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A New Numerical Scheme for Solving Systems of Integro-Differential Equations

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Abstract This paper has been devoted to apply the Reconstruction of Variational Iteration Method (RVIM) to handle the systems of integro-differential equations. RVIM has been induced with Laplace transform from the variational iteration method (VIM) which was developed from the Inokuti method. Actually, RVIM overcome to shortcoming of VIM method to determine the Lagrange multiplier. So that, RVIM method provides rapidly convergent successive approximations to the exact solution. The advantage of the RVIM in comparison with other methods is the simplicity of the computation without any restrictive assumptions. Numerical examples are presented to illustrate the procedure. Comparison with the homotopy perturbation method has also been pointed out.

Keywords. System of integro-differential equations, Volterra equation, Reconstruction of variational iteration method, Homotopy perturbation method.

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1. Introduction

Systems of integral equations, linear or nonlinear, appear in scientific applications in engineering, physics, chemistry and populations growth models[4-6, 14, 17]. Studies of systems of integral equations have attracted much concern in applied sciences. Volterra studied the hereditary influences when he was examining a population growth model. The research resulted in a specific topic, where both differential and integral operators appeared together in the same equation. This new type of equation is named as Volterra integro-differential equation, given in the form:

$$y^{(i)}(x) = f(x) + \int_0^x k(x,t)u(t)dt,$$

where $k(x,t)$ a function of two variables x and t, is called the kernel. In this paper, we will study systems of Volttera integro-differential equations given by:

$$y_j^{(i)}(x) = f_j(x, y_1(x), y_2(x), ..., y_m(x)) + \int_0^x g_j(\tau, y_1(\tau), y_2(\tau), \ldots, y_m(\tau))d\tau,$$

for $j = 1, \ldots, m$. The functions $f_j(x, y_1(x), y_2(x), \ldots, y_m(x))$ are given real valued functions and unknown functions $y_1(x), y_2(x), \ldots, y_m(x)$ will be determined. A variety of numerical and analytical methods such as series solution method [13], homotopy perturbation method [9, 10], Adomian decomposition method [2, 3, 16] and variational iteration method [1, 12, 15] have been used to solve the systems of integro-differential equations. It is important to point out that these methods have been applied for the separable or difference kernels. In this work, we use the reconstruction of variational iteration method for solving systems of integro-differential equations. This method was first proposed by Hesameddini and Latifizadeh [11] and provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists.

2. Preliminaries

For the reader's convenience, we present some necessary definitions which are used further in this paper.

Definition 2.1. The Laplace transform of $f(x)$ is defined as follows:

$$F(s) = \ell\{f(t; s)\} = \int_0^\infty e^{-st} f(t)dt.$$

One of the most important properties of Laplace transform is the convolution of functions f and g. Let the functions $f(t)$ and $g(t)$ be defined for $t \geq 0$, then the convolution of the functions f and g is denoted by $(f * g)(t)$, and is defined as the following integral:

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau.$$

Let $\ell\{f(t)\} = F(s)$, $\ell\{g(t)\} = G(s)$, then $\ell\{(f * g)(t)\} = F(s)G(s)$. Or equivalently, $\ell\{\int_0^t f(\tau)g(t - \tau)d\tau\} = F(s)G(s)$. Conversely,
$\ell^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t - \tau)d\tau$.

Definition 2.2. The Laplace transform for the derivatives of $f(x)$ is given by:

$$\ell\{f^{(i)}(t); s\} = s^i F(s) + \sum_{k=0}^{m-1} s^{(i-k-1)} f^{(k)}(o^+), \quad F(s) = \ell\{f(t); s\}, i = 1, \ldots, n.$$

3. Application of the RVIM Method

In this study we consider the following system of integro-differential equations:

$$\begin{aligned}
y_1^{(i)}(x) &= f_1(x, y_1(x), y_2(x), \dots, y_m(x)) + \int_0^x g_1(\tau, y_1(\tau), Xy_2(\tau), \dots, y_m(\tau))d\tau,\\
y_2^{(i)}(x) &= f_2(x, y_1(x), y_2(x), \dots, y_m(x)) + \int_0^x g_2(\tau, y_1(\tau), y_2(\tau), \dots, y_m(\tau))d\tau,\\
&\vdots\\
y_m^{(i)}(x) &= f_m(x, y_1(x), y_2(x), \dots, y_m(x)) + \int_0^x g_m(\tau, y_1(\tau), y_2(\tau), \dots, y_m(\tau))d\tau,
\end{aligned} \tag{3.1}$$

where g_j's are linear/nonlinear functions of $x, y_1, y_2, \dots, y_m$ and $y_j^{(i)}$ is the derivative of y_j with order i, subject to the initial conditions:

$$y_j^{(k)} = c_k^j, \qquad 1 \le j \le m, \quad 1 \le k < i. \tag{3.2}$$

We summarize system (3.1), in the form

$$y_j^{(i)}(x) = N_j(x, y_1(x), y_2(x), \dots, y_m(x)), \quad j = 1, \dots, m, \tag{3.3}$$

with the zero artificial initial conditions. By taking Laplace transform of the both sides of (3.3), in the usual way and using the artificial initial conditions, the following result is obtained:

$$s^i \ell\{y_j(x)\} = \ell\{N_j(x, y_1(x), y_2(x), \dots, y_m(x))\}, \quad j = 1, \dots, m. \tag{3.4}$$

Therefore, we can conclude that:

$$\ell\{y_j(x)\} = \frac{1}{s^i}\ell\{N_j(x, y_1(x), y_2(x), \dots, y_m(x))\}, \quad j = 1, \dots, m. \tag{3.5}$$

Suppose that $\frac{1}{s^i} = H(s)$, then by using the convolution theorem, one obtains:

$$\ell\{y_j(x)\} = H(s)\ell\{N_j(x, y_1(x), y_2(x), \dots, y_m(x))\} = \ell\{(h * N_j)x\}, \tag{3.6}$$

where $j = 1, \dots, m, \ell^{-1}\{H(s)\} = h(x)$. Taking the inverse Laplace transform to both sides of (3.6), the following result is obtained:

$$y_j(x) = \int_0^x h(x-\tau)N_j(\tau, y_1(\tau), y_2(\tau), \dots, y_m(\tau))d\tau, \quad j = 1, \dots, m. \tag{3.7}$$

Now, we must impose the actual initial conditions to obtain the solution of (3.1). Thus we have the following iteration formulation:

$$y_j^{n+1}(x) = y_j^0(x) + \int_0^x h(x-\tau)N_j(\tau, y_1^n(\tau), y_2^n(\tau), \dots, y_m^n(\tau))d\tau, \tag{3.8}$$

for $j = 1, \ldots, m$. The values $y_1^0(x), y_2^0(x), \ldots, y_m^0(x)$ are given by:

$$y_j^0(x) = y_j(0) + xy_j'(0) + \ldots + \frac{x^n y_j^{(n)}(0)}{n!}. \tag{3.9}$$

Therefore, according to the reconstruction of variational iteration method $y_j(x)$ is obtained as follows:

$$y_j(x) = \lim_{n\to\infty} y_j^n(x), \qquad j = 1, \ldots, n, \tag{3.10}$$

where $y_j^n(x)$ indicates n-th approximation of $y_j(x)$.

4. Numerical Examples

To demonstrate the effectiveness of this method we consider some systems of linear and nonlinear integro-differential equations:

Example 4.1. Consider the following system of Volterra integro-differential equations:

$$\begin{cases} u''(x) = -1 - x^2 - \sin x + \int_0^x (u(t) + v(t))dt, \\ v''(x) = 1 - 2 - \sin x - \cos x + \int_0^x (u(t) - v(t))dt, \end{cases} \tag{4.1}$$

subjected to the initial conditions:

$$u(0) = 1,\ u'(0) = 1,\ v(0) = 0,\ v'(0) = 2. \tag{4.2}$$

Applying the Laplace transform to (4.1), the result is as follows:

$$\begin{cases} \ell\{u(x)\} = \frac{1}{s^2}\ell\{-1 - x^2 - \sin x + \int_0^x (u(t) + v(t))dt\}, \\ \ell\{v(x)\} = \frac{1}{s^2}\ell\{1 - 2 - \sin x - \cos x + \int_0^x (u(t) - v(t))dt\}. \end{cases} \tag{4.3}$$

By applying the inverse Laplace transform to both sides of (4.3), result in:

$$\begin{cases} u(x) = \int_0^x (x-t)(-1 - t^2 - \sin t + \int_0^t (u(\tau) + v(\tau))d\tau)dt, \\ v(x) = \int_0^x (x-t)(1 - 2 - \sin t - \cos t + \int_0^t (u(\tau) - v(\tau))d\tau)dt. \end{cases} \tag{4.4}$$

Considering the initial conditions (4.2), the following iterative relations are obtained as:

$$\begin{cases} u_{n+1}(x) = u_0(x) + \int_0^x (x-t)(-1 - t^2 - \sin t + \int_0^t (u_n(\tau) + v_n(\tau))d\tau)dt, \\ v_{n+1}(x) = v_0(x) + \int_0^x (x-t)(1 - 2 - \sin t - \cos t + \int_0^t (u_n(\tau) - v_n(\tau))d\tau)dt, \end{cases} \tag{4.5}$$

where $u_0(x) = 1 + x$, $v_0(x) = 2x$ and $u_n(x), v_n(x)$ indicates the n-th approximation of $u(x)$ and $v(x)$ respectively. According to (4.5), after some simplification and substitution, the following sets of relations are resulted:

$$u_1(x) = 1 - \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \sin x,$$
$$v_1(x) = -1 + \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + 2\sin x + \cos x,$$
$$u_2(x) = x + (1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots),$$
$$v_2(x) = x + (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots).$$

Thus the closed form solutions are as follows:

$$u(x) = \lim_{n\to\infty} u_n(x) = x + \cos x, \quad v(x) = \lim_{n\to\infty} v_n(x) = x + \sin x.$$

Now, we will solve this example by the homotopy perturbation method (HPM)[7-10]. To do this, we construct a homotopy function as the following form:

$$H(u, p) = u(x) + 1 + x^2 + \sin x - p\int_0^x (u(t) + v(t))dt,$$

$$H(v, p) = v(x) - 1 + 2 + \sin x + \cos x - p\int_0^x (u(t) - v(t))dt. \tag{4.6}$$

The embedding parameter p monotonically increases from 0 to 1. In order to apply this method the following expansion will be used:

$$u(x) = \sum_{n=0}^{\infty} p^n u_n(x), \quad v(x) = \sum_{n=0}^{\infty} p^n v_n(x), \tag{4.7}$$

where $u_n(x)$ and $v_n(x)$, $n \geq 0$ are the components of $u(x)$ and $v(x)$ that will be elegantly determined in the recursive manner. Substituting (4.7), in (4.6), and equating the terms with equal powers, the following sets of relations are resulted:

$$\begin{aligned}
u_0(x) &= 1 - \frac{x^2}{2} - \frac{x^4}{12} + \sin x,\\
v_0(x) &= -1 + \frac{x^2}{2} + 2\sin x + \cos x,\\
u_1(x) &= -3 + x + \frac{3x^2}{2} - \frac{x^7}{2520} + 3\cos x - \sin x,\\
v_1(x) &= 1 - x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{60} - \frac{x^7}{2520} + \sin x - \cos x,\\
u_2(x) &= 2x - \frac{x^3}{3} + \frac{x^5}{60} + \frac{x^6}{360} - \frac{x^8}{20160} - \frac{x^{10}}{907200} - 2\sin x,\\
v_2(x) &= 2 + 4x - x^2 - \frac{2x^3}{3} + \frac{x^4}{12} + \frac{x^5}{30} - \frac{x^6}{360} + \frac{x^8}{20160} - 4\sin x - 2\cos x,\\
u_3(x) &= 6 - 2x - 3x^2 + \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{60} - \frac{x^6}{120} + \frac{x^7}{2520} + \frac{x^8}{6720}\\
&\quad - \frac{x^{13}}{1556755200} - 6\cos x + 2\sin x,\\
v_3(x) &= -2 + 2x + x^2 - \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} - \frac{x^7}{2520} - \frac{x^8}{20160} + \frac{x^9}{90720}\\
&\quad - \frac{x^{11}}{9979200} - \frac{x^{13}}{1556755200} + 2\cos x - 2\sin x,
\end{aligned}$$

and so on. Therefore, the solutions by the HPM with three terms will be determined as:

$$\begin{aligned}
u(x) &= 4 + x - 2x^2 + \frac{x^4}{6} - \frac{x^6}{180} + \frac{x^8}{10080} - \frac{x^{10}}{907200} - \frac{x^{13}}{1556755200}\\
&\quad -3\cos x,\\
v(x) &= 5x - \frac{2x^3}{3} + \frac{x^5}{30} - \frac{x^7}{1260} + \frac{x^9}{90720} - \frac{x^{11}}{9979200} - \frac{x^{13}}{1556755200}\\
&\quad -3\sin x.
\end{aligned}$$

Fig. 1. compares the approximate solutions which are obtained by HPM and RVIM methods. Also the absolute error of these two methods are shown.

Considering the above results we conclude that the RVIM provide the exact solutions of this equation in a few iterations.

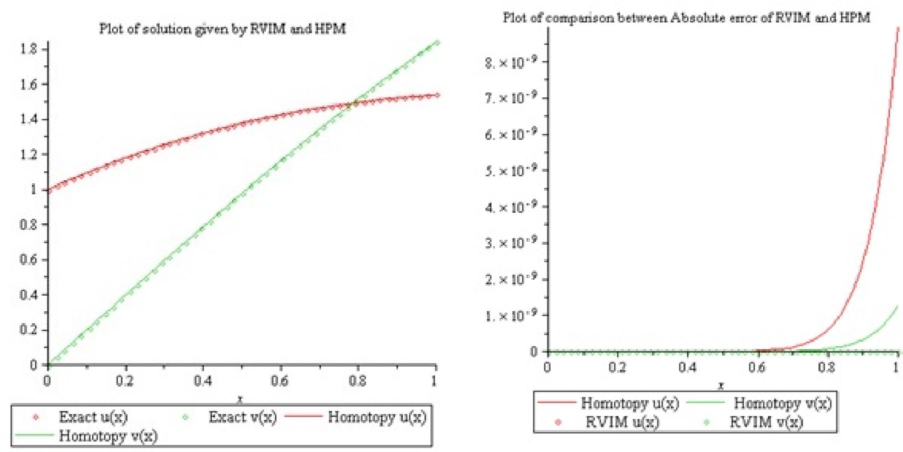

FIGURE 1. Comparison between the solutions of RVIM and HPM

Example 4.2. Now we consider the following system of three Volterra integro-differential equations:

$$\begin{cases} u'(x) = 2 + e^x - 3e^{2x} + e^{3x} + \int_0^x (6v(t) - 3w(t))dt, \\ v'(x) = e^x + 2e^{2x} - e^{3x} + \int_0^x (3w(t) - u(t))dt, \\ w'(x) = -e^x + e^{2x} + 3e^{3x} + \int_0^x (u(t) - 2v(t))dt, \end{cases} \tag{4.8}$$

with the initial conditions:

$$u(0) = 1, \quad v(0) = 1, \quad w(0) = 1. \tag{4.9}$$

Applying the Laplace transform to (4.8), the result is as follows:

$$\begin{cases} \ell\{u(x)\} = \frac{1}{s}\ell\{2 + e^x - 3e^{2x} + e^{3x} + \int_0^x (6v(t) - 3w(t))dt\}, \\ \ell\{v(x)\} = \frac{1}{s}\ell\{e^x + 2e^{2x} - e^{3x} + \int_0^x (3w(t) - u(t))dt\}, \\ \ell\{w(x)\} = \frac{1}{s}\ell\{-e^x + e^{2x} + 3e^{3x} + \int_0^x (u(t) - 2v(t))dt\}. \end{cases} \tag{4.10}$$

Similarly, using the inverse Laplace transform on both sides of (4.10), the following RVIM formula is obtained:

$$\begin{cases} u(x) = \int_0^x (2 + e^t - 3e^{2t} + e^{3t} + \int_0^t (6v(\tau) - 3w(\tau))d\tau)dt, \\ v(x) = \int_0^x (e^t + 2e^{2t} - e^{3t} + \int_0^t (3w(\tau) - u(\tau))d\tau)dt, \\ w(x) = \int_0^x (-e^t + e^{2t} + 3e^{3t} + \int_0^t (u(\tau) - 2v(\tau))d\tau)dt. \end{cases} \tag{4.11}$$

To obtain the approximate solution of (4.8), the iterative relation is considered as

$$\begin{cases} u_{n+1}(x) = u_0(x) + \int_0^x (2 + e^t - 3e^{2t} + e^{3t} + \int_0^t (6v_n(\tau) - 3w_n(\tau))d\tau)dt, \\ v_{n+1}(x) = v_0(x) + \int_0^x (e^t + 2e^{2t} - e^{3t} + \int_0^t (3w_n(\tau) - u_n(\tau))d\tau)dt, \\ w_{n+1}(x) = w_0(x) + \int_0^x (-e^t + e^{2t} + 3e^{3t} + \int_0^t (u_n(\tau) - 2v_n(\tau))d\tau)dt, \end{cases} \tag{4.12}$$

According to (4.9), we consider these initial approximations :

$$u_0(x) = 1, \quad v_0(x) = 1, \quad w_0(x) = 1.$$

Then by means of RVIM technique the successive approximate solutions can be obtained:

$$\begin{aligned} u_1(x) &= \frac{7}{6} + 2x + \frac{3}{2}x^2 + e^x - \frac{3}{2}e^{2x} + \frac{1}{3}e^{3x}, \\ v_1(x) &= -\frac{2}{3} + x^2 + e^x + e^{2x} - \frac{1}{3}e^{3x}, \\ w_1(x) &= \frac{1}{2} - \frac{1}{2}x^2 - e^x + \frac{1}{2}e^{2x} - \frac{1}{3}e^{3x}, \\ u_2(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \\ v_2(x) &= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots, \\ w_2(x) &= 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + ... \end{aligned}$$

Therefore, the exact solutions are given by:

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) = e^x, \\ v(x) &= \lim_{n\to\infty} v_n(x) = e^{2x}, \\ w(x) &= \lim_{n\to\infty} w_n(x) = e^{3x}. \end{aligned}$$

Now, we will use the homotopy perturbation method (HPM) for solving this system. Similarly to the procedure which is explained in example (4.1), one can have the following relations:

$$u_0(x) = 1 + 2x + e^x - \frac{3}{2}e^{2x} + \frac{1}{3}e^{3x},$$
$$v_0(x) = 1 + e^x + e^{2x} - \frac{1}{3}e^{3x},$$
$$w_0(x) = 1 - e^x + \frac{1}{2}e^{2x} + e^{3x},$$
$$u_1(x) = -\frac{689}{72} - \frac{115}{12}x + \frac{3}{2}x^2 + 9e^x + \frac{9}{8}e^{2x} - \frac{5}{9}e^{3x},$$
$$v_1(x) = -\frac{319}{108} + \frac{29}{18}x + x^2 - \frac{1}{3}x^3 - 4e^x + \frac{3}{4}e^{2x} + \frac{8}{27}e^{3x},$$
$$w_1(x) = \frac{127}{72} + \frac{29}{12}x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - e^x - \frac{7}{8}e^{2x} + \frac{1}{9}e^{3x},$$

In the same manner, fourth term approximations of the solutions of (4.8) are computed. By considering these approximations, the solutions are approximated as follows:

$$\begin{aligned} u(x) = &-\frac{321126643}{1119744} - \frac{53233745}{186624}x - \frac{8522971}{62208}x^2 - \frac{1436489}{31104}x^3 \\ &- \frac{212467}{20736}x^4 - \frac{7069}{3456}x^5 - \frac{8023}{17280}x^6 - \frac{509}{20160}x^7 - \frac{3}{640}x^8 \\ &+ \frac{1}{1344}x^9 + 289e^x - \frac{711}{512}e^{2x} + \frac{17}{2187}e^{3x}, \\ v(x) = &-\frac{315159281}{1679616} - \frac{53074939}{279936}x - \frac{8794217}{93312}x^2 - \frac{1486819}{46656}x^3 \\ &- \frac{274865}{31104}x^4 - \frac{42379}{25920}x^5 - \frac{7397}{25920}x^6 - \frac{1219}{30240}x^7 + \frac{19}{6720}x^8 \\ &- \frac{19}{60480}x^9 + 189e^x + \frac{337}{256}e^{2x} - \frac{79}{6561}e^{3x}, \\ w(x) = &\frac{836871971}{3359232} + \frac{138986593}{559872}x + \frac{22766219}{186624}x^2 + \frac{3824473}{93312}x^3 \\ &+ \frac{642467}{62208}x^4 + \frac{101953}{51840}x^5 + \frac{20183}{51840}x^6 + \frac{2293}{60480}x^7 + \frac{1}{13440}x^8 \\ &- \frac{1}{10080}x^9 - 249e^x + \frac{189}{512}e^{2x} + \frac{6593}{6561}e^{3x}. \end{aligned}$$

Fig. 2. shows that the RVIM is more efficient than HPM.

Example 4.3. Finally, let us consider the system of nonlinear Volterra integro-differential equation:

$$\begin{cases} u'(x) = 2x + \frac{1}{6}x^4 + \frac{2}{15}x^6 + \int_0^x (x-2t)(u^2(t) + v(t))dt, \\ v'(x) = -2x - \frac{1}{6}x^4 + \frac{2}{15}x^6 + \int_0^x (x-2t)(u(t) + v^2(t))dt, \end{cases} \tag{4.13}$$

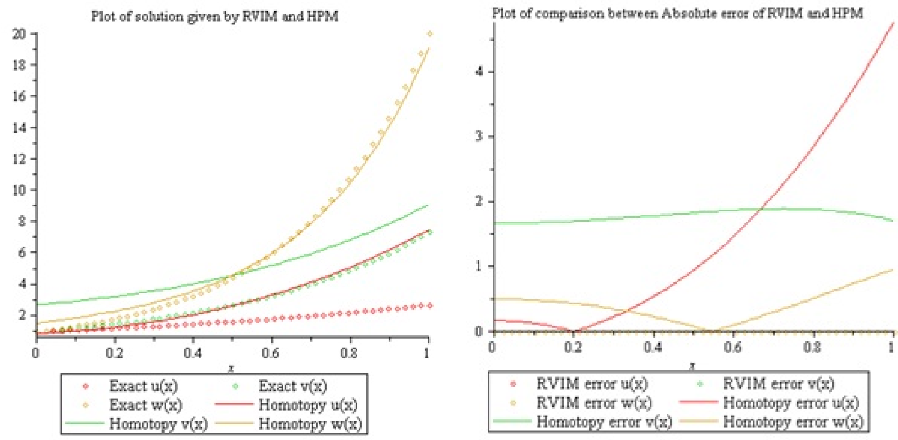

FIGURE 2. Comparison between the RVIM and HPM for system (4.8).

subjected to the initial conditions:

$$u(0) = 1, \quad v(0) = 1. \tag{4.14}$$

Applying the Laplace transform to (4.13), the result is as follows:

$$\begin{cases} \ell\{u(x)\} = \frac{1}{s}\ell\{2x + \frac{1}{6}x^4 + \frac{2}{15}x^6 + \int_0^x (x-2t)(u^2(t)+v(t))dt\}, \\ \ell\{v(x)\} = \frac{1}{s}\ell\{-2x - \frac{1}{6}x^4 + \frac{2}{15}x^6 + \int_0^x (x-2t)(u(t)+v^2(t))dt\}. \end{cases} \tag{4.15}$$

By applying the inverse Laplace transform to both sides of (4.15), result in:

$$\begin{cases} u(x) = \int_0^x (2t + \frac{1}{6}t^4 + \frac{2}{15}t^6 + \int_0^t (t-2\tau)(u^2(\tau)+v(\tau))d\tau)dt, \\ v(x) = \int_0^x (-2t - \frac{1}{6}t^4 + \frac{2}{15}t^6 + \int_0^t (t-2\tau)(u(\tau)+v^2(\tau))d\tau)dt. \end{cases} \tag{4.16}$$

Considering the initial conditions (4.14), the following iterative relations are obtained as:

$$\begin{cases} u_{n+1}(x) = u_0(x) + \int_0^x (2t + \frac{1}{6}t^4 + \frac{2}{15}t^6 + \int_0^t (t-2\tau)(u_n^2(\tau)+v_n(\tau))d\tau)dt, \\ v_{n+1}(x) = v_0(x) + \int_0^x (-2t - \frac{1}{6}t^4 + \frac{2}{15}t^6 + \int_0^t (t-2\tau)(u_n(\tau)+v_n^2(\tau))d\tau)dt, \end{cases} \tag{4.17}$$

According to (4.9), we consider these initial approximations :

$$u_0(x) = 1, \quad v_0(x) = 1.$$

Then by means of RVIM technique the successive approximate solutions can be obtained:

$$u_1(x) = 1 + x^2 + \frac{1}{30}x^5 + \frac{2}{105}x^7,$$
$$v_1(x) = 1 - x^2 - \frac{1}{30}x^5 + \frac{2}{105}x^7,$$
$$u_2(x) = 1 + x^2 + \frac{1}{30}x^5 + \frac{2}{105}x^7,$$
$$v_2(x) = 1 - x^2 - \frac{1}{30}x^5 + \frac{2}{105}x^7.$$

Therefore, the exact solutions are given by:

$$u(x) = \lim_{n\to\infty} u_n(x) = 1 + x^2 + \frac{1}{30}x^5 + \frac{2}{105}x^7,$$
$$v(x) = \lim_{n\to\infty} v_n(x) = 1 - x^2 - \frac{1}{30}x^5 + \frac{2}{105}x^7.$$

As we see in this example, by only two iterations, the method converges to the exact solution and it shows the efficiency of our method for solving nonlinear integro-differential equations.

5. Conclusion

In work, we applied the Reconstruction of Variational Iteration Method (RVIM) for solving the systems of Volterra integro-differential equations. In our method knowing the variational theory is not essential while it was needed in the variational iteration method. It is important to point out that some other methods should be applied for systems with separable or difference kernels. Whereas, the RVIM can be used for solving systems of Volterra integro-differential equations with any kind of kernels. By comparing the results of other numerical methods such as homotopy perturbation method, we conclude that the RVIM is more accurate, fast and reliable. Besides , RVIM does not require small parameters; thus, the limitations of the traditional perturbation methods can be eliminated, and the calculations are also simple and straightforward. These advantages has been confirmed by employing two examples. Therefore, this method is a very effective tool for calculating the exact solutions of systems of integro-differential equations.

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