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Simulations of transport in one dimension

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Abstract Advection-dispersion equation is solved in numerically by using combinations of differential quadrature method (DQM) and various time integration techniques covering some explicit or implicit single and multi step methods. Two different initial boundary value problems modeling conservative and non conservative transports of some substance represented by initial data are chosen as test problems. In the first case, pure advection conservative model problem is studied. The second problem models motion of non conservative substance and simulates fade out of it as time proceeds. The errors between analytical and numerical results are measured by discrete maximum norm. Comparison with some earlier works indicates that the proposed algorithms generate more accurate and valid results for some discretization parameters.

Keywords. Advection-Dispersion equation; Transport; Pollution; Sine cardinal functions; Differential quadrature method.

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1. Introduction

Many physical phenomena in real world are modeled by various linear PDEs. Having both advection and dispersion (diffusion) terms in the Advection-Dispersion Equation (ADE) makes it a useful model for problems in various fields. Isenberg and Gutfinger examined a thin film of incompressible liquid draining down a vertical wall, motion for the film and unsteady heat transfer within the film [13]. Water transport in soils and dispersion in rivers and estuaries are also two well known studies modeled by the ADE [7,32]. Various problems including different types of the equation are used to model for the transient problems associated with flow through wellbores, geothermal production with reinjection, thermal energy storage in porous formations, thermal, hot fluid injection and energy extraction techniques for oil recovery, miscible flooding, oil recovery from hot dry rocks [5]. A type of one dimensional form is used to describe uptake and desorption of solute diffusion into porous soil aggregates, lithofragments in sediments and aquifer materials in the sorptive [10]. Solute transport problem by groundwater flow through isotropic and homogeneous aquifer is also modeled by the ADE [14]. In transport phenomena in food processing, one dimensional unsteady

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diffusion in an isotropic medium, isothermal process, and the moisture content on a dry basis are studied with a different kind of the ADE [1].
Consider the initial boundary value problem (IBVP) for one dimensional form of the ADE

$$\frac{\partial u(x,t)}{\partial t} + \nu \frac{\partial u(x,t)}{\partial x} - \lambda \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \tag{1.1}$$

with initial data

$$u(x,0) = f(x),$$

and boundary data

$$\begin{aligned} u(a,t) &= b_1(t), \\ u(b,t) &= b_2(t), \end{aligned}$$

over a finite interval $[a, b]$. This problem models transport of the quantity $u(x,t)$ of heat, fluid or related substances moving along $x-$axis with a constant flow velocity ν and the dispersion(diffusion) coefficient λ [2, 35].
So far, various numerical methods have been applied to the ADE. Dağ et al. developed the least square finite element algorithm based on low degree B-spline shape functions (FEMLSF and FEMQSF) to solve transport problem modeled by the ADE [8]. Szymkiewicz also solved a model problem described by the ADE via the combination of the spline functions and finite elements [36]. Kadalbajaoo and Arora constructed a Taylor-Galerkin B-spline finite element algorithm to solve various IBVPs for one dimensional ADE [30].
Noye and Tan obtained the numerical solutions of the ADE by the third-order semi explicit finite difference method [31]. Various two-level explicit and implicit finite difference methods covering the upwind explicit, the Lax-Wendroff, the modified Siemieniuch-Gladwell and the fourth-order method have been compared with each other on the numerical solutions of model problems including the ADE. Karahan solved various initial boundary value problems for the ADE by the use of implicit, third-order upwind and explicit finite difference methods [19–21]. Guraslan et al. developed a sixth-order compact finite difference method (CD6) combined with the fourth order Runge-Kutta method for numerical solution of three dynamic model problems [11].
Irk et al. set up a collocation method based on extended cubic B-spline functions (EXCBS) [12]. In that study, pollutant transport through a channel problems modeled by the ADE with mixed boundary conditions were studied. Kaya developed a polynomial based DQM algorithm to obtain numerical solutions of two initial boundary value problems including flood propagation in an open channel [22]. He also compared the obtained results with the explicit and implicit finite difference results. One more DQM based on cubic B-spline functions (CSDQM) was developed for transport of conserved contaminant and fadeout problems in one dimension [25].
Aim of this study is to obtain the numerical solutions of initial boundary value problems for the ADE in one dimension by DQM based on Sinc functions. This technique is a little bit different from common DQM techniques [15–18, 29] owing to the basis function set used while calculating the weight coefficients.

The ODE system obtained by the reduction of the ADE by DQM will be integrated for time variable by using various methods covering forward Euler (FORE), improved polygon (modified Euler) method (IMPOLY), Heun (improved Euler) method (HEUN), classical Runge-Kutta methods of order two to four (RK2,RK3,RK4), implicit Rosenbrock method of third-fourth order (RB34), Gear single step method with Burlirsch-Stoer rational extrapolation (GB), FehlBerg Runge-Kutta method of order fifth order (RKF45), Runge-Kutta method with Cash-Karp coefficients of order four-five (RKCK45), Adams-Bashforth (AB4) and Adams-Moulton methods of order four (AM4). The first three initial steps of the iterations of AB4 and AM4 methods are calculated by RK4. In the predictor-corrector method AM4, the predictor method is chosen as AB4.

2. Numerical Method

The Sinc functions

$$S_m(x) = \begin{cases} \dfrac{\sin([\frac{x-m\Delta x}{\Delta x}]\pi)}{[\frac{x-m\Delta x}{\Delta x}]\pi}, & x \neq m\Delta x, \\ 1, & x = m\Delta x, \end{cases} \tag{2.1}$$

form a basis on the real line where Δx is the equal node size, and m is an integer [6, 9, 33, 34]. The nodal values of sinc functions are described in [9] as

$$S_m(x_j) = \delta_{mj}, \tag{2.2}$$

Consider the series

$$C(u)(x) = \sum_{m=-\infty}^{\infty} u(m\Delta x) S_m(x), \tag{2.3}$$

for the function u defined on $(-\infty, \infty)$. The function $C(u)(x)$ is named the cardinal of u if it converges [28]. First two derivatives of Sinc function $S_m(x)$ are calculated as

$$S'_m(x) = \begin{cases} \dfrac{\frac{\pi}{\Delta x}(x-m\Delta x)\cos\frac{x-m\Delta x}{\Delta x}\pi - \sin\frac{x-m\Delta x}{\Delta x}\pi}{\frac{\pi}{\Delta x}(x-m\Delta x)^2}, & x \neq m\Delta x, \\ 0, & x = m\Delta x, \end{cases} \tag{2.4}$$

$$S''_m(x) = \begin{cases} \dfrac{-\frac{\pi}{\Delta x}\sin\frac{x-m\Delta x}{\Delta x}\pi}{x-m\Delta x} - \dfrac{2\cos\frac{x-m\Delta x}{\Delta x}\pi}{(x-m\Delta x)^2} + \dfrac{2\sin\frac{x-m\Delta x}{\Delta x}\pi}{\frac{\pi}{\Delta x}(x-m\Delta x)^3}, & x \neq m\Delta x, \\ -\dfrac{\pi^2}{3\Delta x^2}, & x = m\Delta x. \end{cases} \tag{2.5}$$

DQM is a derivative approximation technique described as "*the p.th order derivative of a function $u(x)$ at x_m is approximated by finite weighted sum of nodal function values, i.e.,*

$$\left.\frac{\partial u^{(p)}(x)}{\partial x^{(p)}}\right|_{x=x_m} = \sum_{i=1}^{N} w_{mi}^{(p)} u(x_i), \quad m = 1, 2, \ldots, N, \tag{2.6}$$

where the partition of the finite problem interval $[a, b]$ is $x_m = a + (m-1)\Delta x, m = 1, 2, \ldots, N$, $w_{mi}^{(p)}$ are the weights of nodal functional values for the p. th order derivative approximation [3]". The weights $w_{mi}^{(p)}$ are calculated using basis functions spanning the problem interval. So far various basis function sets covering Lagrange interpolation functions and higher order B-splines, etc, have been used to calculate the weights [23, 24, 26].

2.1. **Determination of the first order approximation weights.** Letting $p = 1$ in the fundamental DQM derivative equation will lead to produce the weights of the first order derivative $w_{mi}^{(1)}$. The Sinc functions set $\{S_m(x)\}_{m=1}^{m=N}$ forms a basis for the functions defined on $[x_1 = a, b = x_N]$. In order to calculate the weights $w_{1i}^{(1)}$ of the node x_1, we substitute each Sinc basis functions into the fundamental differential quadrature equation 2.6. Substitution of $S_1(x)$ and using the functional and derivative values of it which can determined by using (2.4) and (2.5) will lead the equation

$$\begin{aligned} S_1'(x_1) &= \sum_{i=1}^{N} w_{1i}^{(1)} S_1(x_i) \\ &= w_{11}^{(1)} S_1(x_1) + w_{12}^{(1)} S_1(x_2) + \ldots + w_{1N}^{(1)} S_1(x_N) \\ &= w_{11}^{(1)} \delta_{11} + w_{12}^{(1)} \delta_{12} + \ldots + w_{1N}^{(1)} \delta_{1N} \\ 0 &= w_{11}^{(1)}, \end{aligned} \tag{2.7}$$

and will generate the weight $w_{11}^{(1)}$. The weight $w_{12}^{(1)}$ can be calculated by substitution of $S_2(x)$ into Eq.(2.6) as

$$\begin{aligned} S_2'(x_1) &= \sum_{i=1}^{N} w_{2i}^{(1)} S_2(x_i) \\ &= w_{11}^{(1)} S_2(x_1) + w_{12}^{(1)} S_2(x_2) + \ldots + w_{1N}^{(1)} S_2(x_N) \\ &= w_{11}^{(1)} \delta_{21} + w_{12}^{(1)} \delta_{22} + \ldots + w_{1N}^{(1)} \delta_{2N} \\ \frac{(-1)^{2+1}}{\Delta x(1-2)} &= w_{12}^{(1)}. \end{aligned} \tag{2.8}$$

It can be concluded that the weights $w_{1i}^{(1)}$ focused on the first node x_1 can be determined by substitution of each Sinc functions $S_m(x), m = 1, 2, \ldots, N$ into the fundamental differential quadrature equation (2.6) as

$$w_{1i}^{(1)} = \frac{(-1)^{1-i}}{\Delta x(1-i)}, \ 1 \neq i, \tag{2.9}$$

$$w_{11}^{(1)} = 0. \tag{2.10}$$

When the weight $w_{mi}^{(1)}$ focused on the node x_m is wanted to be calculated, a general explicit formulation to determine it can be given as [4, 27]

$$w_{mi}^{(1)} = \frac{(-1)^{m-i}}{\Delta x(m-i)},\ m \neq i, \tag{2.11}$$

$$w_{mm}^{(1)} = 0. \tag{2.12}$$

2.2. **Determination of the second order approximation weights.** Assuming $p = 2$ and $m = 1$ in the Eq.(2.6) and using functional and derivative values of $S_1(x)$ will generate the equation

$$\begin{aligned} S_1''(x_1) &= \sum_{i=1}^{N} w_{1i}^{(2)} S_1(x_i) \\ &= w_{11}^{(2)} S_1(x_1) + w_{12}^{(2)} S_1(x_2) + \ldots + w_{1N}^{(2)} S_1(x_N) \\ &= w_{11}^{(2)}\delta_{11} + w_{12}^{(2)}\delta_{12} + \ldots + w_{1N}^{(2)}\delta_{1N} \\ \frac{-\pi^2}{3\Delta x^2} &= w_{11}^{(2)}. \end{aligned} \tag{2.13}$$

Substitution of $S_2(x)$ into the fundamental differential quadrature equation (2.6) will lead the equation

$$\begin{aligned} S_2''(x_1) &= \sum_{i=1}^{N} w_{1i}^{(2)} S_2(x_i) \\ &= w_{11}^{(2)} S_2(x_1) + w_{12}^{(2)} S_2(x_2) + \ldots + w_{1N}^{(2)} S_2(x_N) \\ &= w_{11}^{(2)}\delta_{21} + w_{12}^{(2)}\delta_{22} + \ldots + w_{1N}^{(2)}\delta_{2N} \\ 2\frac{(-1)^{(2+1+1)}}{(\Delta x)^2(1-2)^2} &= w_{12}^{(2)}, \end{aligned} \tag{2.14}$$

and will generate the weight $w_{12}^{(2)}$. In a general case the weights $w_{mi}^{(2)}$ focused on the node x_m of the second order derivative approximation can be written in an explicit form [4, 27]

$$w_{mi}^{(2)} = \frac{2(-1)^{m-i+1}}{\Delta x^2(m-i)^2},\ m \neq i, \tag{2.15}$$

$$w_{mm}^{(2)} = -\frac{\pi^2}{3\Delta x^2}, \tag{2.16}$$

3. Discretization of the ADE

Replacing the space derivative terms by their DQM approximations in ADE (1.1) leads to an ODE system of the form

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{x=x_m} = -\nu \sum_{i=1}^{N} w_{mi}^{(1)} u(x_i,t) + \lambda \sum_{i=1}^{N} w_{mi}^{(2)} u(x_i,t), \tag{3.1}$$

where $w_{mi}^{(1)}$ and $w_{mi}^{(2)}$ are the weights of each $u(x_i,t)$ for the first two derivative approximations at the node x_m. Since the values of the function $u(x,t)$ at x_1 and x_N are boundary data at both ends of the problem interval, then (3.1) can be expressed in the form

$$\begin{aligned}\left.\frac{\partial u(x,t)}{\partial t}\right|_{x=x_m} &= [-\nu+\lambda]\, w_{m1}^{(1)} b_1(t) + [-\nu+\lambda]\, w_{mN}^{(2)} b_2(t) \\ &+ \sum_{i=2}^{N-1}\left[-\nu w_{mi}^{(1)} + \lambda w_{mi}^{(2)}\right] u(x_i,t).\end{aligned} \tag{3.2}$$

The fully space discretized system (3.2) is integrated for the time variable t by using time integration methods.

4. Problems

In the process of application of numerical methods, the numerical error between the results obtained by the proposed algorithms and the analytical solution is measured to indicate the accuracy and the validity of the method. The measure of the error also provides a chance to compare the related method with the other ones. In this study, the discrete maximum norm ${}_{(\Delta x,\Delta t)}L_\infty$ is used to determine the error between the approximate and the analytical solutions. This norm is defined as;

$${}_{(\Delta x,\Delta t)}L_\infty = \max_{2\le m\le N-1} \left|u^{\mathrm{a}}(x_m,t) - u^{\mathrm{n}}(x_m,t)\right|,$$

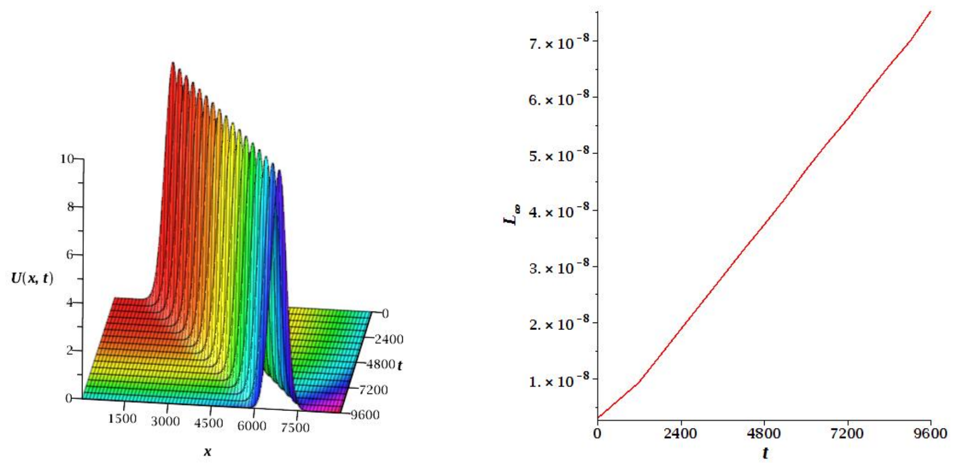
where $u^{\mathrm{a}}(x_m,t)$ and $u^{\mathrm{n}}(x_m,t)$ are the analytical and the numerical solutions, respectively, at the node x_m at a fixed time t for the space and time step sizes Δx and Δt.

4.1. **Transport with only Advection.** The model problem for transport of a quantity of concentration along a channel is described as a pure advection initial boundary value problem for the ADE. The initial condition for the problem is derived by substituting $t = 0$ into the analytical solution

$$u(x,t) = 10\exp\left(-\frac{1}{2\rho^2}(x-\tilde{x}-\nu t)^2\right), \tag{4.1}$$

where ρ and $\tilde{x}$ denote the standard deviation and the initial peak position of the bell-shaped quantity of 10 units height, respectively [8,11,25,36]. The solution represents motion of the initial quantity to the right along the channel of length 9 kilometers with a constant speed ν. For the sake of comparison with the results stated in some earlier studies, the standard deviation $\rho = 264$, the flow velocity $\nu = 0.5\, m/s$ and the initial peak position $\tilde{x} = 2$ referring the 2 kilometers away from the left end of the channel are used as parameters to simulate the solutions. This choice of parameters moves the peak position of the initial quantity to 6.8 kilometers far away from the left end of the channel at the simulation terminating time 9600 seconds. The boundary conditions at both ends are selected as homogeneous Dirichlet conditions over the problem interval $[0, 9]$. The simulation of the transport obtained by SDQM-RKF45 with the parameters $\Delta x = 25$ and $\Delta t = 10$ is graphed in Figure 2(a). The maximum error obtained by SDQM-RKF45 with the same parameters at some specific times are also recorded and depicted in Figure 2(b).

FIGURE 1. Transport of the initial quantity and the maximum errors during simulation.



(a) Simulation of the transport.

(b) Maximum error as time goes with $\Delta x = 25$ and $\Delta t = 10$.

Over the long simulation time, the solutions obtained by the SQDM seem stable and are in very good agreement with the analytical ones. Comparative solutions with the results in some earlier studies for various mesh sizes are tabulated in Table 1.

When $\Delta x = 200$ and $\Delta t = 50$, the error obtained by the SDQM-FORE is too high. The maximum norms are 1.15 and 1.35 for the CSDQM and the FEMLSF, respectively with the same parameters. The results obtained by the methods SDQM-IMPOLY, SDQM-HEUN and SDQM-RK2 methods are accurate to one decimal digit like the results of the FEMQSF, the CD6 and the EXCBS. The RK3 and the AB4 methods have two decimal digits accuracy. The SDQM-RK4, the SDQM-GB, the SDQM-RKF45, the SDQM-RKCK45 and the SDQM-AB4 generate three decimal digit accurate results.

The choice of Δx and Δt as 50 causes to fail the low order the SDQM-FORE, the SDQM-IMPOLY, the SDQM-HEUN and the SDQM-RK2 and multi-step methods the AB4 and the AM4. The FEMLSF and the FEMQSF generates one decimal digit accurate results as the SDQM-RK3 has two decimal digits accuracy. The methods with three decimal digit accurate can be listed as the CSDQM and the EXCBS. The results obtained by the method CD6 are accurate to four decimal digits, the SDQM-RK4, the SDQM-RB34 and SDQM-RKF45 five decimal digits and the RKCK45 six decimal digits. The most accurate results obtained by the method SDQM-GB as eight decimal digits in this case.

Most of the methods applied for the time integration in the present study, covering classical Runge-Kutta methods of order one to four, variations of Euler method and multi step methods, failed when Δx is reduced to 25 with fixed $\Delta t = 50$. The results obtained by the FEMLSF and the FEMQSF are accurate to one decimal digit, the CSDQM three decimal digits, and the CD6 four decimal digits. The accuracy of the results of SDQM-RB34 and the SDQM-RKF45 are five decimal digits as the best results again are obtained by the SDQM-GB as seven decimal digits accuracy.

In the case reduction Δt to 10 with $\Delta x = 25$, the SDQM-FORE method fails. The SDQM-IMPOLY, the SDQM-HEUN and the SDQM-RK2 generate two decimal digits accuracy as the results obtained by the SDQM-RK3 are accurate to four decimal digits, the SDQM-AB4 five decimal digits, the SDQM-RK4 and the SDQM-AM4 six decimal digits. The accuracy of the methods SDQM-RB34 and SDQM-GB are measured in seven decimal digits. The most accurate results for those parameters are produced by the methods SDQM-RK45 and SDQM-RKCK45 to eight decimal digits. Since the better results are obtained by the use of those parameters when compared with the results by EXCBS with $\Delta x = \Delta t = 10$, we do not reduce the step sizes more.

Table 1. Comparison of present results with some earlier ones for pure advection transport.

Method	$_{(200,50)}L_\infty$	$_{(50,50)}L_\infty$	$_{(25,50)}L_\infty$	$_{(25,10)}L_\infty$	$_{(10,10)}L_\infty$
SDQM-FORE	533.5714	∞	∞	∞	
SDQM-IMPOLY	3.9486×10^{-1}	∞	∞	1.7442×10^{-2}	
SDQM-HEUN	3.9486×10^{-1}	∞	∞	1.5005×10^{-2}	
SDQM-RK2	3.9486×10^{-1}	∞	∞	1.7442×10^{-2}	
SDQM-RK3	1.9080×10^{-2}	1.8821×10^{-2}	∞	1.5429×10^{-4}	
SDQM-RK4	1.9151×10^{-3}	7.0186×10^{-5}	∞	1.1436×10^{-6}	
SDQM-RB34	1.9182×10^{-3}	6.1214×10^{-5}	6.1275×10^{-5}	1.1967×10^{-7}	
SDQM-GB	1.9183×10^{-3}	8.7642×10^{-8}	2.0875×10^{-7}	1.1584×10^{-7}	
SDQM-RKF45	1.9186×10^{-3}	1.8497×10^{-5}	1.8834×10^{-5}	7.5235×10^{-8}	
SDQM-RKCK45	1.9183×10^{-3}	3.0192×10^{-6}	23025.3677	7.4091×10^{-8}	
SDQM-AB4	2.8709×10^{-2}	∞	∞	4.6886×10^{-5}	
SDQM-AM4	2.5487×10^{-3}	∞	∞	3.5583×10^{-6}	
CSDQM [25]	1.15	8.00×10^{-3}	1.00×10^{-3}		
FEMLSF [8]	1.35	3.80×10^{-1}	3.77×10^{-1}		
FEMQSF [8]	5.18×10^{-1}	3.73×10^{-1}	3.79×10^{-1}		
CD6 [11]	4.29×10^{-1}	8.00×10^{-4}	7.00×10^{-4}		
EXCBS [12]	6.07×10^{-1}	2.20×10^{-3}			3.44×10^{-6}

4.2. **Transport with both Advection and Dispersion.** The initial boundary value problem, constructed using both advection and dispersion terms together, models the fadeout of an initially solitary wave-shaped quantity while moving to the right of the channel as time proceeds. The exact solution of this model is given as

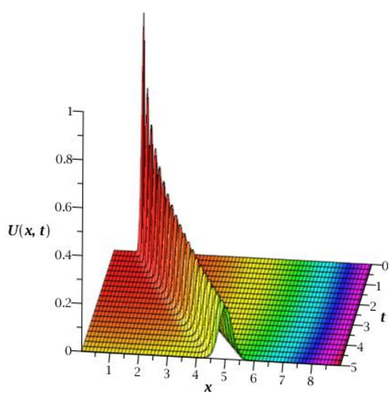
$$u(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(x-\tilde{x}-\nu t)^2}{\lambda(4t+1)}\right), \tag{4.2}$$

where $\tilde{x}$ is the initial peak position of the quantity of unit height moving with a constant velocity ν [30, 31]. The initial data are chosen as

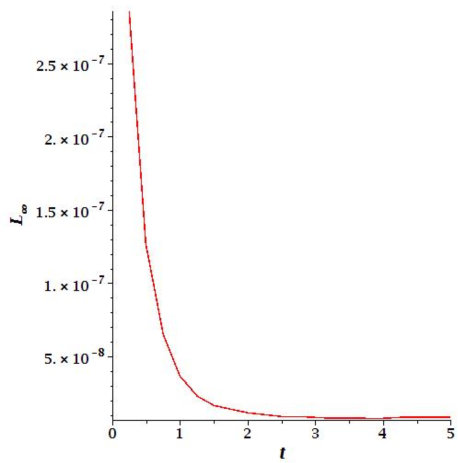
$$u(x,0) = \exp\left(-\frac{(x-\tilde{x})^2}{\lambda}\right), \tag{4.3}$$

which can be obtained by substitution of $t = 0$ into the analytical solution. The simulation is accomplished by assuming homogeneous Dirichlet boundary data at both ends of the channel of length 9 kilometers. The algorithm to simulate the solution of the problem is run up to the compatible test time $t = 5$ seconds with the dispersion coefficient $\lambda = 0.005$, the transport velocity $\nu = 0.8m/s$ and the initial peak position of the quantity $\tilde{x} = 1$. The simulation of the motion and the maximum error-time graph are depicted in Figure 3(a) and in Figure 3(b), respectively. The peak of the quantity reaches the fifth kilometers of the channel at the end of the simulation. This situation corresponds to the theoretical aspects of the solution owing to the value of ν.

FIGURE 2. The fadeout of an initial quantity and the error at $t = 5$.



(a) Fadeout of quantity as time goes.



(b) Maximum error as time goes.

A comparison of the results obtained by SQDM methods with the ones from the CSDQM method is also summarized for some various mesh sizes and fixed $\Delta t = 0.0125$, Table 2. When $\Delta x = 0.2$, the results of all methods given in the table are as accurate as each other, namely to one decimal digit.

When the mesh size is chosen as 0.1, the results obtained from SDQM-FORE are one decimal digit accurate. This choice of Δx causes two decimal digits accuracy for the

method CSDQM (Method II). The results obtained by the CSDQM (Method I) has three decimal digits accuracy like all SDQM methods except SDQM-FORE.
In the case $\Delta x = 0.05$, the results of SDQM-FORE has one decimal digit accuracy as the SDQM-AB4 fails. The accuracy of the results of the SDQM-IMPOLY, the SDQM-HEUN, the SDQM-RK2 and the CSDQM (Method I) are to three decimal digits. The methods SDQM-RK3, SDQM-RK4, SDQM-RB34, SDQM-GB, SDQM-RKF45, SDQM-RKCK45, SDQM-AM4, and CSDQM (Method II) have four decimal digits accurate.
In the last case, we choose Δx as 0.0025. This choice of Δx causes the methods SDQM-FORE, SDQM-RB34 and multi step methods to fail. The results obtained by the SDQM-IMPOLY, the SDQM-HEUN, and the SDQM-RK2 are accurate to three decimal digits accurate results, the CSDQM (Method I) four decimal digits, the SDQM-RK3 and the CSDQM (Method II) five decimal digits, the SDQM-RK4 seven decimal digits and the SDQM-GB and the SDQM-RKF45 eight decimal digits. The most accurate results are obtained by the SDQM-RKCK45 as nine decimal digits for this case.

Table 2. Comparison of the results with some earlier studies on the maximum error at $t = 5$ for the fadeout problem.

Method	$(0.2,0.0125)\, L_\infty$	$(0.1,0.0125)\, L_\infty$	$(0.05,0.0125)\, L_\infty$	$(0.025,0.0125)\, L_\infty$
SDQM-FORE	4.7876×10^{-1}	2.2734×10^{-1}	2.2243×10^{-1}	∞
SDQM-IMPOLY	1.3818×10^{-1}	9.9836×10^{-3}	1.6755×10^{-3}	1.6842×10^{-3}
SDQM-HEUN	1.3818×10^{-1}	9.9836×10^{-3}	1.6755×10^{-3}	1.6842×10^{-3}
SDQM-RK2	1.3855×10^{-1}	9.9836×10^{-3}	1.7655×10^{-3}	1.6842×10^{-3}
SDQM-RK3	1.3848×10^{-1}	9.9843×10^{-3}	1.1087×10^{-4}	3.9909×10^{-5}
SDQM-RK4	1.3855×10^{-1}	9.9863×10^{-3}	1.1070×10^{-4}	8.8121×10^{-7}
SDQM-RB34	1.3855×10^{-1}	9.9863×10^{-3}	1.1071×10^{-4}	∞
SDQM-GB	1.3855×10^{-1}	9.9863×10^{-3}	1.1071×10^{-4}	1.9130×10^{-8}
SDQM-RKF45	1.3855×10^{-1}	9.9863×10^{-3}	1.1071×10^{-4}	1.1869×10^{-8}
SDQM-RKCK45	1.3855×10^{-1}	9.9863×10^{-3}	1.1071×10^{-4}	8.6012×10^{-9}
SDQM-AB4	1.3856×10^{-1}	9.9860×10^{-3}	∞	∞
SDQM-AM4	1.3855×10^{-1}	9.9864×10^{-3}	1.1073×10^{-4}	∞
CSDQM(Method I) [25]	1.25×10^{-1}	6.95×10^{-3}	1.21×10^{-3}	3.07×10^{-4}
CSDQM(Method II) [25]	1.36×10^{-1}	1.45×10^{-2}	2.88×10^{-4}	1.81×10^{-5}

5. Conclusion

In the study, DQM based on sine cardinal functions is setup to solve the advection-dispersion equation numerically. The weight coefficients required for differential quadrature derivative approximations are computed in an explicit form. After discretization of the ADE in space by the DQM, and application of boundary conditions, the resultant ODE system is integrated with respect to the time variable t using various methods covering single step methods of different orders, and explicit Adams-Bahsforth and implicit Adams-Moulton multistep methods of order four. In order to show the validity and accuracy of the numerical results, two IBVPs are studied. The simulations and error distributions at the terminating times for both problems are depicted. The discrete maximum error norms are computed for various mesh and time step sizes. A comparison of the results with each other and some results from different studies in literature is also accomplished by the comparison of norms. Comparisons also show that Sinc DQM generates acceptable, accurate and valid, better for some cases, solutions like the earlier solutions obtained by various methods in literature.

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