Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 6, No. 1, 2018, pp. 1-11

Solving nonlinear space-time fractional differential equations via ansatz method

Ozkan Guner*
Cankiri Karatekin University, Faculty of Economics and Administrative Sciences,
Department of International Trade, Cankiri, Turkey.
E-mail: ozkanguner@karatekin.edu.tr

Ahmet Bekir
Eskisehir Osmangazi University, Art-Science Faculty,
Department of Mathematics-Computer, Eskisehir, Turkey.
E-mail: abekir@ogu.edu.tr

Abstract In this paper, the fractional partial differential equations are defined by modified Riemann-Liouville fractional derivative. With the help of fractional derivative and fractional complex transform, these equations can be converted into the nonlinear ordinary differential equations. By using solitay wave ansatz method, we find exact analytical solutions of the space-time fractional Zakharov Kuznetsov Benjamin Bona Mahony (ZK-BBM) equation, the space-time fractional Klein-Gordon equation and the space-time fractional modified Regularized Long Wave (RLW) equation. This method can be suitable and more powerful for solving other kinds of nonlinear FDEs arising in mathematical physics.

Keywords. Ansatz method, Exact solution, Space-time fractional differential equations.

2010 Mathematics Subject Classification. 26A33, 35R11, 83C15.

1. Introduction

Fractional differential equations are generalizations of differential equations of integer order. So, in recent years, nonlinear fractional differential equations (FDEs) have gained importance and popularity in various fields of science. These equations appear in a great array of contexts such as in plasma physics, fluid mechanics, nonlinear optics, geochemistry, acoustic waves, hydrodynamics, chemical kinematics, control theory, optical fibers, chemical physics, signal processing, systems identification and many other fields [45, 48, 54].

The study of solitary wave has made remarkable advances in the past decades. Soliton is one of the major areas of research in nonlinear dispersive media. There are two different types of envelope solitons bright and dark. This area of research has made an enormous progress especially in recent years [6, 7, 10, 14–18, 43, 60]. The existence of soliton-type solution for nonlinear PDEs is of particular interest because of their extensive applications in many physics areas. This paper addresses the dynamics

Received: 6 April 2017 ; Accepted: 28 November 2017.
$*$ Corresponding author.

of soliton propagation through soliton solutions. This leads to a different kind of nonlinear FDEs that describes the dynamics of soliton propagation [29, 30, 49, 50, 52].

There are, in fact, various modern methods of integrability of a variety of nonlinear fractional differential equations. Some of these methods are the exp-function method, the $(G\prime/G)$-expansion method, the first integral method, the fractional sub-equation method, the functional variable method, the fractional modified trial equation method, the ansatz method, the modified simple equation method and the modified Kudryashov method [2, 5, 8, 9, 12, 13, 19, 21–28, 31, 32, 34–37, 47, 51, 57, 62].

There are several approaches to the generalization of the notion of differentiation to fractional orders e.g. Grünwald–Letnikow, Caputo and Riemann–Liouville [20, 58]. Modified Riemann-Liouville derivative is defined a local fractional derivative by Jumarie [40]. The definition and some properties for the Jumarie's derivative of order α are listed as follows [41]

$$D_w^\alpha f(w) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dw} \int_0^w \frac{f(\tau)-f(0)}{(w-\tau)^\alpha} d\tau, \ 0<\alpha<1, \\ (f^{(n)}(w))^{(\alpha-n)}, \ n \le \alpha \le n+1, \ n \ge 1. \end{cases} \tag{1.1}$$

$$D_w^\alpha w^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} w^{\gamma-\alpha}, \ \gamma > 0, \tag{1.2}$$

$$D_w^\alpha (c) = 0, \quad c = \text{constant}, \tag{1.3}$$

$$D_w^\alpha \{a f(w) + b g(w)\} = a D_w^\alpha f(w) + b D_w^\alpha g(w), \tag{1.4}$$

where $a \neq 0$ and $b \neq 0$ are constants. Now, we will give main steps of methodology.

Step 1: We consider the following general nonlinear space-time FDE of the type

$$H(u, D_t^\alpha u, D_x^\alpha u, D_t^{2\alpha} u, D_t^\alpha D_x^\alpha u, D_x^{2\alpha} u, ...) = 0, \quad 0<\alpha<1, \tag{1.5}$$

where u is an unknown function, and H is a polynomial of u and its partial fractional derivatives.

Step 2: By using fractional complex transform

$$\begin{gathered} u(x,t) = f(\tau), \\ \tau = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \end{gathered} \tag{1.6}$$

where $k \neq 0$ and $c \neq 0$ are constants and by using the chain rule

$$\begin{gathered} D_t^\alpha u = \sigma_t \frac{\partial f}{d\tau} D_t^\alpha \tau, \\ D_x^\alpha u = \sigma_x \frac{\partial f}{d\tau} D_x^\alpha \tau, \end{gathered} \tag{1.7}$$

where σ_t, σ_x are called the sigma indexes [33] and it can take $\sigma_t = \sigma_x = L$, where L is a constant.

Step 3: When we substitute (1.6) with (1.2) and (1.7) into (1.5), we get following nonlinear ODE,

$$N(U, \frac{df}{d\tau}, \frac{d^2 f}{d\tau^2}, \frac{d^3 f}{d\tau^3}, ...) = 0. \tag{1.8}$$

2. Applications

2.1. **The space-time fractional ZK-BBM equation.** Let us consider, the space-time fractional ZK-BBM equation [3]

$$D_t^{\alpha} u + D_x^{\alpha} u - 2au D_x^{\alpha} u - b D_t^{\alpha} (D_x^{2\alpha} u) = 0, \tag{2.1}$$

where a and b are arbitrary constants. It arises as a description of gravity water waves in the long-wave regime. Alzaidy solved this equation by a fractional sub-equation method in [3] and obtained three types of exact analytical solutions. Bekir et al. have applied the functional variable method to obtain new periodic and hyperbolic solutions of Eq.(2.1) in [11]. We will use the ansatz method to obtain the exact solutions with the help of ansatz method. In order to solve Eq.(2.1), we use the transformation (1.6) then integrating Eq.(2.1) with respect to τ and setting the integration constant equal to zero, we have

$$(k-c)f - akf^2 + bck^2L^2f^{''} = 0. \tag{2.2}$$

To obtain bright soliton solution of Eq.(2.2),

$$f(\tau) = A \operatorname{sech}^p \tau, \tag{2.3}$$

where

$$\tau = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{2.4}$$

which k, c and A are constant coefficients. From the ansatz (2.3) and (2.4), we get

$$\frac{d^2 f(\tau)}{d\tau^2} = Ap^2 \operatorname{sech}^p \tau - Ap(p+1) \operatorname{sech}^{p+2} \tau, \tag{2.5}$$

and

$$f^2(\tau) = A^2 \operatorname{sech}^{2p} \tau. \tag{2.6}$$

Thus, substituting the ansatz (2.3)-(2.6) into Eq.(2.2), yields to

$$\begin{aligned} &(k-c)A \operatorname{sech}^p \tau - akA^2 \operatorname{sech}^{2p} \tau \\ &- bck^2L^2Ap^2 \operatorname{sech}^p \tau - bck^2L^2Ap(p+1) \operatorname{sech}^{p+2} \tau \\ &= 0. \end{aligned} \tag{2.7}$$

From (2.7), when we equate exponents $p+2$ and $2p$, that leads to $p = 2$. From (2.7), setting the coefficients of $\operatorname{sech}^{p+2} \tau$ and $\operatorname{sech}^{2p} \tau$ terms to zero,

$$-akA^2 - bck^2L^2Ap(p+1) = 0, \tag{2.8}$$

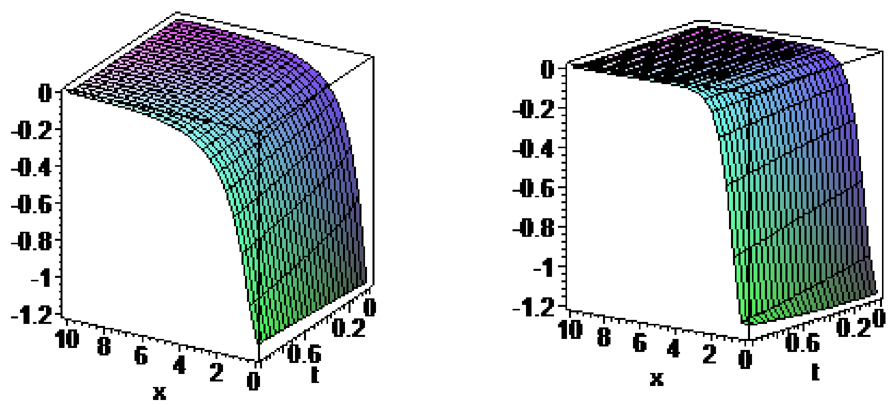
then we obtain

$$A = -\frac{bckL^2p(p+1)}{a}. \tag{2.9}$$

We find, from setting the coefficients of $\operatorname{sech}^p \tau$ terms in Eq.(2.7) to zero

$$(k-c)A - bck^2L^2Ap^2 = 0, \tag{2.10}$$

FIGURE 1. Shape of solution for (2.12) with $k = 1, a = 1, b = 1, L = 1$.



(a) $\alpha = 0.5$ (b) $\alpha = 1$

also we get

$$c = \frac{k}{bk^2L^2p^2 + 1}. \tag{2.11}$$

Finally; when we use $p = 2$, we get the bright soliton solution for the space-time fractional ZK-BBM equation as follow:

$$u(x,t) = -\frac{6bckL^2A}{a}\operatorname{sech}^2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{kt^\alpha}{(4bk^2L^2+1)\,\Gamma(1+\alpha)}\right). \tag{2.12}$$

The solution.(2.12) is represented in Figure 1, within the interval $0 < x < 10$ and $0 < t < 1$.

Remark 1. The solution (2.12) is not given in [3, 11] and have not been reported by other authors in the literature.

2.2. **The nonlinear space-time fractional Klein-Gordon equation.** We next consider the nonlinear space-time fractional Klein-Gordon (KG) equation [39]

$$D_{tt}^{2\alpha}u - D_{xx}^{2\alpha}u + \gamma u - \beta u^2 = 0. \tag{2.13}$$

where α, β are nonzero constant. This equation describes many types of nonlinearities and plays a significant role in several real world applications such as the solid state physics, nonlinear optics and quantum field theory. Baleanu and his colleagues have found many new types of exact travelling wave solutions of KG equation by using the auxiliary equation method by using sub-equation method, and obtained new exact solutions of equation (2.13) containing hyperbolic, trigonometric and rational functions. In [11], the functional variable method successfully applied to finding periodic and hyperbolic solutions of the fractional KG equation by Bekir at al. When

$\alpha = 1$, equation (2.13) is called the quadratic nonlinear Klein–Gordon equation and there are a lot of studies for this equation [4, 38, 46, 53, 59, 61, 63].

Now we consider the nonlinear fractional KG equation. With the same process as in the previous example, we obtain following ODE

$$(c^2L^2 - k^2L^2)f^{''} + \gamma U - \beta U^2 = 0, \tag{2.14}$$

where $f' = \frac{df}{d\tau}$.

From the ansatz (2.3) and (2.4), we obtain necessary derivatives. Then, substituting them into Eq.(2.14), yields to

$$\begin{aligned}&(c^2L^2 - k^2L^2)Ap^2 \operatorname{sech}^p \tau - (c^2L^2 - k^2L^2)Ap(p+1)\operatorname{sech}^{p+2}\tau\\ &+ \gamma A \operatorname{sech}^p \tau - \beta A^2 \operatorname{sech}^{2p}\tau \\ &= 0.\end{aligned} \tag{2.15}$$

From (2.15), equating exponents $p+2$ and $2p$, that gives $p = 2$. From (2.15), setting the coefficients of $\operatorname{sech}^{p+2}\tau$ and $\operatorname{sech}^{2p}\tau$ terms to zero,

$$(c^2L^2 - k^2L^2)Ap(p+1) + \beta A^2 = 0, \tag{2.16}$$

by use (4.10), we obtain

$$A = -\frac{(c^2L^2 - k^2L^2)p(p+1)}{\beta}. \tag{2.17}$$

Analogously, from setting the coefficients of $\operatorname{sech}^p \tau$ terms in Eq.(2.15) to zero, we have

$$(c^2L^2 - k^2L^2)Ap^2 + \gamma A = 0, \tag{2.18}$$

then we get

$$c = \mp\frac{\sqrt{p^2k^2L^2 + \gamma}}{pL}. \tag{2.19}$$

Consequently, we can determine the bright soliton solution of (2.13) as with $p = 2$,

$$u(x,t) = -\frac{6(c^2L^2 - k^2L^2)A}{\beta}\operatorname{sech}^2\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} \pm \frac{\sqrt{4k^2L^2+\gamma}t^\alpha}{2L\Gamma(1+\alpha)}\right). \tag{2.20}$$

Also, Eq.(2.19) implies the domain restrictions $4k^2L^2 + \gamma > 0$.

We plot the solutions of Eq.(2.20) for this equation in Figure 2 within the interval $0 < x < 100$ and $0 < t < 100$.

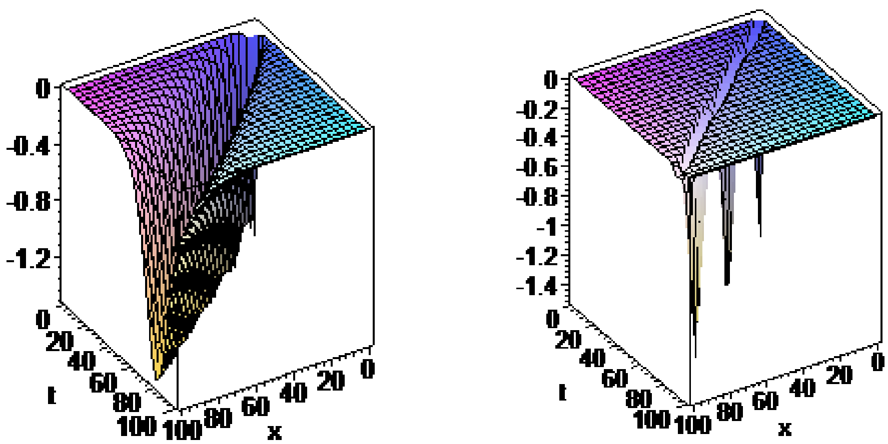
Remark 2. Comparing our result with Baleanu's and Bekir's [11, 39] results, it can be seen that the result is new.

2.3. **The space-time fractional modified RLW equation.** This equation has the form [42]

$$D_t^\alpha u + vD_x^\alpha u + \mu u^2 D_x^\alpha u - \varepsilon D_t^\alpha D_x^{2\alpha} u = 0, \tag{2.21}$$

where α describing the order of the fractional derivatives $0 < \alpha \leq 1$ and v, μ and ε are all constants. Kaplan et al. solved this equation by the modified simple equation

Figure 2. Shape of solution for (2.20) with $k = 1, L = 1, \beta = 1, \gamma = 1$.



(a) $\alpha = 0.5$

(b) $\alpha = 1$

method [42]. Abdel-Salam and Gumma have obtained abundant types of exact analytical solutions including generalized trigonometric and hyperbolic functions solutions of this equation with the improved fractional Riccati expansion method in [1]. The modified RLW equation is considered as an alternative to the modified KdV equation. This equation is modeled to govern a large number of physical phenomena such as transverse waves in shallow water and magneto hydrodynamic waves in plasma and phonon packets in nonlinear crystals [44, 55, 56].

When we substitute (1.6) with (1.2) and (1.7) into (2.21), integrating Eq.(2.1) with respect to τ and setting the integration constant equal to zero, Eq.(2.21) can reduced into an ODE

$$(kv - c)f + \tfrac{\mu k}{3} f^3 + \varepsilon c k^2 L^2 f'' = 0, \tag{2.22}$$

where $f' = \frac{df}{d\tau}$.

From the ansatz (2.3) and (2.4), we obtain necessary derivatives. Then, substituting them into Eq.(2.22), yields to

$$\begin{aligned}&(kv - c)A \operatorname{sech}^p \tau + \frac{\mu k}{3} A^3 \operatorname{sech}^{3p} \tau \\ &+ \varepsilon c k^2 L^2 A p^2 \operatorname{sech}^p \tau - \varepsilon c k^2 L^2 A p(p+1) \operatorname{sech}^{p+2} \tau \\ &= 0.\end{aligned} \tag{2.23}$$

From (2.23), if we equate the exponents $p + 2$ and $3p$, we have

$$p = 1. \tag{2.24}$$

When we set, coefficients of $\operatorname{sech}^{p+2}\tau$ and $\operatorname{sech}^{3p}\tau$ terms to zero in Eq.(2.23), we get

$$\frac{\mu k}{3}A^3 - \varepsilon ck^2L^2Ap(p+1) = 0, \tag{2.25}$$

by use (2.24) and after some calculations, we have

$$A = \mp L\sqrt{\frac{6\varepsilon kc}{\mu}},\ \mu \neq 0. \tag{2.26}$$

Again from setting coefficients of $\operatorname{sech}^{p}\tau$ terms in Eq.(2.23) to zero

$$(kv - c)A + \varepsilon ck^2L^2Ap^2 = 0, \tag{2.27}$$

we obtain

$$c = \frac{vk}{1 - \varepsilon k^2L^2}. \tag{2.28}$$

From (2.28) it is important to note that

$$4\varepsilon k^2L^2 \neq 1. \tag{2.29}$$

Thus finally, we can determine the bright soliton solution of (2.21) as with $p = 1$,

$$u(x,t) = A\operatorname{sech}\left(\frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right), \tag{2.30}$$

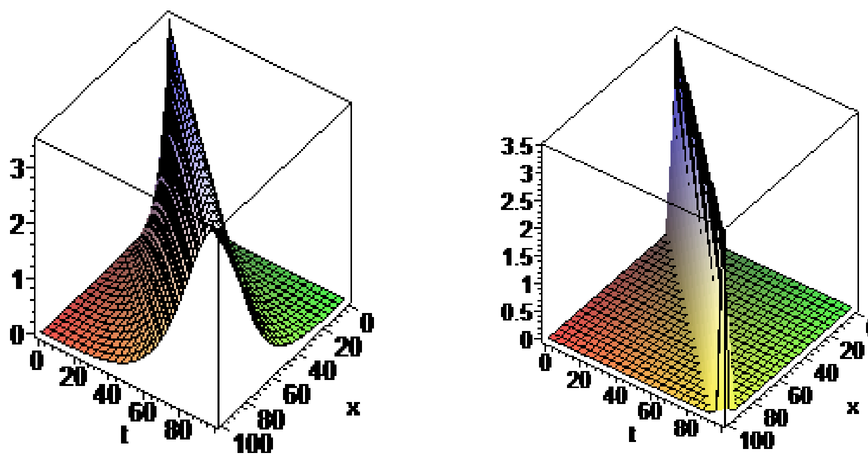
where the A is given by (2.26) and the c is given by (2.28). We plot the solutions of Eq.(2.30) for this equation in Figure 3 within the interval $0 < x < 100$ and $0 < t < 100$.

Remark 3. Note that solution (2.30) is quite different from the travelling wave solutions found in [1, 42].

3. Conclusion

In this paper, the ansatz method is used to obtain the bright soliton solution of the nonlinear FDEs with Jumarie's modified Riemann–Liouville derivative. In general, there exist no method that yields soliton solutions for fractional differential equations. But, a fractional complex transform is adopted in this paper to convert such equations into classical partial differential equations. We succeeded in extracting soliton solutions for the space-time fractional ZK-BBM equation, the space-time fractional Klein-Gordon equation and the space-time fractional modified RLW equation. As a result, some new exact solutions for them have been succesfully found. Being concise and powerful, the proposed method can be applied to solve other nonlinear FDEs and systems. All the solutions reported above have been verified using the symbolic computation system Maple.

Figure 3. Shape of solution for (2.30) with $k = 1, \tau = 2, v = -1, L = 1, \mu = 1$.



(a) $\alpha = 0.5$ (b) $\alpha = 1$

References

[1] E. A. B. Abdel-Salam and E. A. E. Gumma, *Analytical solution of nonlinear space-time fractional differential equations using the improved fractional Riccati expansion method*, Ain Shams Eng. J., *6* (2015), 613-620.

[2] E. Aksoy, A. C. Cevikel, and A. Bekir, *Soliton solutions of (2+1)-dimensional time-fractional Zoomeron equation*, Optik, *127*(17) (2016), 6933-6942.

[3] J. F. Alzaidy, *Fractional Sub-Equation Method and its Applications to the Space–Time Fractional Differential Equations in Mathematical Physics*, British Journal of Mathematics & Computer Science, *3* (2013), 153-163.

[4] J. F. Alzaidy, *The Fractional Sub-Equation Method and Exact Analytical Solutions for Some Nonlinear Fractional PDEs*, American Journal of Mathematical Analysis, *1*(1) (2013), 14-19.

[5] A. H. Arnous, M. Mirzazadeh, Q. Zhou, S. P. Moshokoa, A. Biswas, and M. Belic, *Soliton solutions to resonant nonlinear schrodinger's equation with time-dependent coefficients by modified simple equation method*, Optik, *127*(23) (2016), 11450-11459.

[6] A. Bekir, E. Aksoy and O. Guner, *Optical soliton solutions of the Long-Short-Wave interaction system*, Journal of Nonlinear Optical Physics & Materials, *22*(2) (2013), 1350015(11 pages).

[7] A. Bekir and O. Guner, *Bright and dark soliton solutions of the (3 + 1)-dimensional generalized Kadomtsev-Petviashvili equation and generalized Benjamin equation*, Pramana-J. Phys., *81*(2) (2013), 203-214.

[8] A. Bekir and O. Guner, *Exact solutions of distinct physical structures to the fractional potential Kadomtsev-Petviashvili equation*, Comput. Methods Differ. Equ., *2*(1) (2014), 26-36.

[9] A. Bekir and O. Guner, *Exact solutions of nonlinear fractional differential equations by* (G'/G)*-expansion method*, Chin. Phys. B, *22*(11) (2013), 110202(6 pages).

[10] A. Bekir and O. Guner, *Topological (dark) soliton solutions for the Camassa-Holm type equations*, Ocean Eng., *74* (2013), 276-279.

[11] A. Bekir, O. Guner, and E. Aksoy, *Periodic and hyperbolic solutions of nonlinear fractional differential equations*, Appl. Comput. Math., *15*(1) (2016), 88-95.

[12] A. Bekir, O. Guner, A. H. Bhrawy, and A. Biswas, *Solving nonlinear fractional differential equations using exp-function and (G'/G)-expansion methods*, Rom. Journ. Phys., *60*(3-4) (2015), 360-378.

[13] A. Bekir, O. Guner, and A. C. Cevikel, *Fractional Complex Transform and exp-Function Methods for Fractional Differential Equations*, Abstr. Appl. Anal., *2013* (2013), Article ID 426462, 8 pages.

[14] A. Biswas, *1-Soliton solution of the B(m,n) equation with generalized evolution*, Commun. Nonlinear Sci. Numer. Simul., *14* (2009), 3226-3229.

[15] A. Biswas, *1-Soliton solution of the K(m,n) equation with generalized evolution*, Phys. Lett. A, *372* (2008), 4601-4602.

[16] A. Biswas, *Optical solitons with time-dependent dispersion, nonlinearity and attenuation in a power-law media*, Commun. Nonlinear Sci. Numer. Simulat., *14* (2009), 1078-1081.

[17] A. Biswas and D. Milovic, *Bright and dark solitons of the generalized nonlinear Schrödinger's equation*, Commun. Nonlinear Sci. Numer. Simulat., *15* (2010), 1473-1484.

[18] A. Biswas, H. Triki, T. Hayat, and O. M. Aldossary, *1-Soliton solution of the generalized Burgers equation with generalized evolution*, Applied Mathematics and Computation, *217* (2011), 10289-10294.

[19] H. Bulut, H. M. Baskonus, and Y. Pandir, *The Modified Trial Equation Method for Fractional Wave Equation and Time Fractional Generalized Burgers Equation*, Abstr. Appl. Anal., *2013* (2013), Article ID 636802, 8 pages.

[20] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent II*, Geophys. J. Royal Astronom. Soc, *13* (1967), 529-539.

[21] S. T. Demiray and H. Bulut, *Generalized Kudryashov method for nonlinear fractional double sinh-Poisson equation*, J. Nonlinear Sci. Appl., *9* (2016), 1349-1355.

[22] S. T. Demiray, Y. Pandir, and H. Bulut, *Generalized Kudryashov Method for Time-Fractional Differential Equations*, Abstr. Appl. Anal., *2014* (2014), Article ID 901540, 13 pages.

[23] S. M. Ege and E. Mısırlı, *The modified Kudryashov method for solving some fractional-order nonlinear equations*, Adv. Differ. Equ., *2014* (2014), Article number 135, 13 pages.

[24] M. Ekici, M. Mirzazadeh, M. Eslami, Q. Zhou, S.P. Moshokoa, A. Biswas, and M. Belic, *Optical soliton perturbation with fractional-temporal evolution by first integral method with conformable fractional derivatives*, Optik, *127* (2016), 10659-10669.

[25] M. Eslami, V. B. Fathi, M. Mirzazadeh, and A. Biswas, *Application of first integral method to fractional partial differential equations*, Indian J. Phys., *88* (2014), 177-184.

[26] M. Eslami and M. Mirzazadeh, *First integral method to look for exact solutions of a variety of Boussinesq-like equations*, Ocean Eng., *83* (2014), 133-137.

[27] K. A. Gepreel and S. Omran, *Exact solutions for nonlinear partial fractional differential equations*, Chin. Phys. B, *21*(11) (2012), 110204(7 pages).

[28] M. Gubes, Y. Keskin, and G. Oturanc, *Numerical solution of time-dependent Foam Drainage Equation (FDE)*, Comput. Methods Differ. Equ., *3*(2) (2015), 111-122.

[29] O. Guner, *Singular and non-topological soliton solutions for nonlinear fractional differential equations*, Chin. Phys. B, *24*(10) (2015), 100201(6 pages).

[30] O. Guner and A. Bekir, *Bright and dark soliton solutions for some nonlinear fractional differential equations*, Chin. Phys. B, *25*(3) (2016), 030203(8 pages).

[31] O. Guner and A. C. Cevikel, *A Procedure to Construct Exact Solutions of Nonlinear Fractional Differential Equations*, Sci. World J., *2014* (2014), Article ID 489495, 10 pages.

[32] O. Guner and D. Eser, *Exact Solutions of the Space Time Fractional Symmetric Regularized Long Wave Equation Using Different Methods*, Advances in Mathematical Physics, *2014* (2014), Article ID 456804, 8 pages.

[33] J. H. He, S. K. Elegan, and Z. B. Li, *Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus*, Physics Letters A, *376* (2012), 257-259.

[34] K. Hosseini and R. Ansari, *New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method*, Waves in Random and Complex Media, 2017. DOI 10.1080/17455030.2017.1296983.

[35] K. Hosseini, A. Bekir, and R. Ansari, *New exact solutions of the conformable time-fractional Cahn-Allen and Cahn-Hilliard equations using the modified Kudryashov method*, Optik, *132* (2017), 203-209.

[36] K. Hosseini, A. Bekir, and M. Kaplan, *New exact traveling wave solutions of the Tzitzéica-type evolution equations arising in non-linear optics*, Journal of Modern Optics, *64*(16) (2017), 1688-1692.

[37] K. Hosseini, P. Mayeli, and R. Ansari, *Modified Kudryashov method for solving the conformable time-fractional Klein-Gordon equations with quadratic and cubic nonlinearities*, Optik, *130* (2017), 737-742.

[38] H. Jafari, A. Borhanifar, and S. A. Karimi, *New solitary wave solutions for generalized regularized long-wave equation*, International Journal of Computer Mathematics, *87* (2009), 509-514.

[39] H. Jafari, H. Tajadodi, N. Kadkhoda, and D. Baleanu, *Fractional Subequation Method for Cahn-Hilliard and Klein-Gordon Equations*, Abstr. Appl. Anal., *2013* (2013), Article ID 587179, 5 pages.

[40] G. Jumarie, *Modified Riemann–Liouville derivative and fractional Taylor series of nondifferentiable functions further results*, Comput. Math. Appl., *51* (2006), 1367-1376.

[41] G. Jumarie, *Table of some basic fractional calculus formulae derived from a modified Riemann-Liouvillie derivative for nondifferentiable functions*, Appl. Maths. Lett., *22* (2009), 378-385.

[42] M. Kaplan, A. Bekir, A. Akbulut, and E. Aksoy, *The modified simple equation method for nonlinear fractional differential equations*, Rom. Journ. Phys., *60*(9-10) (2015), 1374-1383.

[43] C. M. Khalique and A. Biswas, *Optical solitons with parabolic and dual-power law nonlinearity via Lie group analysis*, Journal of Electromagnetic Waves and Applications, *23*(7) (2009), 963-973.

[44] A. K. Khalifaa, K. R. Raslana, and H. M. Alzubaidi, *A collocation method with cubic B-splines for solving the MRLW equation*, Journal of Computational and Applied Mathematics, *212*(2) (2008), 406-418.

[45] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[46] S. Liu, Z. Fu, S. Liu, and Z. Wang, *Combinability of Travelling Wave Solutions to Nonlinear Evolution Equation*, Verlag der Zeitschriftfür Naturforschung, *59a* (2004), 623-628.

[47] B. Lu, *The first integral method for some time fractional differential equations*, J. Math. Anal. Appl., *395* (2012), 684-693.

[48] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[49] M. Mirzazadeh, *Topological and non-topological soliton solutions to some time-fractional differential equations*, Pramana-J. Phys., *85*(1) (2015), 17-29.

[50] M. Mirzazadeh, M. Eslami, B. F. Vajargah, and A. Biswas, *Solitons and Periodic Solutions to a Couple of Fractional Nonlinear Evolution Equations*, Pramana-J. Phys., *82*(3) (2014), 465-476.

[51] M. Mirzazadeh, M. Eslami, E. Zerrad, M. F. Mahmood, A. Biswas, and M. Belic, *Optical solitons in nonlinear directional couplers by sine-cosine function method and Bernoulli's equation approach*, Nonlinear Dynamics, *81*(4) (2015), 1933-1949.

[52] S. T. Mohyud-Din, Y. Khan, F. Faraz, and A. Yıldırım, *Exp-function method for solitary and periodic solutions of Fitzhugh-Nagumo equation*, International Journal of Numerical Methods for Heat & Fluid Flow, *22*(3) (2012), 335-341.

[53] E. J. Parkes, B. R. Duffy, and P. C. Abbott, *The Jacobi elliptic-function method for finding periodic-wave solutions to nonlinear evolution equations*, Physics Letters A, *295* (2002), 280-286.

[54] I. Podlubny, *Fractional Differential Equations*, Academic Press, California, 1999.

[55] K. R. Raslan, *Numerical study of the Modified Regularized Long Wave (MRLW) equation*, Chaos, Solitons & Fractals, *42*(3) (2009), 1845-1853.

[56] K. R. Raslan and S. M. Hassan, *Solitary waves for the MRLW equation*, Applied Mathematics Letters, *22*(7) (2009), 984-989.

[57] S. Sahoo and S. S. Ray, *Improved fractional sub-equation method for (3+1)-dimensional generalized fractional KdV-Zakharov-Kuznetsov equations*, Computers and Mathematics with Applications, *70* (2015), 158-166.

[58] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Switzerland, 1993.

[59] Sirendaoreji, *Auxiliary equation method and new solutions of Klein-Gordon equations*, Chaos, Solitons and Fractals, *31*(4) (2007), 943-950.

[60] H. Triki and A. M. Wazwaz, *Bright and dark soliton solutions for a K(m,n) equation with t-dependent coefficients*, Phys. Lett. A, *373* (2009), 2162-2165.

[61] F. Xu, *Application of Exp-function method to Symmetric Regularized Long Wave (SRLW) equation*, Physics Letters A, *372* (2008), 252-257.

[62] E. M. E. Zayed and K. A. E. Alurrfi, *The modified Kudryashov method for solving some seventh order nonlinear PDEs in mathematical physics*, World Journal of Modelling and Simulation, *11*(4) (2015), 308-319.

[63] E. M. E. Zayed, Y. A. Amer, and R. M. A. Shohib, *Exact traveling wave solutions for nonlinear fractional partial differential equations using the improved* (G'/G)*-expansion method*, International Journal of Engineering and Applied Science, *4*(7) (2014), 18-31.