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Solution to time fractional generalized KdV of order 2q+1 and system of space fractional PDEs

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Abstract In this work, it has been shown that the combined use of exponential operators and integral transforms provides a powerful tool to solve time fractional generalized KdV of order 2q+1 and certain fractional PDEs. It is shown that exponential operators are an effective method for solving certain fractional linear equations with non-constant coefficients. It may be concluded that the combined use of integral transforms and exponential operator method is very efficient tool in finding exact solutions for ordinary and partial differential equations with fractional order. Finally, illustrative examples are also provided.

Keywords. Fractional partial differential equations; Exponential operational method; Modified Bessel's functions; Riemann - Liouville fractional derivative; Laplace transform; Caputo fractional derivative.

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1. Introduction

In this study, we present a general method of operational nature to obtain solutions for several types of partial differential equations.
Until now, two methods, have been more extensively used for solving partial fractional differential equations, the Laplace and Fourier transformations on the one hand and separation of variables on the other hand. Many powerful and efficient methods have been proposed so far, including the first integral method, the fractional $(G^{'}/G)$-expansion method, [5], [6], [9]. Let us mention also solution in the form of a series of functions. Mathematical physics and population growth models characterized by systems of partial differential equations, such as shallow water waves, Lotka - Volterra model, Brusselator model are of wide applicability. The main purpose of this work has been to employ the integral transforms and exponential operator method for studying certain models. The goal has been achieved by formally deriving the exact analytic solutions.

Definition 1.1. The Laplace transform of function $f(t)$ is defined as follows [4]

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s). \tag{1.1}$$

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If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \tag{1.2}$$

where $F(s)$ is analytic in the region $\mathrm{R}e(s) > c$.
Definition 1.2. If the function $\Phi(t)$ belongs to $C[a,b]$ and $a < t < b$, the left Riemann-Liouville fractional integral of order $0 < \alpha < 1$ is defined as

$$I_a^{RL,\alpha}\{\Phi(t)\} = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\Phi(\xi)}{(t-\xi)^{1-\alpha}}d\xi. \tag{1.3}$$

Definition 1.3. The left Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined as follows [7]

$$D_a^{RL,\alpha}\phi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^t \frac{\Phi(\xi)}{(t-\xi)^{\alpha}}d\xi. \tag{1.4}$$

It follows that $D_a^{RL,\alpha}\phi(x)$ exists for all $\Phi(t)$ belongs to $C[a,b]$, and $a < t < b$.
Note. A very useful fact about the R- L operators is that they satisfy semi group properties of fractional integrals.The special case of fractional derivative when $\alpha = 0.5$ is called semiderivative.
Definition 1.4. The left Caputo fractional derivative of order α $(0 < \alpha < 1)$ of $\phi(t)$ is as follows

$$D_a^{c,\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{1}{(t-\xi)^{\alpha}}\phi'(\xi)d\xi. \tag{1.5}$$

Note. Let us recall the following important lemmas that will be used throughout the paper.
Lemma 1.1. Let $\mathcal{L}\{f(t)\} = F(s)$ then, the following identities hold true.

(1) $\mathcal{L}^{-1}(e^{-k\sqrt{s}}) = \frac{k}{(2\sqrt{\pi})}\int_0^\infty e^{-t\xi-\frac{k^2}{4\xi}}d\xi,$
(2) $e^{-\omega s^\beta} = \frac{1}{\pi}\int_0^\infty e^{-r^\beta(\omega cos\beta\pi)}sin(\omega r^\beta sin\beta\pi)(\int_0^\infty e^{-s\tau-r\tau}d\tau)dr,$
(3) $\mathcal{L}^{-1}(F(s^\alpha)) = \frac{1}{\pi}\int_0^\infty f(u)\int_0^\infty e^{-tr-ur^\alpha cos\alpha\pi}sin(ur^\alpha sin\alpha\pi)drdu,$
(4) $\mathcal{L}^{-1}(F(\sqrt{s}) = \frac{1}{2t\sqrt{\pi t}}\int_0^\infty ue^{-\frac{u^2}{4t}}f(u)du.$

Proof. See [1], [2],[3].
Lemma 1.2. The following exponential identities hold true.

(1) $\exp(\pm\lambda\frac{d}{dt})\Phi(t) = \Phi(t\pm\lambda),$
(2) $\exp(\pm\lambda t\frac{d}{dt})\Phi(t) = \Phi(te^{\pm\lambda}),$
(3) $\exp(\lambda q(t)\frac{d}{dt})\Phi(t) = \Phi(Q(F(t)+\lambda)),$

where $F(t)$ is primitive of $(q(t))^{-1}$ and $Q(t)$ is inverse of $F(t)$.
Proof. See [5],[6].
The most important use of the Caputo fractional derivative is treated in initial value

problems where the initial conditions are expressed in terms of integer order derivatives. In this respect, it is interesting to know the Laplace transform of this kind of derivative.

$$\mathcal{L}\{D_a^{c,\alpha} f(t)\} = sF(s) - f(0+), 0 < \alpha < 1, \tag{1.6}$$

and generally [7]

$$\mathcal{L}\{D_a^{c,\alpha} f(t)\} = s^\alpha F(s) - \sum_{k=0}^{k=m-1} s^{\alpha-1-k} f^k(0+), m-1 < \alpha < m. \tag{1.7}$$

The Laplace transform provides a useful technique for the solution of such fractional singular integro-differential equations.

Example 1.1. Let us solve the following fractional Volterra integral equation of convolution type.

$$\lambda \int_0^t \sqrt[m]{(t-\xi)} D^\alpha \phi(\xi) d\xi = (\frac{t}{a})^{\frac{\mu}{2}} I_\mu(2\sqrt{at}) \quad \phi(0) = 0, \alpha + \mu \geq 1 + \frac{1}{m}. \tag{1.8}$$

Solution: Upon taking the Laplace transform of the given integral equation, we obtain

$$s^\alpha \Phi(s) \frac{\Gamma(1+\frac{1}{m})}{s^{1+\frac{1}{m}}} = \frac{e^{\frac{a}{s}}}{s^{1+\mu}}, \tag{1.9}$$

solving the above equation, leads to

$$\Phi(s) = \frac{e^{\frac{a}{s}}}{\Gamma(1+\frac{1}{m}) s^{\alpha - \frac{1}{m} + \mu}}, \tag{1.10}$$

at this point, taking the inverse Laplace transform term wise, after simplifying we obtain

$$\phi(t) = \frac{1}{\Gamma(1+\frac{1}{m})} (\frac{t}{a})^{\frac{\alpha - \frac{1}{m} + \mu - 1}{2}} I_{\alpha - \frac{1}{m} + \mu - 1}(2\sqrt{at}). \tag{1.11}$$

Note. In the above relation $I_\eta(.)$, stands for the modified Bessel's function of the first kind of order η.

Example 1.2. Let us solve the following impulsive fractional differential equation.

$$D^{R.L,\alpha} y(t) + \beta y(t) = t^k \delta(t-\xi), \qquad 0 < \alpha < 1. \tag{1.12}$$

Solution: The above fractional differential equation can be written as follows

$$y(t) = \frac{1}{\beta + D^{R.L,\alpha}} t^k \delta(t-\xi), \tag{1.13}$$

let us recall the following well- known identity from Laplace transform of the exponential function

$$\frac{1}{\beta + s^\alpha} = \int_0^{+\infty} e^{-\beta u - s^\alpha u} du, \tag{1.14}$$

by choosing $s = D_t$ and using integral representation for the exponential fraction, we get

$$y(t) = \int_0^{+\infty} du(e^{-\lambda u - uD_t^\alpha} t^k \delta(t-\xi)), \tag{1.15}$$

at this point, in order to evaluate the result of the action of the exponential operator over Dirac delta function, we may use part 2 of the Lamma (1.1) to obtain

$$y(t) = \int_0^{+\infty} e^{-\beta u} \frac{1}{\pi} \int_0^{\infty} e^{-r^\alpha(ucos\alpha\pi)} sin(ur^\alpha sin\alpha\pi)...$$

$$... \int_0^{\infty} (e^{-r\tau - \tau D_t} t^k \delta(t-\xi)) d\tau) dr) du, \tag{1.16}$$

after simplifying of the integrals, we arrive at

$$y(t) = \frac{\xi^k}{\pi} \int_0^{+\infty} e^{-\beta u} (\int_0^{\infty} e^{-r(t-\xi) - r^\alpha(u\cos\alpha\pi)} \sin(ur^\alpha sin\alpha\pi) dr) du. \tag{1.17}$$

Let us consider the special case $\alpha = 0.5$, after simplifying, we have

$$y(t) = \frac{\xi^k}{\pi} \int_0^{+\infty} e^{-\beta u} (\int_0^{\infty} e^{-r(t-\xi)} \sin(u\sqrt{r}) dr) du, \tag{1.18}$$

by changing the order of integration, we obtain

$$y(t) = \frac{\xi^k}{\pi} \int_0^{+\infty} \frac{\sqrt{r} e^{-r(t-\xi)}}{r+\beta^2} dr. \tag{1.19}$$

2. Evaluation of Certain Integrals via the Laplace Transforms

The Laplace transform is especially well- suited for evaluation of the integrals. Let us recall some important properties of the Laplace transform, useful lemmas and corollaries, that will be considered in the next part of this article.

Lemma 2.1. Let us assume that,

$$\mathcal{L}\{\frac{I_\mu(\lambda t)}{t}\} = \int_0^{\infty} e^{-st} \frac{I_\mu(\lambda t)}{t} dt := \frac{((\sqrt{s^2-\lambda^2}) + s)^{-\mu}}{\lambda^{-\mu}\sqrt{s^2-\lambda^2}}, \tag{2.1}$$

then, we have the following integral identities

$$\int_0^{\infty} K_\mu^*(\lambda t) dt := \frac{\sin\frac{\mu\pi}{2} - \frac{\mu\pi}{2}\cos\frac{\mu\pi}{2}}{\lambda^2 \sin^2\frac{\mu\pi}{2}}, \tag{2.2}$$

$$\int_0^{\infty} K_{\pm 1}^*(\lambda t) dt := \pm\frac{1}{\lambda^2}, \tag{2.3}$$

where $K_\nu(.)$ stands for the modified Bessel's function of the second kind of order ν

or Mac donald's function.

Proof. By definition of the Laplace transform, we have

$$\mathcal{L}\{\frac{I_\mu(\lambda t)}{t}\} = \int_0^\infty \frac{I_\mu(\lambda t)e^{-st}}{t}dt := \frac{((\sqrt{s^2-\lambda^2})+s)^{-\mu}}{\lambda^{-\mu}\sqrt{s^2-\lambda^2}}. \tag{2.4}$$

Let us introduce a change of parameter $s = \lambda \cosh\phi$ in the above integral and simplifying, we obtain

$$\int_0^\infty \frac{I_\mu(\lambda t)e^{-(\lambda\cosh\phi)t}}{t}dt := \frac{e^{-\phi\mu}}{\lambda\sinh\phi}. \tag{2.5}$$

Using the well- known identity for the modified Bessel functions of the first and second kinds as below

$$\frac{K_\mu(\lambda t)}{t} = (\frac{\pi}{2})\frac{I_{-\mu}(\lambda t) - I_\mu(\lambda t)}{t sin \mu\pi}. \tag{2.6}$$

Therefore, we have

$$\int_0^\infty \frac{K_\mu(\lambda t)e^{-(\lambda\cosh\phi)t}}{t}dt := \frac{e^{\phi\mu}-e^{-\phi\mu}}{\lambda\sin\mu\pi\sinh\phi}. \tag{2.7}$$

In relation (2.7), let us first differentiate with respect to ϕ after simplifying, we obtain

$$\int_0^\infty K_\mu(\lambda t)e^{-(\lambda\cosh\phi)t}dt := \frac{(\mu e^{\phi\mu}+\mu e^{-\phi\mu})\sinh\phi - (e^{\phi\mu}-e^{-\phi\mu})\cosh\phi}{-\lambda^2\sin\mu\pi\sinh^3\phi}, \tag{2.8}$$

at this point, let us choose $\phi = \frac{i\pi}{2}$,

$$\int_0^\infty K_\mu(\lambda t)dt := \frac{\mu}{\lambda^2\sin\frac{\mu\pi}{2}}, \tag{2.9}$$

now, let us differentiate (2.9) with respect to order μ to get

$$\int_0^\infty K^*_\mu(\lambda t)dt := \frac{\sin\frac{\mu\pi}{2} - \frac{\mu\pi}{2}\cos\frac{\mu\pi}{2}}{\lambda^2\sin^2\frac{\mu\pi}{2}}, \tag{2.10}$$

in the above relation, let us put $\mu = \pm 1$, we arrive at

$$\int_0^\infty K^*_{\pm 1}(\lambda t)dt := \pm\frac{1}{\lambda^2}. \tag{2.11}$$

Corollary 2.1. The following identity holds true,

$$\int_0^\infty K^*_1(t)dt := 1. \tag{2.12}$$

Proof. In relation (2.10), let us take $\lambda = \mu = 1$ then, we get the desired result.

Lemma 2.2. Let us assume that,

$$\mathcal{L}\{\frac{J_\mu(\lambda t)}{t}\} = \int_0^\infty e^{-st}\frac{J_\mu(\lambda t)}{t}dt := \frac{((\sqrt{s^2+\lambda^2})-s)^\mu}{\mu\lambda^\mu}, \tag{2.13}$$

then, we have the following integral identities

$$\int_0^\infty Y^*_\mu(\lambda t)dt := \frac{\pi(cos\mu\pi - 1)}{\lambda\sin^2\mu\pi}, \tag{2.14}$$

$$\int_0^\infty Y_{\pm 0.5}(\lambda t)dt := \mp\frac{1}{\lambda}. \tag{2.15}$$

Where$Y_\nu(.)$ stands for the Bessel's function of the second kind of order ν or the Weber's function.

Proof. By definition of the Laplace transform, we have

$$\mathcal{L}\{\frac{J_\mu(\lambda t)}{t}\} = \int_0^\infty \frac{J_\mu(\lambda t)e^{-st}}{t}dt := \frac{((\sqrt{s^2+\lambda^2}) - s)^\mu}{\mu\lambda^\mu}. \tag{2.16}$$

Let us introduce a change of parameter $s = \lambda\sinh\phi$ in the above integral and simplifying, we obtain

$$\int_0^\infty \frac{J_\mu(\lambda t)e^{-(\lambda\sinh\phi)t}}{t}dt := \frac{e^{-\phi\mu}}{\mu}. \tag{2.17}$$

Using the well- known identity for the Bessel's functions of the first and second kinds as below

$$\frac{Y_\mu(\lambda t)}{t} = \frac{J_\mu(\lambda t)cos\mu\pi - J_{-\mu}(\lambda t)}{t}. \tag{2.18}$$

Therefore, we have

$$\int_0^\infty \frac{Y_\mu(\lambda t)e^{-(\lambda\sinh\phi)t}}{t}dt := \frac{e^{-\phi\mu}\cos\mu\pi - e^{\phi\mu}}{\mu\sin\mu\pi}. \tag{2.19}$$

In relation (2.19), let us first differentiate with respect to ϕ and choose $\phi = 0$ after simplifying, we obtain

$$\int_0^\infty Y_\mu(\lambda t)dt := \frac{cos\mu\pi - 1}{\lambda\sin\mu\pi}, \tag{2.20}$$

at this point, if we differentiate with respect to order μ, we get

$$\int_0^\infty Y^*_\mu(\lambda t)dt := \frac{\pi(cos\mu\pi - 1)}{\lambda\sin^2\mu\pi}. \tag{2.21}$$

Note. By setting $\mu = \pm 0.5$ in (2.18), we have

$$\int_0^\infty Y_{\pm .5}(\lambda t)dt := \mp\frac{1}{\lambda}. \tag{2.22}$$

Corollary 2.2. The following integral identities hold true

$$\int_0^\infty K_{\frac{1}{3}}(\lambda t)dt := \frac{2}{3\lambda^2}, \tag{2.23}$$

$$\int_0^{\infty} K_{\frac{2}{3}}(\lambda t)dt := \frac{4}{3\lambda^2}. \tag{2.24}$$

Proof. In relation (2.9), let us substitute $\mu = \frac{1}{3}$ and $\mu = \frac{2}{3}$ respectively, after simplifying we get the following

$$\int_0^{\infty} K_{\frac{1}{3}}(\lambda t)dt := \frac{2}{3\lambda^2}, \tag{2.25}$$

and

$$\int_0^{\infty} K_{\frac{2}{3}}(\lambda t)dt := \frac{4}{3\lambda^2}. \tag{2.26}$$

3. Solution to Time Fractional KdV of Order 2q+1

Fractional calculus has been used to model physical and engineering processes which are found to be best described by fractional differential equations. It is worth noting that the standard mathematical models of integer order derivatives, including non-linear models do not work adequately in many cases. In this section, theorems and the results which has been introduced are used to solve a variety of the time fractional KdV.

The author implemented the joint Laplace- Fourier transform technique for solving generalized time fractional KdV of order 2q+1, where the fractional derivative is in the Caputo sense.

Problem 3.1. Let us consider the following time fractional KdV, with the initial condition

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + k\frac{\partial^{2q+1} u(x,t)}{\partial x^{2q+1}} = 0, \tag{3.1}$$

where $-\infty < x < \infty$, $t > 0$ and $0 < \alpha < 1$ subject to the initial condition

$$u(x,0) = \phi(x), \ -\infty < x < \infty.$$

Note. Fractional derivative is in the Caputo sense.

Solution: Let us define the joint Laplace - Fourier transform as following

$$\mathcal{F}\{\mathcal{L}\{u(x,t)\} = (\tfrac{1}{\sqrt{2\pi}}) \textstyle\int_{-\infty}^{+\infty} e^{i\omega x} \int_0^{\infty} e^{-st} u(x,t)dt)dx := U(\omega, s),$$

taking the joint the Laplace - Fourier transform of PDE term wise and the Fourier transforms of the boundary condition leads to the following relation

$$U(\omega, s) = \tfrac{s^{\alpha-1}\Phi(\omega)}{s^{\alpha}+(ik\omega)^{2q+1}},$$

upon inverting the joint Laplace - Fourier transform, we get

$$\mathcal{F}^{-1}\{\mathcal{L}^{-1}\{u(x,t)\} = (\tfrac{1}{\sqrt{2\pi}}) \textstyle\int_{-\infty}^{+\infty} e^{-i\omega x}(\int_{c-i\infty}^{c+i\infty} \tfrac{s^{\alpha-1}\Phi(\omega)e^{st}}{s^{\alpha}+(ik\omega)^{2q+1}} ds)d\omega := u(x,t),$$

or, equivalently

$$u(x,t) = (\tfrac{1}{\sqrt{2\pi}}) \textstyle\int_{-\infty}^{+\infty} e^{-i\omega x}\Phi(\omega)(\int_{c-i\infty}^{c+i\infty} \tfrac{e^{st}}{s^{1-\alpha}(s^{\alpha}+(ik\omega)^{2q+1})} ds)d\omega.$$

Let us take $\alpha = 0.5$ (semi - derivative), after evaluating the inner integral, we get the following formal solution

$$u(x,t) = (\tfrac{1}{\sqrt{2\pi}}) \int_{-\infty}^{+\infty} e^{-i\omega x + t(ik\omega)^{2(2q+1)}} Erfc((ik\omega)^{2q+1}\sqrt{t})\Phi(\omega)d\omega,$$

obviously, we have

$$u(x,0) = (\tfrac{1}{\sqrt{2\pi}}) \int_{-\infty}^{+\infty} e^{-i\omega x}\Phi(\omega)d\omega = \phi(x).$$

4. Main Results

In this section, we implemented the exponential operational method for solving a system of space fractional partial differential equations with non-constant coefficients.

Problem 4.1. Let us solve the following coupled space-fractional PDE with non-constant

coefficients, where fractional derivative is in the Riemann-Liouville sense

$$t^{-b}\frac{\partial u(x,t)}{\partial t} + \beta t^k v(x,t) + \lambda(b+1)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \sigma t^k u(x,t), \tag{4.1}$$

$$t^{-b}\frac{\partial v(x,t)}{\partial t} - \beta t^k u(x,t) + \lambda(b+1)\frac{\partial^\alpha v(x,t)}{\partial x^\alpha} = \sigma t^k v(x,t), \tag{4.2}$$

where $-\infty < x < \infty$, $t > 0$, and subject to the boundary conditions and the initial condition

$$u(x,0) = \phi(x), v(x,0) = \psi(x), -\infty < x < \infty.$$

Solution: Let us define the function $w(x,t) = u(x,t) + iv(x,t)$ and the initial condition $w(x,0) = \theta(x)$ we get the following space fractional partial differential equation

$$t^{-b}\frac{\partial w(x,t)}{\partial t} - (\sigma + i\beta)t^k w(x,t) + \lambda(b+1)\frac{\partial^\alpha w(x,t)}{\partial x^\alpha} = 0, \tag{4.3}$$

with the initial condition $w(x,0) = \theta(x)$. At this point, in order to solve the above linear space fractional PDE, we may rewrite the equation in the following exponential operator form

$$\frac{\partial w(x,t)}{\partial t} = ((\sigma + i\beta)t^{b+k} - \lambda(b+1)t^b\frac{\partial^\alpha}{\partial x^\alpha})w(x,t). \tag{4.4}$$

In order to obtain a solution for the equation (4.4) first by solving the first order PDE with respect to variable t, and applying the initial condition, we get the following

$$w(x,t) = \exp(\tfrac{(\sigma+i\beta)t^{b+k+1}}{b+k+1})\exp(-\lambda t^{b+1}\tfrac{\partial^\alpha}{\partial x^\alpha})\theta(x),$$

by virtue of Lemma (1.1), we get the following solution

$$w(x,t) = \tfrac{1}{\pi}\exp(\tfrac{(\sigma+i\beta)t^{b+k+1}}{b+k+1})\int_0^\infty e^{-r^\alpha(\lambda t^{b+1}\cos\alpha\pi)}\ldots$$

$$\sin(\lambda t^{b+1}r^\alpha \sin\alpha\pi)\int_0^\infty (e^{-r\tau-\tau D_x}\theta(x))d\tau dr,$$

finally, we obtain the solution to the system as below

$$w(x,t) = \frac{1}{\pi}\exp(\frac{(\sigma+i\beta)t^{b+k+1}}{b+k+1})\int_0^\infty e^{-r^\alpha(\lambda t^{b+1}\cos\alpha\pi)}\ldots$$
$$\sin(\lambda t^{b+1}r^\alpha\sin\alpha\pi)\int_0^\infty e^{-r\tau}\theta(x-\tau)d\tau)dr,$$

from which we obtain

$$u(x,t) = \exp(\frac{\sigma t^{b+k+1}}{b+k+1})\cos(\frac{\beta t^{b+k+1}}{b+k+1})\frac{1}{\pi}\int_0^\infty e^{-r^\alpha(\lambda t^{b+1}\cos\alpha\pi)}\ldots$$
$$(\sin(\lambda t^{b+1}r^\alpha sin\alpha\pi)\int_0^\infty e^{-r\tau}\phi(x-\tau)d\tau)dr$$
$$+$$
$$\exp(\frac{\sigma t^{b+k+1}}{b+k+1})\sin(\frac{\beta t^{b+k+1}}{b+k+1})\frac{1}{\pi}\int_0^\infty e^{-r^\alpha(\lambda t^{b+1}\cos\alpha\pi)}\sin(\lambda t^{b+1}r^\alpha sin\alpha\pi)\ldots$$
$$(\int_0^\infty e^{-r\tau}\psi(x-\tau)d\tau)dr,$$

$$v(x,t) = \exp(\frac{\sigma t^{b+k+1}}{b+k+1})\cos(\frac{\beta t^{b+k+1}}{b+k+1})\frac{1}{\pi}\int_0^\infty e^{-r^\alpha(\lambda t^{b+1}\cos(\alpha\pi)}\ldots$$
$$(\sin(\lambda t^{b+1}r^\alpha)\sin(\alpha\pi))\int_0^\infty e^{-r\tau}\psi(x-\tau)d\tau)dr$$
$$-\exp(\frac{\sigma t^{b+k+1}}{b+k+1})\sin(\frac{\beta t^{b+k+1}}{b+k+1})\frac{1}{\pi}\int_0^\infty e^{-r^\alpha(\lambda t^{b+1}\cos\alpha\pi)}\ldots$$
$$\sin(\lambda t^{b+1}r^\alpha)\sin(\alpha\pi)\int_0^\infty e^{-r\tau}\phi(x-\tau)d\tau dr.$$

Note. It is easy to verify that $u(x,0+)=\phi(x)$, $v(x,0+)=\psi(x)$.

5. Conclusion

Operational methods provide fast and universal mathematical tool for obtaining the solution of PDEs or even FPDEs. The combination of the integral transforms, operational methods and the special functions give more powerful analytical instrument for solving a wide range of engineering and physical problems. The paper is devoted to study the Laplace transform, exponential operators and their applications in solving certain systems of boundary value problems. The present method can be readily applied to more complicated fractional differential equations. The results of these developments will be published in the future articles.

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