minimum and maximum favorable response of the structures using interval analysis or convex models (Ben-Haim and Elishakoff [13]; Genzerli and Pantelides [28]). The connection between the fuzzy analysis and the interval analysis is very well known (Zadeh [64], Moore and Lodwick [44], Pedrycz and Gomide [48]). Interval analysis and fuzzy analysis were introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. The main theoretical and practical results in the fields of fuzzy analysis and the interval analysis can be found in several works (Moore [42, 43], Alefeld and Herzberger [5], Kolev [38], Alefed and Mayer [4], Baker Kearfott and Kreinovich [11], and Nguyen et al. [45]).

The concept of solution for fractional differential equations with uncertainty was introduced by Agarwal, Lakshmikantham and Nieto [3]. This concept has been studied and developed in several papers. Arshad and Lupulescu [8, 7] developed the new results about fractional differential equations with uncertainty , Agarwal et al. [1, 2] developed a schauder fixed point theorem and fuzzy fractional integral equations, Allahviranloo et al. [6] proposed explicit solutions of fractional differential equation with uncertainty, Salahshour et al. [54, 55] proved existence and uniqueness for fuzzy fractional differential equation and solved these equations by Laplace transforms, a modified fractional Euler method was proposed by Mazandarani and Kamyad [41] for fuzzy fractional differential equations. The concepts of fractional derivatives for a fuzzy function are based on the notion of Hukuhara derivative (H-derivative) that the concept of Hukuhara derivative is well known (Hukuhara [32], Banks and Jacobs [12], Puri and Ralescu [50]).

Most of the literature deals with the different methods for the uncertain fractional differential equations to obtain the approximate solutions. Not much work has been made to determine the approximate solution of the fuzzy fractional Bagley-Torvik equation. Not work has been carried out when uncertainty has been taken into consideration for the fractional differential equation by using the RBF collocation method. As both fractional and fuzzy plays an important role in the structural modeling and design for the structural safety (like the fuzzy fractional Bagley-Torvik equation), hence an attempt has been made to combined the both for a better reliable analysis.

The fundamental aim of this paper is to extend the application of spectral collocation method based on radial basis functions to solve the fractional Bagley-Torvik equation under uncertainty, in the fuzzy Caputo's H-differentiability sense with order $(1 < \nu < 2)$. So, in the beginning, we define the fuzzy Caputo's H-differentiability sense with order $(1 < \nu < 2)$ which is a direct extension of Caputo derivatives with respect to Hukuhara difference, and then, employing the collocation RBF method for upper and lower solutions. In this way, the RBF collocation method reduces the problem of solving the fuzzy fractional Bagley-Torvik equation to two systems of linear equations. To eliminate the effects of the radial basis function shape parameters, we use thin plate spline radial basis functions which have no shape parameter , and also we use the variable shape parameter for Matérn radial basis function which give almost optimal shape parameter in this work. Numerical results will show the capabilities and improved efficiency of the RBF collocation method.

The layout of the paper is as follows: In Section 2 we show that how we use the thin plate spline radial basis functions to approximate the solution and recall some basic concepts. In Section 3, Caputo H-differentiability is introduced and some of its properties are considered. We introduce the fuzzy fractional Bagley-Torvik equation, in section 4. Then we briefly present the collocation RBF method for the fuzzy fractional Bagley-Torvik equation in next section. The results of numerical experiments are presented in Section 7. Section 8 is dedicated to a brief conclusion.

2. Preliminaries

In this section, some basic concepts of radial basis functions and fuzzy sets are presented.

TABLE 1. Examples of some popular RBFs, where $r = ||\cdot||$ is the Euclidean norm, and $\nu \in \mathbb{N}$.

RBF	$\phi_\epsilon(r)$
Gaussian C^∞ (GA)	$e^{-\epsilon^2 r^2}$
MultiQuadric C^∞ (MQ)	$(1+\epsilon^2 r^2)^{1/2}$
Inverse MultiQuadric C^∞ (IMQ)	$(1+\epsilon^2 r^2)^{-1/2}$
Thin Plate Spline $C^{\nu+1}$ (TPS)	$(-1)^{\nu+1} r^{2\nu} \log r$
Matérn C^6 (M6)	$e^{-\epsilon r}(\epsilon^3 r^3 + 6\epsilon^2 r^2 + 15\epsilon r + 15)$
Matérn C^4 (M4)	$e^{-\epsilon r}(\epsilon^2 r^2 + 3\epsilon r + 3)$
Matérn C^2 (M2)	$e^{-\epsilon r}(\epsilon r + 1)$

2.1. **Radial basis function approximation.** In the interpolation of the scattered data using radial basis functions the approximation of a function $u(x)$ at the centers $X = \{x_1, \ldots, x_N\}$, may be written as a linear combination of N RBFs; usually it takes the following form:

$$s_{u,X}(x) = \sum_{j=1}^{N} \alpha_j \phi(x - x_j) + \sum_{k=1}^{Q} \beta_k p_k(x). \tag{2.1}$$

Here, Q denotes the dimension of the polynomial space $\pi_{m-1}(\mathbb{R}^d)$, $p_1, \ldots, p_Q$ denote a basis of $\pi_{m-1}(\mathbb{R}^d)$, $x = (x_1, x_2, \ldots, x_d)$, d is the dimension of the problem, α's and β's are coefficients to be determined, ϕ is the RBF. To cope with additional degrees of freedom, the interpolation conditions

$$s_{u,X}(x_j) = u(x_j), \qquad 1 \le j \le N, \tag{2.2}$$

are completed by the additional conditions

$$\sum_{j=1}^{N} \alpha_j p_k(x_j) = 0, \qquad 1 \le k \le Q. \tag{2.3}$$

Solvability of this system is therefore equivalent to solvability of the system

$$\begin{pmatrix} A_{\phi,X} & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} u\,|X \\ 0 \end{pmatrix}, \tag{2.4}$$

where $A_{\phi,X} = (\phi(x_j - x_k)) \in \mathbb{R}^{N\times N}$ and $P = (p_k(x_j)) \in \mathbb{R}^{N\times Q}$. This last system is obviously solvable if the coefficient matrix on the left-hand side is invertible. Eq. (2.1) can be written without the additional polynomial $\sum_{k=1}^{Q} \beta_k p_k(x)$. In that case, ϕ must be unconditionally positive definite to guarantee the solvability of the resulting system (e.g. Gaussian, inverse multiquadrics, or Matérn). However $\sum_{k=1}^{Q} \beta_k p_k(x)$ is usually required when ϕ is conditionally positive definite, i.e. when ϕ has a polynomial growth towards infinity. For instance, suppose ϕ is thin plate splines. Moreover, since these functions are globally supported, the interpolation matrix is full and may be very ill–conditioned for some RBFs.

Although to improve the conditioning of the system of collocation equations Compactly supported RBFs (CSRBFs) have been applied, but the CSRBFs are vanish beyond a user defined threshold distance σ. Therefore, only the entries in the collocation matrix corresponding to collocation nodes lying closer than σ to a given CSRBF center are nonzero, leading to a sparse matrix. In fact, the interest in CSRBFs waned as it became evident that, in order to obtain a good accuracy, the overlap distance σ should cover most nodes in the point set, thus resulting in a populated matrix again [39].

In a similar representation as (2.1), for any linear partial differential operator $\mathcal{L}$, $\mathcal{L}u$ may be approximated by [20]

$$\mathcal{L}u(x) \simeq \sum_{j=1}^{N} \alpha_j \mathcal{L}\phi(x - x_j) + \sum_{k=1}^{Q} \beta_k \mathcal{L}p_k(x). \tag{2.5}$$

At first, we use the thin plate spline RBFs. The reason is that it has been shown by Franke [26], that MQ and thin plate spline give the most accurate results for scattered data approximations. Furthermore, the accuracy of the MQ method depends on a shape parameter and as yet there is no mathematical theory about how to choose its optimal value. Hence, most applications of the MQ use experimental tuning parameters or expensive optimization techniques to evaluate the optimum shape parameter [16]. While the thin plate spline method gives good agreement without requiring such additional parameters and is based on sound mathematical theory [15]. $\phi(x) = (-1)^{k+1}\|x\|_2^{2k} \log \|x\|_2,\ k \in \mathbb{N}$, from $\mathbb{R}^d$ to $\mathbb{R}$ that generates thin plate spline RBFs is conditionally positive definite of order $m = k + 1$, [62]. Since ϕ is C^{2k-1} continuous, a higher-order thin plate spline must be used, for higher-order partial differential operators. To avoid problems at $x = 0$ (since $log(0) = -\infty$), we implement $\phi(x) = (-1)^3\|x\|_2^3 \log \|x\|_2^{\|x\|_2}$ for $k = 2$. Also, we use the Matérn RBFs. Furthermore, we use the variable shape parameter to choose optimal shape parameter [21]. the Matérn RBFs are unconditionally positive definite, so Eq. (2.1) can be written without the additional polynomial $\sum_{k=1}^{Q} \beta_k p_k(x)$.

Definition 2.1. The points $X = \{x_1, \dots, x_N\} \subseteq \mathbb{R}^d$ with $N \geq Q = \dim \pi_m(\mathbb{R}^d)$ are called $\pi_m(\mathbb{R}^d)$-unisolvent if the zero polynomial is the only polynomial from $\pi_m(\mathbb{R}^d)$ that vanishes on all of them.

Theorem 2.2. *Suppose that ϕ is conditionally positive definite of order m and X is a $\pi_{m-1}(\mathbb{R}^d)$–unisolvent set of centers. Then the system (2.4) is uniquely solvable.*

Proof. [62]. □

The numerical solution of the fuzzy fractional Bagley-Torvik equation by RBF method is based on a scattered data interpolation problem which was reviewed in this section.

2.2. **Basic concepts of fuzzy set theory.**

Definition 2.3. A fuzzy number u is a fuzzy subset of the real line with a normal, convex and upper semi-continuous membership function of bounded support. The family of fuzzy numbers will be denoted by $\mathbb{E}$. An arbitrary fuzzy number is represented by an ordered pair of functions $[\underline{u}(\alpha), \overline{u}(\alpha)]$, $0 \leq \alpha \leq 1$, that satisfies the following requirements:

• $\underline{u}(\alpha)$ is a bounded left continuous nondecreasing function over $[0,1]$, with respect to any α.

• $\overline{u}(\alpha)$ is a bounded right continuous nonincreasing function over $[0,1]$, with respect to any α.

• $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, $0 \leq \alpha \leq 1$.

The α-level set

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ cl(\text{supp } u(x)), & \alpha = 0, \end{cases}$$

is a closed bounded interval, denoted $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$, where supp $u(x) = \{x \in \mathbb{R} : u(x) \geq 0\}$ is the support of the $u(x)$ and $cl(\text{supp } u(x))$ is its clusure.

A crisp number u is simply represented by $u = (\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \leq \alpha \leq 1$, we recall that for $a < b < c$ which $a, b, c \in \mathbb{R}$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}(\alpha) = a + (b-a)\alpha$ and $\overline{u}(\alpha) = c - (c-b)\alpha$ are the end points of the α–level sets. For all $\alpha \in [0,1]$ for arbitrary $u = (\underline{u}(\alpha), \overline{u}(\alpha))$, $v = (\underline{v}(\alpha), \overline{v}(\alpha))$ and $k > 0$ we define addition $u \oplus v$ and scalar multiplication by Kaleva [33].

• Addition:

$$u \oplus v = (\underline{u}(\alpha) + \underline{v}(\alpha), \overline{u}(\alpha) + \overline{v}(\alpha)).$$

• Scalar multiplication:

$$k \odot u = \begin{cases} (k\underline{u}(\alpha), k\overline{u}(\alpha)), & k \geq 0, \\ (k\overline{u}(\alpha), k\underline{u}(\alpha)), & k < 0, \end{cases}$$

if $k = -1$ then $k \odot u = -u$.

Definition 2.4. For arbitrary numbers $u = (\underline{u}(\alpha), \overline{u}(\alpha))$ and $v = (\underline{v}(\alpha), \overline{v}(\alpha))$,

$$D(u, v) = \max\{\sup_{0 \leq \alpha \leq 1} |\overline{u}(\alpha) - \overline{v}(\alpha)|, \sup_{0 \leq \alpha \leq 1} |\underline{u}(\alpha) - \underline{v}(\alpha)|\},$$

is the distance between u and v.

Definition 2.5. Let $u, v \in \mathbb{E}$. If there exists $w \in \mathbb{E}$ such that $u = v + w$, then w is called the H-difference of u and v, and it is denoted by $u \ominus v$.

Definition 2.6. Let $f : (a, b) \longrightarrow \mathbb{E}$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable on x_0, if there exists an element $f'(x_0) \in \mathbb{E}$, such that

(i) for all $h > 0$ sufficiently small, there exist $f(x_0+h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0-h)$ and the limits (in the metric D)

$$\lim_{h\searrow 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\searrow 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0),$$

or

(ii) for all $h > 0$ sufficiently small, there exist $f(x_0) \ominus f(x_0+h)$, $f(x_0-h) \ominus f(x_0)$ and the limits

$$\lim_{h\searrow 0} \frac{f(x_0) \ominus f(x_0+h)}{-h} = \lim_{h\searrow 0} \frac{f(x_0-h) \ominus f(x_0)}{-h} = f'(x_0),$$

(h and $-h$ at denominators mean $\frac{1}{h}\odot$ and $-\frac{1}{h}\odot$, respectively).

Theorem 2.7. *Let $f(x) : [a,\infty) \longrightarrow \mathbb{E}$ be a fuzzy valued function demonstrated by $(\underline{f^\alpha}(x), \overline{f^\alpha}(x))$. For any fixed $\alpha \in [0,1]$, consider $\underline{f^\alpha}(x)$ and $\overline{f^\alpha}(x)$ are Riemann-integrable on $[a,b]$ for every $b \geq a$, and assume there are two positive functions $\underline{M^\alpha}(x)$ and $\overline{M^\alpha}(x)$ such that $\int_a^b |\underline{f^\alpha}(x)|dx \leq \underline{M^\alpha}(x)$ and $\int_a^b |\overline{f^\alpha}(x)|dx \leq \overline{M^\alpha}(x)$ for every $b \geq a$. Then, $f(x)$ is improper fuzzy Riemann-integrable on $[a,\infty)$ and the improper fuzzy Riemann-integral is a fuzzy number. Furthermore, we have*

$$\left[\int_a^\infty |f(x)|dx\right]^\alpha = \left[\int_a^\infty \underline{f^\alpha}(x)dx, \int_a^\infty \overline{f^\alpha}(x)dx\right].$$

Proof. [63]. □

3. Fuzzy Caputo-type fractional differentiability

The fuzzy fractional differentiability of order $1 < \nu < 2$, particularly Caputo type, is investigated in this section. Some basic definitions and theorems are presented and introduced the necessary notation, which will be used in the rest of paper. See, for example, [3, 41, 54].

At first, some notations are presented which are put to use throughout the remaining sections:

- $\mathbf{L}^{\mathbb{E}}[a,b]$ is the set of all fuzzy-valued measurable functions f on $[a,b]$.
- $\mathbf{C}^{\mathbb{E}}[a,b]$ is the space of fuzzy-valued functions which are continuous on $[a,b]$.

The next step is to describe the fuzzy Riemann-Liouville integral of fuzzy-valued function as

Definition 3.1. Let $f(x) \in \mathbf{C}^{\mathbb{E}}[0,b] \cap \mathbf{L}^{\mathbb{E}}[0,b]$. The fuzzy Riemann-Liouville integral of the fuzzy valued function $f(x)$ is described as follows:

$$J^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x \frac{f(t)}{(x-t)^{1-\nu}} \mathrm{d}t, \quad x, \nu \in (0,\infty),$$

where J^ν is the Riemann-Liouville integral operator of order ν, and $\Gamma(\nu)$ is the famous Gamma function.

Theorem 3.2. *Let $f(x) \in \mathbf{C}^{\mathbb{E}}[0,b] \cap \mathbf{L}^{\mathbb{E}}[0,b]$ be a fuzzy valued function. The Riemann-Liouville integral of the $f(x)$, based on its α-level representation can be expressed as follows:*

$$[J^\nu f(x)]^\alpha = \left[J^\nu \underline{f^\alpha}(x), J^\nu \overline{f^\alpha}(x)\right], \quad 0 \leq \alpha \leq 1,$$

where

$$J^{\nu}\underline{f}^{\alpha}(x) = \frac{1}{\Gamma(\nu)}\int_0^x \frac{\underline{f}^{\alpha}(t)}{(x-t)^{1-\nu}}\,dt, \qquad x,\nu \in (0,\infty),$$

$$J^{\nu}\overline{f^{\alpha}}(x) = \frac{1}{\Gamma(\nu)}\int_0^x \frac{\overline{f^{\alpha}}(t)}{(x-t)^{1-\nu}}\,dt, \qquad x,\nu \in (0,\infty),$$

Proof. The proof of the theorem can be found in [55]. □

Now, we define Caputos fuzzy differentiable of fuzzy-valued function as

Definition 3.3. Let $f(x) \in \mathbf{C}^{\mathbb{E}}[0,b] \cap \mathbf{L}^{\mathbb{E}}[0,b]$ be a fuzzy valued function. Then f is said to be Caputo's fuzzy differentiable at x when

$$D^{\nu}f(x) = J^{2-\nu}D^2 f(x) = \frac{1}{\Gamma(2-\nu)}\int_0^x \frac{f''(t)}{(x-t)^{\nu-1}}\mathrm{d}t,$$

where $1 < \nu < 2$.

Definition 3.4. Let $f(x) \in \mathbf{C}^{\mathbb{E}}[0,b] \cap \mathbf{L}^{\mathbb{E}}[0,b]$, $\Phi(x) = \dfrac{1}{\Gamma(2-\nu)}\int_0^x \dfrac{f(t)}{(x-t)^{\nu-1}}\mathrm{d}t$, $G(x_0) = \lim_{h\searrow 0}\dfrac{\Phi(x_0+h)\ominus\Phi(x_0)}{h} = \lim_{h\searrow 0}\dfrac{\Phi(x_0)\ominus\Phi(x_0-h)}{h}$, and $H(x_0) = \lim_{h\searrow 0}\dfrac{\Phi(x_0)\ominus\Phi(x_0+h)}{-h} = \lim_{h\searrow 0}\dfrac{\Phi(x_0-h)\ominus\Phi(x_0)}{-h}$. $f(x)$ is the Caputo-type fuzzy fractional differentiable function of order $1 < \nu < 2$ at $x_0 \in (0,b)$, if there exists an element $D^{\nu}f(x) \in \mathbf{C}^{\mathbb{E}}$ such that for all $0 \le \alpha \le 1$ and for $h > 0$ sufficiently near zero, either.

(a) $D^{\nu}f(x_0) = \lim_{h\searrow 0}\dfrac{G(x_0+h)\ominus G(x_0)}{h} = \lim_{h\searrow 0}\dfrac{G(x_0)\ominus G(x_0-h)}{h}$.

(b) $D^{\nu}f(x_0) = \lim_{h\searrow 0}\dfrac{G(x_0)\ominus G(x_0+h)}{-h} = \lim_{h\searrow 0}\dfrac{G(x_0-h)\ominus G(x_0)}{-h}$.

(c) $D^{\nu}f(x_0) = \lim_{h\searrow 0}\dfrac{H(x_0+h)\ominus H(x_0)}{h} = \lim_{h\searrow 0}\dfrac{H(x_0)\ominus H(x_0-h)}{h}$.

(d) $D^{\nu}f(x_0) = \lim_{h\searrow 0}\dfrac{H(x_0)\ominus H(x_0+h)}{-h} = \lim_{h\searrow 0}\dfrac{H(x_0-h)\ominus H(x_0)}{-h}$.

In this paper, Definition 3.4(a,c) and 3.4(b,d) will be referred to as the Caputo-type fuzzy differentiable of the first type (i) and the Caputo-type fuzzy differentiable of the second type (ii), respectively.

Theorem 3.5. *Let $f(x) \in \mathbf{C}^{\mathbb{E}}[0,b] \cap \mathbf{L}^{\mathbb{E}}[0,b]$ be a fuzzy valued function. $[f(x)]^{\alpha} = \left[\underline{f}^{\alpha}(x), \overline{f^{\alpha}}(x)\right]$, $0 \le \alpha \le 1$ and $x_0 \in (0,b)$. Then*

(a) *If $G(x)$ is a Caputo-type fuzzy fractional differentiable function in the first type, then*

$$[D^{\nu}f(x_0)]^{\alpha} = \left[D^{\nu}\underline{f}^{\alpha}(x_0), D^{\nu}\overline{f^{\alpha}}(x_0)\right], \qquad 1 < \nu < 2.$$

(b) *If $G(x)$ is a Caputo-type fuzzy fractional differentiable function in the second type, then*

$$[D^{\nu}f(x_0)]^{\alpha} = \left[D^{\nu}\overline{f^{\alpha}}(x_0), D^{\nu}\underline{f}^{\alpha}(x_0)\right], \qquad 1 < \nu < 2.$$

(c) *If $H(x)$ is a Caputo-type fuzzy fractional differentiable function in the first type, then*

$$[D^{\nu} f(x_0)]^{\alpha} = \left[D^{\nu}\overline{f^{\alpha}}(x_0), D^{\nu}\underline{f^{\alpha}}(x_0)\right], \qquad 1 < \nu < 2.$$

(d) *If $H(x)$ is a Caputo-type fuzzy fractional differentiable function in the second type, then*

$$[D^{\nu} f(x_0)]^{\alpha} = \left[D^{\nu}\underline{f^{\alpha}}(x_0), D^{\nu}\overline{f^{\alpha}}(x_0)\right], 1 < \nu < 2.$$

where

$$D^{\nu}\underline{f^{\alpha}}(x_0) = \left[\frac{1}{\Gamma(2-\nu)}\int_0^x \frac{\underline{f^{\alpha}}''(t)}{(x-t)^{\nu-1}}\,dt\right]_{x=x_0},$$

$$D^{\nu}\overline{f^{\alpha}}(x_0) = \left[\frac{1}{\Gamma(2-\nu)}\int_0^x \frac{\overline{f^{\alpha}}''(t)}{(x-t)^{\nu-1}}\,dt\right]_{x=x_0}.$$

Proof. It is easy to prove using Definition 3.4 and Theorem 3.2. □

4. Fuzzy fractional Bagley-Torvik equation

Torvik and Bagley [10] derived a fractional differential equation of degree $3/2$ for the description of the motion of an immersed plate in a Newtonian fluid [60]. The motion of a rigid plate of mass m and area A connected by a mass less spring of stiffness k, immersed in a Newtonian fluid, was originally introduced by Bagley and Torvik. A rigid plate of mass m immersed into an infinite Newtonian fluid as displayed in the Figure 1 . The plate is held at a fixed point by means of a spring of stiffness k. It is supposed that the motions of spring do not influence the motion of the fluid and that the area A of the plate is very large, such that the stress-velocity relationship is valid on both sides of the plate.

Let μ be the viscosity and ρ be the fluid density. The displacement of the plate u is described by

$$A\widetilde{u}''(t) \oplus BD^{3/2}\widetilde{u}(t) \oplus C\widetilde{u}(t) = \widetilde{f}(t), \qquad 0 \le t \le T, \tag{4.1}$$

with the initial conditions

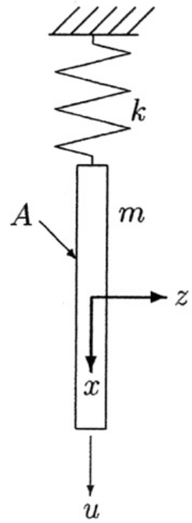
$$\widetilde{u}(0) = \widetilde{b}_0, \qquad \widetilde{u}'(0) = \widetilde{b}_1, \tag{4.2}$$

where $\widetilde{b}_0$, $\widetilde{b}_1$ are the triangular fuzzy number and $D^{3/2}$ is Caputo-type fractional derivative.

If $\widetilde{f}(t)$ is a crisp function, then the solutions of Eq. (4.1) are crisp too. However, if $\widetilde{f}(t)$ is a fuzzy function, these equations may only possess fuzzy solutions. In this paper, the fuzzy fractional Bagley-Torvik equation is discussed. Introducing the parametric forms of $\widetilde{f}(t)$ and $\widetilde{u}(t)$, we have the parametric form of the fuzzy fractional Bagley-Torvik equation as follows

$$\begin{aligned}&\left[\underline{f}(t;\alpha), \overline{f}(t;\alpha)\right] = \\ &\left[A\underline{u}''(t;\alpha) + BD^{3/2}\underline{u}(t;\alpha) + C\underline{u}(t;\alpha), A\overline{u}''(t;\alpha) + BD^{3/2}\overline{u}(t;\alpha) + C\overline{u}(t;\alpha)\right]\end{aligned} \tag{4.3}$$

FIGURE 1. Rigid plate of mass m immersed into a Newtonian fluid [51, 60].



where $0 \leq \alpha \leq 1$, $\widetilde{f}(t) = \left[\underline{f}(t;\alpha), \overline{f}(t;\alpha)\right]$ is a predetermined data function, and $\widetilde{u}(t) = [\underline{u}(t;\alpha), \overline{u}(t;\alpha)]$ is the solution that will be determined.

5. Application of collocation method to fuzzy fractional Bagley-Torvik equation using RBFs

In this section, a meshfree method namely, the RBF collocation method is presented to fuzzy fractional Bagley-Torvik equation. Consider the fuzzy fractional Bagley-Torvik equation (4.3) with $A = B = C = 1$, and the initial conditions

$$\begin{cases} \underline{u}(0;\alpha) = \underline{b}_0(\alpha), & \underline{u}'(0;\alpha) = \underline{b}_1(\alpha), \\ \overline{u}(0;\alpha) = \overline{b}_0(\alpha), & \overline{u}'(0;\alpha) = \overline{b}_1(\alpha), \end{cases} \tag{5.1}$$

Buckly-Feuring method of solution is to fuzzify the crisp solution to obtain a fuzzy function, and then check to see if it satisfies the differential equation with fuzzy initial conditions. In this paper, we proposed the collocation method for solving fuzzy fractional differential equation. This method is to seek approximate solutions as

$$\begin{cases} \underline{u}_N(t;\alpha) = \displaystyle\sum_{j=0}^{N} \underline{\lambda}_j(\alpha)\varphi_j(t), \\ \overline{u}_N(t;\alpha) = \displaystyle\sum_{j=0}^{N} \overline{\lambda}_j(\alpha)\varphi_j(t), \end{cases} \tag{5.2}$$

where $\phi_k(t)$ are radial basis function. Now, the aim is to compute the coefficients $\underline{\lambda}_j(\alpha)$ and $\overline{\lambda}_j(\alpha)$ in (5.2) using collocation method.
Because we use thin plate spline radial basis functions, assuming that there are a total of $(N-3)$ interpolation points, $\underline{u}(t;\alpha)$ and $\overline{u}(t;\alpha)$ can be approximated by

$$\begin{cases} \underline{u}_N(t;\alpha) \simeq \sum_{j=1}^{N-3} \underline{\lambda}_j(\alpha)\varphi_j(t) + \left(\underline{\lambda}_{N-2}(\alpha)t^2\right) + \left(\underline{\lambda}_{N-1}(\alpha)t\right) + \underline{\lambda}_N(\alpha), \\ \overline{u}_N(t;\alpha) \simeq \sum_{j=1}^{N-3} \overline{\lambda}_j(\alpha)\varphi_j(t) + \left(\overline{\lambda}_{N-2}(\alpha)t^2\right) + \left(\overline{\lambda}_{N-1}(\alpha)t\right) + \overline{\lambda}_N(\alpha). \end{cases} \tag{5.3}$$

To determine the interpolation coefficients $\left(\underline{\lambda}_1(\alpha), \underline{\lambda}_2(\alpha), \ldots, \underline{\lambda}_{N-1}(\alpha), \underline{\lambda}_N(\alpha)\right)$ and $\left(\overline{\lambda}_1(\alpha), \overline{\lambda}_2(\alpha), \ldots, \overline{\lambda}_{N-1}(\alpha), \overline{\lambda}_N(\alpha)\right)$, the collocation method is used by applying Eq. (5.3) at every point $t_i,\ i = 1, 2, \ldots, N-3$. Thus, we have

$$\begin{cases} \underline{u}_N(t_i;\alpha) \simeq \sum_{j=1}^{N-3} \underline{\lambda}_j(\alpha)\varphi_j(t_i) + \left(\underline{\lambda}_{N-2}(\alpha)t_i^2\right) + \left(\underline{\lambda}_{N-1}(\alpha)t_i\right) + \underline{\lambda}_N(\alpha), \\ \overline{u}_N(t_i;\alpha) \simeq \sum_{j=1}^{N-3} \overline{\lambda}_j(\alpha)\varphi_j(t_i) + \left(\overline{\lambda}_{N-2}(\alpha)t_i^2\right) + \left(\overline{\lambda}_{N-1}(\alpha)t_i\right) + \overline{\lambda}_N(\alpha). \end{cases} \tag{5.4}$$

The additional conditions due to Eq. (2.3) are written as

$$\begin{cases} \sum_{j=1}^{N-3} \underline{\lambda}_j(\alpha)t_j^2 = \sum_{j=1}^{N-3} \underline{\lambda}_j(\alpha)t_j = \sum_{j=1}^{N-3} \underline{\lambda}_j(\alpha) = 0, \\ \sum_{j=1}^{N-3} \overline{\lambda}_j(\alpha)t_j^2 = \sum_{j=1}^{N-3} \overline{\lambda}_j(\alpha)t_j = \sum_{j=1}^{N-3} \overline{\lambda}_j(\alpha) = 0. \end{cases} \tag{5.5}$$

Writing Eq. (5.4) together with Eq. (5.5) in a matrix form, we have

$$\begin{cases} [\underline{U}] = \mathbf{A}\,[\underline{\Lambda}]\,, \\ \left[\overline{U}\right] = \mathbf{A}\left[\overline{\Lambda}\right], \end{cases} \tag{5.6}$$

where $[\underline{U}] = \left[\underline{u}_1(\alpha)\ \cdots\ \underline{u}_{N-3}(\alpha)\ 0\ 0\ 0\right]^T$, $[\underline{\Lambda}] = [\underline{\lambda}_1(\alpha)\ \cdots\ \underline{\lambda}_N(\alpha)]^T$, $\left[\overline{U}\right] = [\overline{u}_1(\alpha)\ \cdots\ \overline{u}_{N-3}(\alpha)\ 0\ 0\ 0]^T$, $\left[\overline{\Lambda}\right] = \left[\overline{\lambda}_1(\alpha)\ \cdots\ \overline{\lambda}_N(\alpha)\right]^T$, and $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1(N-3)} & t_1^2 & t_1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{(N-3)1} & \cdots & \varphi_{(N-3)(N-3)} & t_{N-3}^2 & t_{N-3} & 1 \\ x_1^2 & \cdots & x_{N-3}^2 & 0 & 0 & 0 \\ x_1 & \cdots & x_{N-3} & 0 & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 & 0 \end{pmatrix}_{N\times N}. \tag{5.7}$$

Now, substituting approximation (5.4) in Eq. (4.1) for all points except the initial point $t_i,\ i = 2, 3, \ldots, N-3$, and the initial conditions (4.2). we get the following systems of linear equation in a matrix form

$$\begin{cases} \mathbf{B}\,[\underline{\Lambda}] = [\underline{F}]\,, \\ \mathbf{B}\,[\overline{\Lambda}] = [\overline{F}]\,, \end{cases} \tag{5.8}$$

where

$$\begin{aligned}
&\mathbf{B} = \nabla^2 A_d + D^{3/2} A_d + A_d + A_b + \nabla A_b + A_e, \\
&\mathbf{A} = A_d + A_b + A_e, \text{ where} \\
&A_d = [a_{ij}, \text{ for } (2 \le i \le N-3,\ 1 \le j \le N) \text{ and } 0, \text{ elsewhere}]\,, \\
&A_b = [a_{ij}, \text{ for } (i = 1,\ 1 \le j \le N) \text{ and } 0, \text{ elsewhere}]\,, \\
&A_e = [a_{ij}, \text{ for } (N-2 \le i \le N,\ 1 \le j \le N) \text{ and } 0, \text{ elsewhere}]\,, \\
&[\underline{F}] = \left[\underline{b}_0(\alpha) + \underline{b}_1(\alpha),\ \underline{f}(t_2;\alpha),\ \ldots,\ \underline{f}(t_{N-3};\alpha),\ 0,\ 0,\ 0\right]^T, \\
&[\overline{F}] = \left[\overline{b}_0(\alpha) + \overline{b}_1(\alpha),\ \overline{f}(t_2;\alpha),\ \ldots,\ \overline{f}(t_{N-3};\alpha),\ 0,\ 0,\ 0\right]^T.
\end{aligned} \tag{5.9}$$

The parameters $\underline{\lambda}_1(\alpha)\ \cdots\ \underline{\lambda}_N(\alpha)$ and $\overline{\lambda}_1(\alpha)\ \cdots\ \overline{\lambda}_N(\alpha)$ are obtained by solving the systems of linear equation (5.8). These parameters yield the fuzzy approximate solution $[\underline{u}(t;\alpha), \overline{u}(t;\alpha)]$.

Remark 5.1. Finding a closed form analytic expression for the fractional derivative of a radial basis function can be challenging. We are often bound to having to represent functions as Taylor series expansions before applying the fractional derivative operator term by term. Here we derive these series expansions for the Caputo fractional derivative of the thin plate spline and Matérn radial function and truncate the infinite sum once the terms are smaller in magnitude than machine precision.

It should also be noted that we can applied the boundary conditions instead of the initial conditions. We will use this condition in example 3 in section 7. Also, we use Matérn radial functions instead of the thin plate spline radial function. In this case, we use Eqs. (5.2) instead of Eqs. (5.3), and also additional conditions (5.5) don't need anymore.
When we use the Matérn radial functions, we have to choose an optimal shape parameter.

6. Choosing the shape parameter and the variable shape parameter (VSP) strategy

Meshless methods which are based on radial basis functions (RBFs) contain a free shape parameter that plays an important role for the accuracy and condition number of the coefficient matrix of the method. Most authors use the trial and error method for obtaining a good shape parameter that results in best accuracy. The simplest strategy that is named trial and error, is to perform Matlab program with varying shape parameters and then to pick the best one that has the least error. Relating to choosing a good shape parameter several excellent articles have also been written by different authors such as in [21].

In this section, we express the variable shape parameter strategy. The VSP strategy considers a different value of the shape parameter at each center. Applying the VSP strategy the coefficient matrix of the linear system of algebra equations will be non-symmetric. In the following using some VSP strategies based on ideas presented in [57].

1) Exponentially strategy
   Kansa introduced the VPS to the following form:

$$\epsilon_j = \left[\epsilon_{min}^2 \left(\frac{\epsilon_{max}^2}{\epsilon_{min}^2} \right)^{\dfrac{j-1}{N-1}} \right]^{\frac{1}{2}}, \quad j = 1, 2, \ldots, N.$$

   The above VSP strategy was introduced for radial basis function MQ.
2) Linearly strategy
   Other possible procedure is as follows:

$$\epsilon_j = \epsilon_{min} + \left(\frac{\epsilon_{max} - \epsilon_{min}}{N-1} \right) j, \quad j = 0, 1, 2, \ldots, N-1.$$

   This procedure is introduced in [57] and is named a linearly variable shape parameter.
3) Randomly strategy
   Another VSP strategy that named a randomly variable shape parameter is [57]

$$\epsilon_j = \epsilon_{min} + (\epsilon_{max} - \epsilon_{min}) \times \mathrm{rand}(1, N), \quad j = 0, 1, 2, \ldots, N-1.$$

In which $\epsilon_{min} < \epsilon_{max}$ are arbitrary non-negative real numbers. The function `rand` is a Matlab command that returns N uniformly distributed pseudo-random numbers on the unit interval.
We will apply randomly variable shape parameter for our examples in next section, because most researchers believe that randomly strategy is the best.

7. Numerical Results

The aim of this section is to demonstrate the RBF method described earlier for the solution of the fuzzy fractional Bagley-Torvik equation under the Caputo-type fuzzy fractional derivatives. For this purpose, we use three examples to illustrate the performance of this method. The first and second examples are the fuzzy fractional Bagley-Torvik equations with the non-homogeneous and homogeneous initial conditions, respectively, and third example is the fuzzy fractional Bagley-Torvik equation with the homogeneous boundary conditions. All these results illustrated in some tables and figures have been carried out in Matlab on a laptop with a 2.6 GHz Intel Core i5 processor. Two types of norms are used to measure the error of approximation. The L_∞ and the RMS are described below:

$$L_\infty = \max \left| u(X_i) - \tilde{u}(X_i) \right|,$$

Table 2. Max and RMS errors, CPU Time, and condition number (CN) for lower and upper solutions on $t \in [0, 10]$ using TPS with $N = 5$ in example 7.1.

			Errors for $\underline{u}(t;\alpha)$		Errors for $\overline{u}(t;\alpha)$	
α	CPU(s)	CN	RMS	Max Error	RMS	Max Error
0.0	0.01	$4.68 \times 10^{+08}$	6.98×10^{-14}	1.69×10^{-13}	1.39×10^{-13}	3.37×10^{-13}
0.1	0.01	$4.68 \times 10^{+08}$	4.93×10^{-14}	1.26×10^{-13}	1.25×10^{-13}	3.02×10^{-13}
0.2	0.01	$4.68 \times 10^{+08}$	6.61×10^{-14}	1.12×10^{-13}	1.16×10^{-13}	1.76×10^{-13}
0.3	0.01	$4.68 \times 10^{+08}$	5.68×10^{-14}	1.48×10^{-13}	1.40×10^{-13}	3.66×10^{-13}
0.4	0.01	$4.68 \times 10^{+08}$	2.75×10^{-14}	4.62×10^{-14}	1.34×10^{-13}	2.18×10^{-13}
0.5	0.01	$4.68 \times 10^{+08}$	2.27×10^{-14}	5.06×10^{-14}	1.15×10^{-13}	1.65×10^{-13}
0.6	0.01	$4.68 \times 10^{+08}$	1.28×10^{-14}	1.95×10^{-14}	8.52×10^{-14}	1.28×10^{-13}
0.7	0.01	$4.68 \times 10^{+08}$	2.46×10^{-15}	3.69×10^{-15}	1.26×10^{-13}	1.99×10^{-13}
0.8	0.01	$4.68 \times 10^{+08}$	2.21×10^{-14}	3.13×10^{-14}	9.88×10^{-14}	1.37×10^{-13}
0.9	0.01	$4.68 \times 10^{+08}$	3.23×10^{-14}	4.62×10^{-14}	9.09×10^{-14}	2.71×10^{-13}
1.0	0.01	$4.68 \times 10^{+08}$	6.98×10^{-14}	1.69×10^{-13}	6.98×10^{-14}	1.69×10^{-13}

$$RMS = \sqrt{\frac{1}{N}\sum_{i=1}^{N} \left|u\left(X_i\right) - \tilde{u}\left(X_i\right)\right|^2},$$

where $u(x)$ is the exact solution, $\tilde{u}(x)$ is the approximate solution and N is the total number of evaluation points. A uniform distribution containing $(0 : 0.01 : T)$ nodes in the domain is used to evaluate the L_∞ and the RMS error norms in all examples. We use two thin plate spline (TPS) and Matérn (M6) basis functions in all examples. Also, the stability is studied by evaluating the condition number (CN) obtained by using the Matlab command `cond`.

Example 7.1. Consider the fuzzy fractional Bagley-Torvik equation (4.1) with $A = B = C = 1$ and

$$\begin{cases} \underline{f}(t;\alpha) = \left(\alpha^3 + \alpha - 1\right)(t+1), \\ \overline{f}(t;\alpha) = \left(2 - \alpha^2\right)(t+1). \end{cases}$$

The initial condition is given by

$$\begin{cases} \underline{u}(0;\alpha) = \underline{u}'(0;\alpha) = \left(\alpha^3 + \alpha - 1\right), \\ \overline{u}(0;\alpha) = \overline{u}'(0;\alpha) = \left(2 - \alpha^2\right), \end{cases}$$

and the exact solution [17] is

$$\begin{cases} \underline{u}(t;\alpha) = \left(\alpha^3 + \alpha - 1\right)(t+1), \\ \overline{u}(t;\alpha) = \left(2 - \alpha^2\right)(t+1). \end{cases}$$

Table 3. Max and RMS errors, CPU Time, and condition number (CN) for lower and upper solutions on $t \in [0, 10]$ using M6 with randomly variable shape parameter ($\epsilon_{min} = 0.00001$, $\epsilon_{max} = 0.0001$) and $N = 5$ in example 7.1.

			Errors for $\underline{u}(t;\alpha)$		Errors for $\overline{u}(t;\alpha)$	
α	CPU(s)	CN	RMS	Max Error	RMS	Max Error
0.0	0.02	$2.77 \times 10^{+17}$	6.00×10^{-08}	1.95×10^{-07}	1.20×10^{-07}	3.90×10^{-07}
0.1	0.02	$1.81 \times 10^{+17}$	3.50×10^{-08}	1.19×10^{-07}	4.96×10^{-07}	7.64×10^{-07}
0.2	0.02	$1.37 \times 10^{+17}$	7.79×10^{-08}	1.46×10^{-07}	1.15×10^{-07}	2.64×10^{-07}
0.3	0.02	$2.04 \times 10^{+17}$	4.40×10^{-08}	1.01×10^{-07}	1.54×10^{-07}	3.18×10^{-07}
0.4	0.02	$1.20 \times 10^{+17}$	8.67×10^{-08}	1.45×10^{-07}	2.08×10^{-07}	3.64×10^{-07}
0.5	0.02	$3.45 \times 10^{+17}$	3.12×10^{-08}	8.00×10^{-08}	1.03×10^{-07}	2.25×10^{-07}
0.6	0.02	$4.32 \times 10^{+17}$	1.34×10^{-08}	3.35×10^{-08}	9.23×10^{-08}	2.90×10^{-07}
0.7	0.02	$1.25 \times 10^{+17}$	5.41×10^{-09}	9.05×10^{-09}	9.82×10^{-08}	3.18×10^{-07}
0.8	0.02	$2.59 \times 10^{+17}$	3.18×10^{-08}	5.74×10^{-08}	1.48×10^{-07}	2.48×10^{-07}
0.9	0.02	$3.42 \times 10^{+17}$	9.71×10^{-08}	1.79×10^{-07}	9.45×10^{-08}	2.00×10^{-07}
1.0	0.02	$1.13 \times 10^{+18}$	1.01×10^{-07}	1.86×10^{-07}	1.01×10^{-07}	1.86×10^{-07}

Table 4. Max and RMS errors, CPU Time, and condition number (CN) for lower and upper solutions on $t \in [0, 10]$ using TPS with $N = 5$ in example 7.2.

			Errors for $\underline{u}(t;\alpha)$		Errors for $\overline{u}(t;\alpha)$	
α	CPU(s)	CN	RMS	Max Error	RMS	Max Error
0.0	0.02	$4.68 \times 10^{+08}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	5.49×10^{-13}	1.53×10^{-12}
0.1	0.02	$4.68 \times 10^{+08}$	3.21×10^{-14}	1.03×10^{-13}	7.80×10^{-13}	1.96×10^{-12}
0.2	0.02	$4.68 \times 10^{+08}$	6.43×10^{-14}	2.06×10^{-13}	6.64×10^{-13}	1.14×10^{-12}
0.3	0.02	$4.68 \times 10^{+08}$	1.13×10^{-13}	3.23×10^{-13}	1.26×10^{-12}	3.47×10^{-12}
0.4	0.02	$4.68 \times 10^{+08}$	1.28×10^{-13}	4.12×10^{-13}	5.14×10^{-13}	1.65×10^{-12}
0.5	0.02	$4.68 \times 10^{+08}$	1.37×10^{-13}	3.84×10^{-13}	5.03×10^{-13}	7.10×10^{-13}
0.6	0.02	$4.68 \times 10^{+08}$	2.26×10^{-13}	6.47×10^{-13}	3.36×10^{-13}	1.25×10^{-12}
0.7	0.02	$4.68 \times 10^{+08}$	1.68×10^{-13}	6.25×10^{-13}	6.83×10^{-13}	2.44×10^{-12}
0.8	0.02	$4.68 \times 10^{+08}$	2.57×10^{-13}	8.24×10^{-13}	4.53×10^{-13}	1.29×10^{-12}
0.9	0.02	$4.68 \times 10^{+08}$	3.23×10^{-13}	5.68×10^{-13}	2.60×10^{-13}	9.52×10^{-13}
1.0	0.02	$4.68 \times 10^{+08}$	2.75×10^{-13}	7.67×10^{-13}	2.75×10^{-13}	7.67×10^{-13}

TABLE 5. Max and RMS errors, CPU Time, and condition number (CN) for lower and upper solutions on $t \in [0, 10]$ using M6 with randomly variable shape parameter ($\epsilon_{min} = 0.001$, $\epsilon_{max} = 0.01$) and $N = 5$ in example 7.2.

			Errors for $\underline{u}(t;\alpha)$		Errors for $\overline{u}(t;\alpha)$	
α	CPU(s)	CN	RMS	Max Error	RMS	Max Error
0.0	0.02	$7.16 \times 10^{+13}$	0.00×10^{-00}	0.00×10^{-00}	2.08×10^{-03}	2.09×10^{-03}
0.1	0.02	$4.92 \times 10^{+13}$	6.92×10^{-04}	6.93×10^{-04}	3.60×10^{-02}	3.61×10^{-02}
0.2	0.02	$1.61 \times 10^{+14}$	1.75×10^{-03}	1.75×10^{-03}	3.01×10^{-03}	3.06×10^{-03}
0.3	0.02	$4.73 \times 10^{+13}$	1.51×10^{-02}	1.51×10^{-02}	2.38×10^{-02}	2.38×10^{-02}
0.4	0.02	$3.68 \times 10^{+14}$	3.57×10^{-03}	3.57×10^{-03}	1.43×10^{-02}	1.43×10^{-02}
0.5	0.02	$2.12 \times 10^{+15}$	1.39×10^{-04}	1.39×10^{-04}	2.33×10^{-04}	2.33×10^{-04}
0.6	0.02	$1.03 \times 10^{+13}$	2.02×10^{-03}	2.02×10^{-03}	9.11×10^{-03}	9.13×10^{-03}
0.7	0.02	$3.76 \times 10^{+14}$	5.11×10^{-02}	5.11×10^{-02}	3.46×10^{-03}	3.57×10^{-03}
0.8	0.02	$7.40 \times 10^{+15}$	2.63×10^{-05}	2.68×10^{-05}	2.11×10^{-04}	2.12×10^{-04}
0.9	0.02	$1.83 \times 10^{+14}$	5.12×10^{-03}	5.12×10^{-03}	1.48×10^{-03}	1.48×10^{-03}
1.0	0.02	$1.87 \times 10^{+15}$	1.59×10^{-06}	3.73×10^{-06}	1.59×10^{-06}	3.73×10^{-06}

TABLE 6. Max and RMS errors, CPU Time, and condition number (CN) for lower and upper solutions on $t \in [0, 1]$ using TPS with $N = 5$ in example 7.3.

			Errors for $\underline{u}(t;\alpha)$		Errors for $\overline{u}(t;\alpha)$	
α	CPU(s)	CN	RMS	Max Error	RMS	Max Error
0.0	0.01	$2.06 \times 10^{+03}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	3.50×10^{-15}	6.34×10^{-15}
0.1	0.01	$2.06 \times 10^{+03}$	1.75×10^{-16}	2.97×10^{-16}	2.71×10^{-15}	4.25×10^{-15}
0.2	0.01	$2.06 \times 10^{+03}$	3.49×10^{-16}	5.93×10^{-16}	3.44×10^{-15}	6.04×10^{-15}
0.3	0.01	$2.06 \times 10^{+03}$	6.06×10^{-16}	1.06×10^{-15}	2.72×10^{-15}	4.84×10^{-15}
0.4	0.01	$2.06 \times 10^{+03}$	6.98×10^{-16}	1.19×10^{-16}	2.79×10^{-15}	4.75×10^{-15}
0.5	0.01	$2.06 \times 10^{+03}$	8.75×10^{-16}	1.58×10^{-15}	2.35×10^{-15}	4.09×10^{-15}
0.6	0.01	$2.06 \times 10^{+03}$	1.21×10^{-15}	2.13×10^{-15}	2.46×10^{-15}	3.83×10^{-15}
0.7	0.01	$2.06 \times 10^{+03}$	1.23×10^{-15}	1.91×10^{-15}	2.84×10^{-15}	5.29×10^{-15}
0.8	0.01	$2.06 \times 10^{+03}$	1.40×10^{-15}	2.37×10^{-15}	2.42×10^{-15}	4.25×10^{-15}
0.9	0.01	$2.06 \times 10^{+03}$	1.72×10^{-15}	3.02×10^{-15}	2.43×10^{-15}	4.21×10^{-15}
1.0	0.01	$2.06 \times 10^{+03}$	1.75×10^{-15}	3.17×10^{-15}	1.75×10^{-15}	3.17×10^{-15}

TABLE 7. Max and RMS errors, CPU Time, and condition number (CN) for lower and upper solutions on $t \in [0, 1]$ using M6 with randomly variable shape parameter ($\epsilon_{min} = 0.001,\ \epsilon_{max} = 0.01$) and $N = 5$ in example 7.3.

			Errors for $\underline{u}(t;\alpha)$		Errors for $\overline{u}(t;\alpha)$	
α	CPU(s)	CN	RMS	Max Error	RMS	Max Error
0.0	0.02	$9.57 \times 10^{+16}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	5.46×10^{-05}	5.46×10^{-05}
0.1	0.02	$7.99 \times 10^{+16}$	1.75×10^{-06}	1.75×10^{-06}	4.83×10^{-05}	4.83×10^{-05}
0.2	0.02	$1.34 \times 10^{+17}$	8.04×10^{-06}	8.04×10^{-06}	5.89×10^{-05}	5.89×10^{-05}
0.3	0.02	$8.06 \times 10^{+16}$	3.53×10^{-06}	3.53×10^{-06}	2.56×10^{-05}	2.56×10^{-05}
0.4	0.02	$1.89 \times 10^{+17}$	1.08×10^{-05}	1.08×10^{-05}	4.33×10^{-05}	4.33×10^{-05}
0.5	0.02	$9.47 \times 10^{+16}$	5.29×10^{-05}	5.29×10^{-05}	1.25×10^{-04}	1.25×10^{-04}
0.6	0.02	$1.02 \times 10^{+17}$	1.02×10^{-05}	1.02×10^{-05}	5.00×10^{-05}	5.00×10^{-05}
0.7	0.02	$8.35 \times 10^{+16}$	3.45×10^{-05}	3.45×10^{-05}	6.77×10^{-05}	6.77×10^{-05}
0.8	0.02	$7.71 \times 10^{+16}$	4.26×10^{-05}	4.26×10^{-05}	2.53×10^{-04}	2.53×10^{-04}
0.9	0.02	$1.04 \times 10^{+17}$	9.19×10^{-06}	9.19×10^{-06}	2.14×10^{-05}	2.14×10^{-05}
1.0	0.02	$6.84 \times 10^{+16}$	6.37×10^{-05}	6.37×10^{-05}	6.37×10^{-05}	6.37×10^{-05}

FIGURE 2. The approximate solution for example 7.1 on $t \in [0, 10]$ using TPS ($N = 5$)

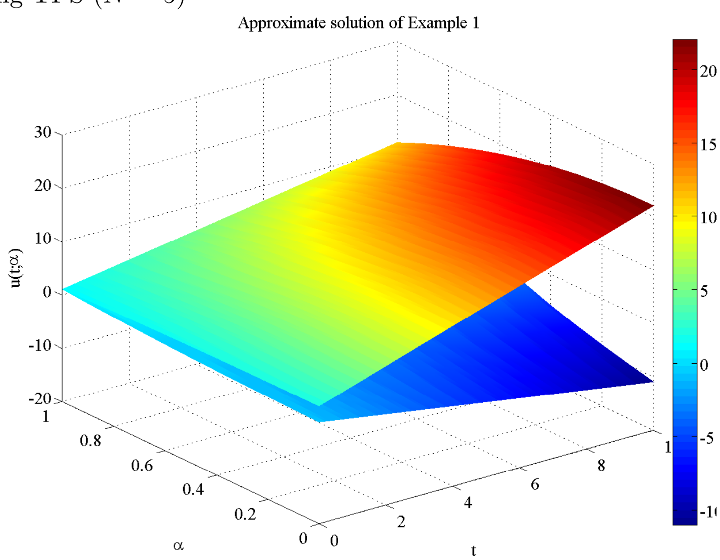


FIGURE 3. The analytical solution for example 7.1 on $t \in [0, 10]$ using TPS ($N = 5$)

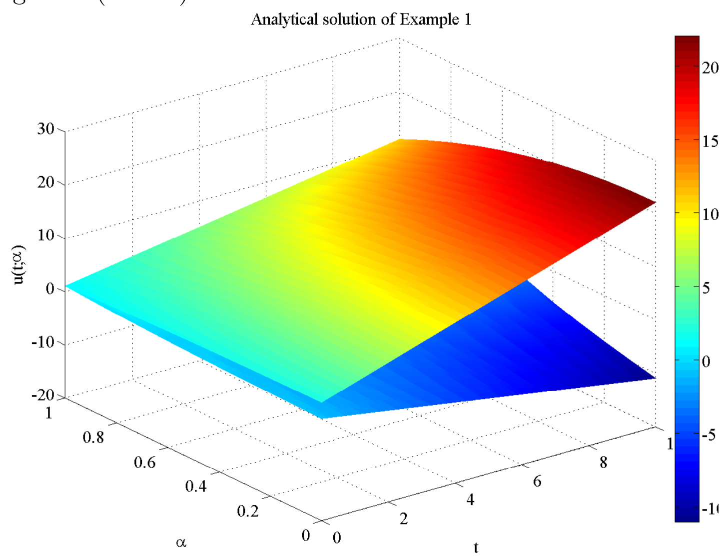


FIGURE 4. The absolute error of lower solution for example 7.1 on $t \in [0, 10]$ using TPS ($N = 5$)

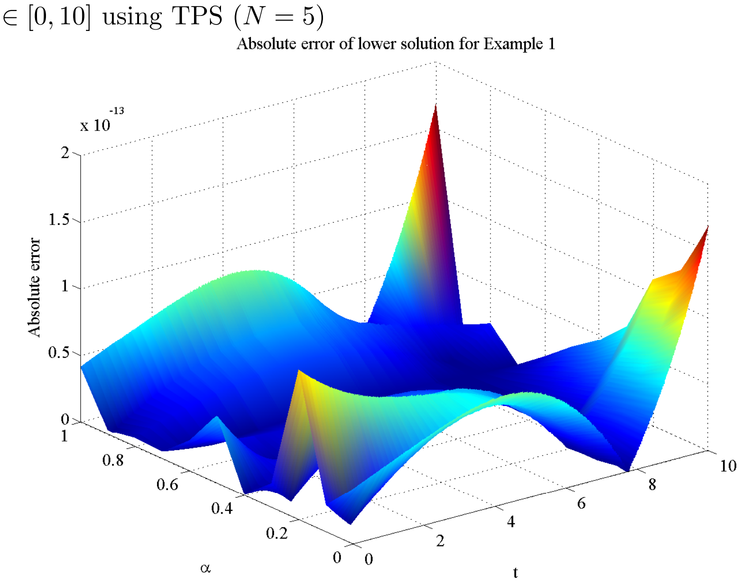
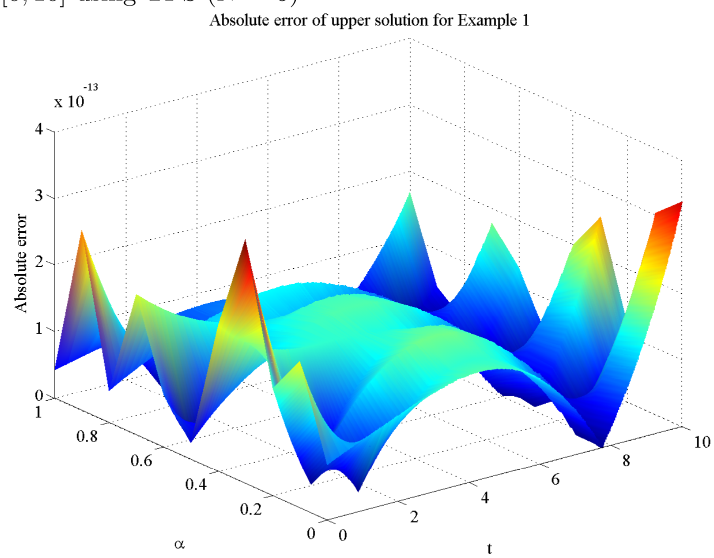


FIGURE 5. The absolute error of upper solution for example 7.1 on $t \in [0, 10]$ using TPS ($N = 5$)

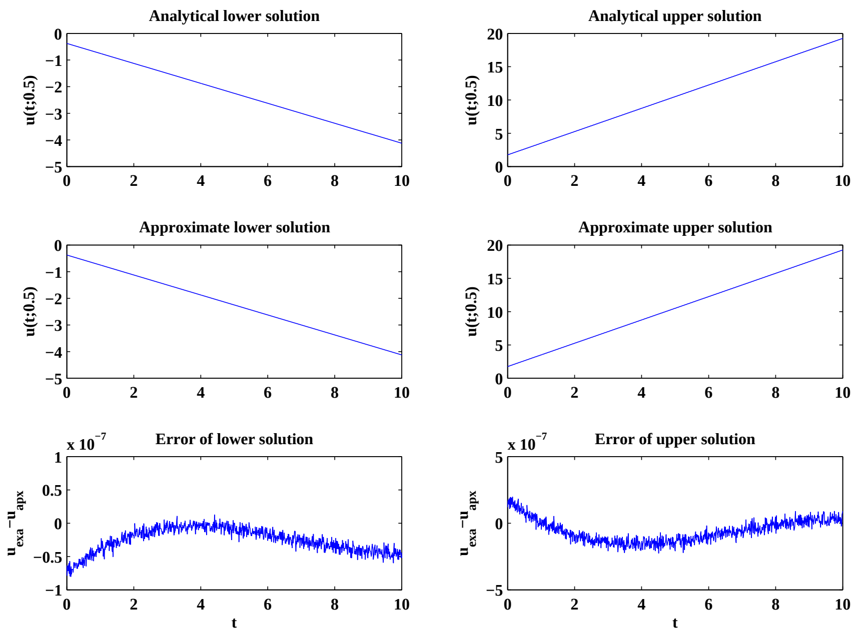


RMS and Maximum errors (accuracy), CPU time (efficiency), and condition number (stability) of our method are reported in Tables 2 and 3 for lower and upper solutions using TPS and M6 basis functions, respectively. Figures 2 and 3 show the profiles of the approximate solution (with 5 collocation points) and the analytical solution using TPS basis function on $t \in [0, 10]$, respectively. From these tables and figures, it is clearly seen that the RBF collocation approximation and the analytical solution are in good agreement. Figures 4 and 5 represent the absolute error graphs of the lower and upper solutions for our scheme using TPS basis function, respectively. Also, we report the profiles of the exact solution, the approximate solution, and error of lower and upper solutions, in Figure 6, using M6 basis function with $\alpha = 0.5$. In this figure, we use the randomly variable shape parameter with $\epsilon_{min} = 0.00001$ and $\epsilon_{max} = 0.0001$. From this study we can note a quite uniform behavior: on the one hand, efficiency (CPU time) of the two basis functions (TPS vs M6) is almost similar, with usually a slight advantage for the TPS basis function, but on the other hand we observe a significant reduction of accuracy and stability for the M6 function compared to the TPS one.

Example 7.2. In this example, consider the fuzzy fractional Bagley-Torvik equation (4.1) with $A = B = C = 1$ and

$$\begin{cases} \underline{f}(t;\alpha) = 2\alpha + \alpha t^2 + \dfrac{4\alpha}{\sqrt{\pi}}\sqrt{t}, \\ \overline{f}(t;\alpha) = 2\,(2-\alpha) + (2-\alpha)\,t^2 + \dfrac{4\,(2-\alpha)}{\sqrt{\pi}}\sqrt{t}. \end{cases}$$

FIGURE 6. The analytical solution, approximate solution, and error of lower and upper solutions for example 7.1 on $t \in [0, 10]$ using M6 with $\alpha = 0.5$ $(N = 5)$



The initial condition is given by

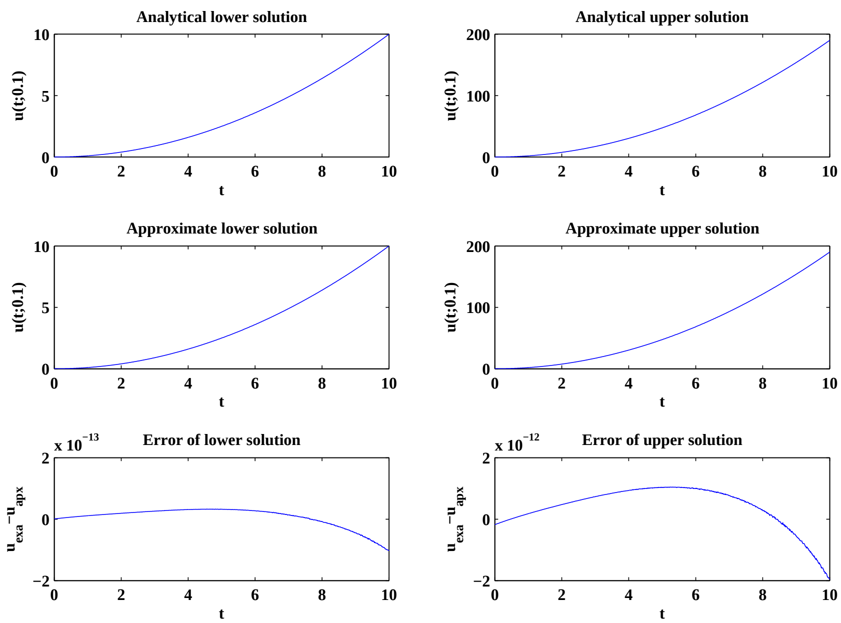
$$\begin{cases} \underline{u}(0;\alpha) = \underline{u}'(0;\alpha) = (\alpha - 1)\,, \\ \overline{u}(0;\alpha) = \overline{u}'(0;\alpha) = (1 - \alpha)\,, \end{cases}$$

and the exact solution [14] is

$$\begin{cases} \underline{u}(t;\alpha) = \alpha t^2, \\ \overline{u}(t;\alpha) = (2 - \alpha)\, t^2. \end{cases}$$

In Tables 4 and 5, L_∞ and RMS error norms, CPU time, and condition number of coefficient matrix using TPS and M6 basis functions are given, respectively. Figures 7 and 8, with $\alpha = 0.1$ and $\alpha = 0.9$, show the profiles of the exact solution, the approximate solution, and error of lower and upper solutions for our method using TPS function, respectively. Also, we report the profiles of the exact solution, the approximate solution, and error of lower and upper solutions, in Figure 9, using M6 basis function with $\alpha = 0.5$. In this figure, we use the randomly variable shape parameter with $\epsilon_{min} = 0.001$ and $\epsilon_{max} = 0.01$. From these tables and figures, it is

FIGURE 7. The analytical solution, approximate solution, and error of lower and upper solutions for example 7.2 on $t \in [0, 10]$ using TPS with $\alpha = 0.1$ $(N = 5)$



clearly seen that the RBF collocation approximation and the analytical solution are in good agreement.

Example 7.3. In this example, consider the fuzzy fractional Bagley-Torvik equation (4.1) with $A = 0, B = C = 1$, and

$$\begin{cases} \underline{f}(t;\alpha) = \alpha t^2 - \alpha t + \dfrac{4\alpha}{\sqrt{\pi}}\sqrt{t}, \\ \overline{f}(t;\alpha) = (2-\alpha)\,t^2 - (2-\alpha)\,t + \dfrac{4\,(2-\alpha)}{\sqrt{\pi}}\sqrt{t}. \end{cases}$$

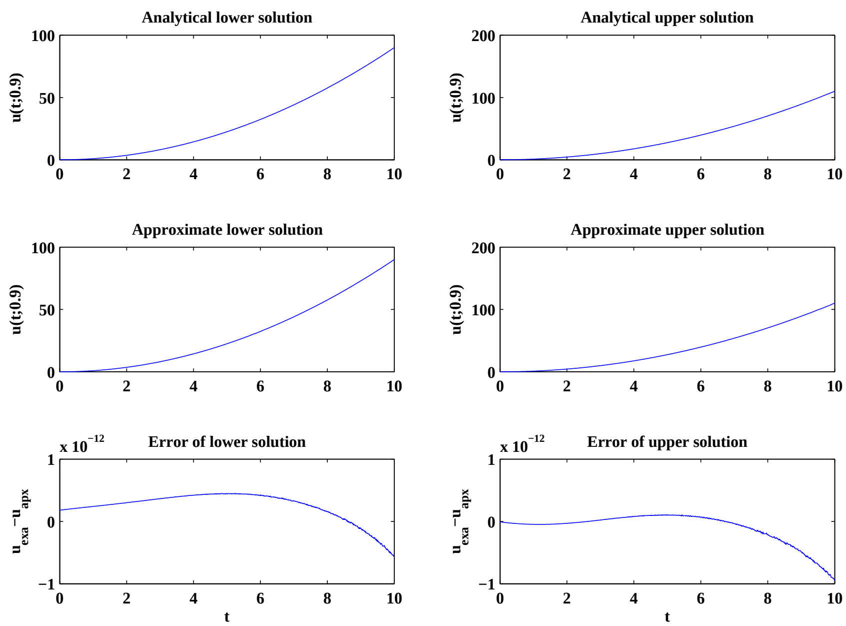
The boundary conditions is given by

$$\begin{cases} \underline{u}(0;\alpha) = \underline{u}(1;\alpha) = (\alpha - 1)\,, \\ \overline{u}(0;\alpha) = \overline{u}(1;\alpha) = (1-\alpha)\,, \end{cases}$$

and the exact solution [40] is

$$\begin{cases} \underline{u}(t;\alpha) = \alpha t^2 - \alpha t, \\ \overline{u}(t;\alpha) = (2-\alpha)\,t^2 - (2-\alpha)\,t. \end{cases}$$

FIGURE 8. The analytical solution, approximate solution, and error of lower and upper solutions for example 7.2 on $t \in [0, 10]$ using TPS with $\alpha = 0.9$ $(N = 5)$

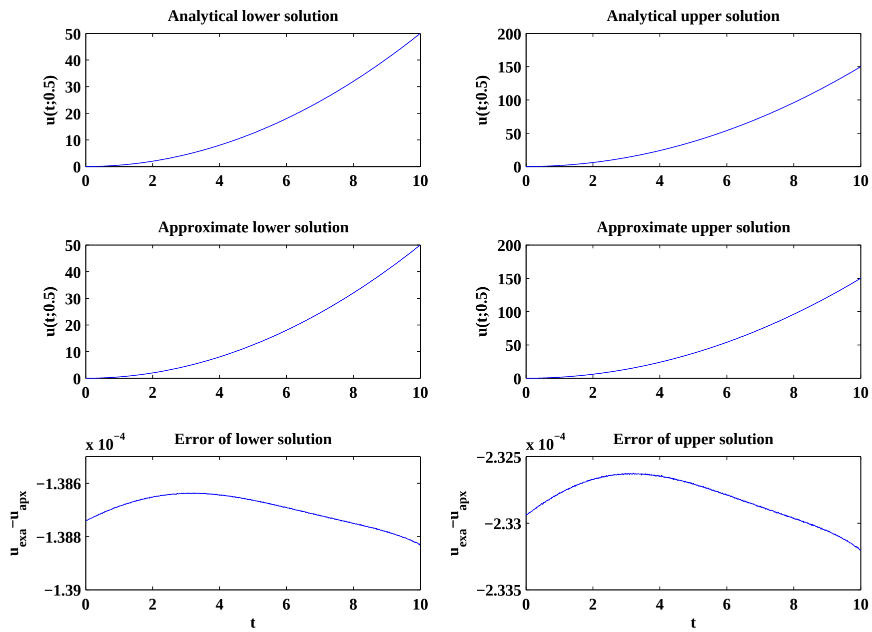


In Tables 6 and 7, L_∞ and RMS error norms, CPU time, and condition number of coefficient matrix using TPS and M6 basis functions are given, respectively. Figures 10 and 11, with $\alpha = 0.5$, show the profiles of the exact solution, the approximate solution, and error of lower and upper solutions for our method using TPS and M6 basis functions, respectively. In Figure 11, we use the randomly variable shape parameter with $\epsilon_{min} = 0.001$ and $\epsilon_{max} = 0.01$. From these results, we can deduce considerations similar to those explained at two previous examples.

8. Conclusions

In this paper, we proposed the spectral collocation method based on radial basis functions to solve the fractional Bagley-Torvik equation under uncertainty, in the fuzzy Caputo's H-differentiability sense with order $(1 < \nu < 2)$. We defined the fuzzy Caputo's H-differentiability sense with order $(1 < \nu < 2)$ which is a direct extension of Caputo derivatives with respect to Hukuhara difference, and then, employed the collocation RBF method for upper and lower solutions. The main advantage of this approach is that the fuzzy fractional Bagley-Torvik equation was reduced to the

FIGURE 9. The analytical solution, approximate solution, and error of lower and upper solutions for example 7.2 on $t \in [0, 10]$ using M6 with $\alpha = 0.5$ ($N = 5$)



problem of solving two systems of linear equations. In many meshfree methods, radial basis functions with shape parameter have been used. Determining a good shape parameter is still an outstanding research topic. To eliminate the effects of the radial basis function shape parameters, we used thin plate spline radial basis functions which have no shape parameters in our scheme. Also, we used Matérn radial function instead of the thin plate spline radial function. In this case, we applied randomly variable shape parameter for our examples. The numerical investigation presented in this paper shows that excellent accuracy can be obtained even when few nodes are used in analysis. In contrast, many more nodes are needed to achieve relatively good accuracy in other methods. Numerical examples are included to demonstrate the validity and applicability of the technique, and are performed on a computer using a code written in MATLAB. Illustrative examples with the satisfactory results are used to demonstrate the application of this method. It is seen from the numerical examples that this method is very attractive and contributed to the excellent agreement between approximate and exact values in the numerical example. The method can be implemented for solving partial and ordinary differential equations in higher dimensions.

FIGURE 10. The analytical solution, approximate solution, and error of lower and upper solutions for example 7.3 on $t \in [0,1]$ using TPS with $\alpha = 0.5$ $(N = 5)$

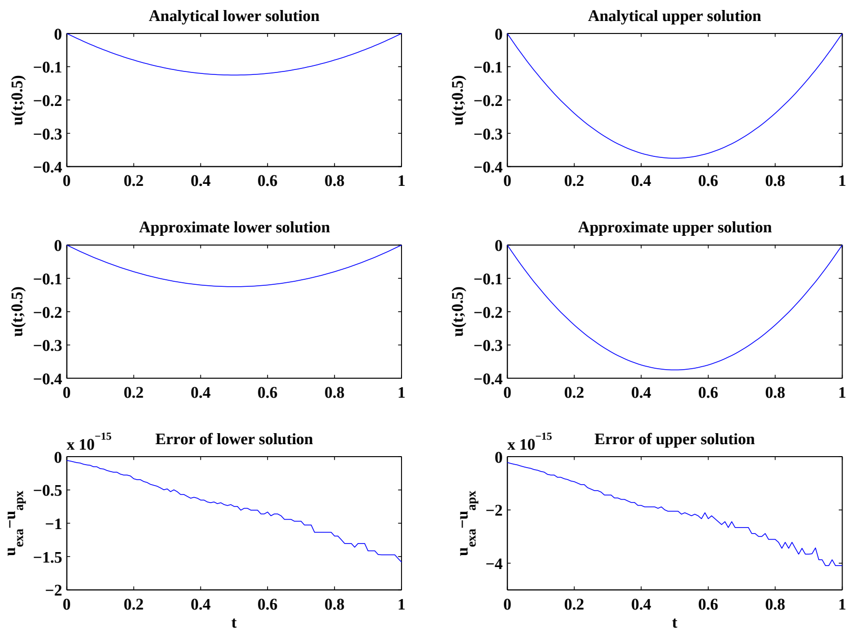


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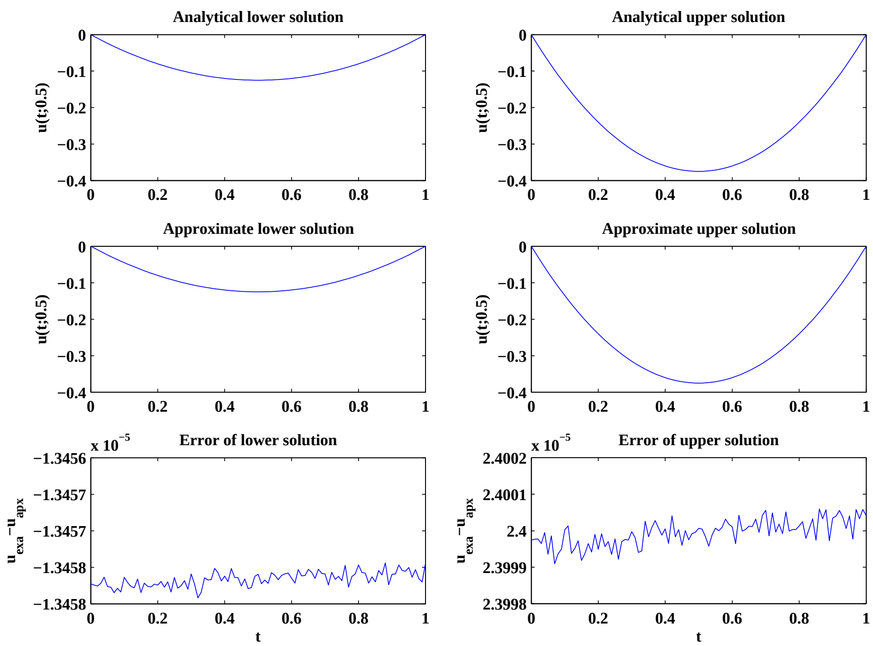
FIGURE 11. The analytical solution, approximate solution, and error of lower and upper solutions for example 7.3 on $t \in [0, 1]$ using M6 with $\alpha = 0.5$ $(N = 5)$



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