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Application of linear combination between cubic B-spline collocation methods with different basis for solving the KdV equation

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Abstract In the present article, a numerical method is proposed for the numerical solution of the KdV equation by using a new approach by combining cubic B-spline functions. In this paper we convert the KdV equation to system of two equations. The method is shown to be unconditionally stable using von-Neumann technique. To test accuracy the error norms L_2, L_∞ are computed. Three invariants of motion are predestined to determine the preservation properties of the problem and the numerical scheme leads to careful and active results. Furthermore, interaction of two and three solitary waves is shown. These results show that the technique introduced here is easy to apply.

Keywords. Collocation Method, cubic B-Spline methods, KdV equation.

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1. Introduction

We will solve the KdV equation in this form [9]

$$u_t + \varepsilon\, u u_x + \mu\, u_{xxx} = 0, \tag{1.1}$$

where ε, μ are positive parameters and the subscripts x and t denote differentiation. The boundary conditions of (1.1) are given by.

$$\begin{aligned} &u(a,t) = f_1(a,t), \qquad u(b,t) = f_2(b,t), \\ &u_x(a,t) = g_1(a,t), \qquad u_x(b,t) = g_2(b,t), \quad 0 \le t \le T. \end{aligned} \tag{1.2}$$

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And initial conditions of (1.1) are given by

$$\begin{aligned} &u(x,0) = f(x), \\ &u_x(x,0) = f'(x) = g(x), \qquad a \leq x \leq b. \end{aligned} \tag{1.3}$$

KdV equation is prototypical example of exactly solvable mathematical paradigm of waves on shallow water superficies. It grow for evolution, interaction of waves, and generation in physics. Due to the term u_t, (1.1) is called the evolution equation, the nonlinear term causes the steepness of the wave, and the dispersive term defines the spreading of the wave. It is known that the influence of the steepness and spreading results in soliton solutions for the KdV equation.

The KdV equation is a one-dimensional nonlinear partial differential equation of third order, which plays a big role in the discussion of nonlinear dispersive waves. This equation was primarily derived by Korteweg-de Vries [4] to characterize the action of one dimensional shallow water solitary waves. Solitary waves are wave packets or pulses which diffuse in nonlinear dispersive media. For stable solitary wave solutions the nonlinear and dispersive terms in the KdV equation must equilibrium and in this status the KdV equation has wandering wave solutions called solitons. A soliton is a very particular type of solitary waves which save its waveform after inconsistency with other solitons. A small time solutions using a heat balance integral method to solve the KdV equation was gained by Kutluay et al. [9]. In their article, comprehensive comparisons with the analytical values over the acquaint interval are given. Bahadir [1] studied the exponential finite-difference technique to solve the KdV equation. This method has been shown to supply higher accuracy than the classical explicit finite difference and the heat balance integral method. Ozer and Kutluay [13] applied an analytical–numerical method, for solving the KdV equation and the obtained results are compared with that of the heat balance integral method and the corresponding analytical solution. Irk et al. [8] studied a second order spline approximation technique and made comparisons with earlier methods. Ozdes and Aksan [12] applied the method of lines for solving the KdV equation. A. Ozdes and E.N. Aksan [11] used a quadratic B-spline Galerkin finite element method and compared these techniques with the analytical solutions and other numerical solutions that are obtained earlier using various numerical techniques. O. Ersoy and I. Dag [7] applied the exponential cubic B-spline algorithm for solving the KdV equation. B. Saka [15] used cosine expansion-based differential quadrature method for numerical solution of the KdV equation. Dag and Y. Dereli [3] applied numerical solutions of KdV equation using radial basis functions. A. Canıvar et al. [2] applied A Taylor-galerkin finite element method for the KdV equation using cubic B-splines and also G. Micula and M. Micula [10] used on the numerical approach of Korteweg-de Vries-Burger equations by spline finite element and collocation methods.

The paper is organized as follows. In section 2, we convert the KdV equation to system of nonlinear equations. In section 3, we introduced the description of method. In section 4, we introduced the decoction of the liner combination between cubic B-spline collocation method, dissection of initial state and stability. In section 5, numerical results for problem and some related figures are given in order to show the efficiency as well as the accuracy of the proposed method and we introduced the interaction of two and three solitary waves. Finally, conclusions are followed in section 6.

2. The KdV equation

Now we can convert the Eq. (1.1) to system of equations as the following process: We take $u_x = v$ in the Eq. (1.1), so we get

$$\begin{aligned} &u_t + \varepsilon\, u u_x + \mu\, v_{xx} = 0,\\ &u_x = v, \end{aligned} \tag{2.1}$$

where ε, μ are positive parameters and the subscripts x and t denote differentiation.
The boundary conditions of (2.1) are given by

$$\begin{aligned} &u(a,t) = f_1(a,t), \qquad u(b,t) = f_2(b,t),\\ &v(a,t) = g_1(a,t), \qquad v(b,t) = g_2(b,t), \quad 0 \le t \le T. \end{aligned} \tag{2.2}$$

And initial conditions of (2.1) are given by

$$\begin{aligned} &u(x,0) = f(x),\\ &v(x,0) = g(x), \qquad a \le x \le b. \end{aligned} \tag{2.3}$$

3. Liner combination between cubic B-spline collocation method

To construct numerical solution, consider nodal points (x_j, t_n) defined in the region $[a,b] \times [0,T]$, where

$$a = x_0 < x_1 < ... < x_N = b, \qquad h = x_{j+1} - x_j = \frac{b-a}{N}, \;\; j = 0,1,...,N,$$

$$0 = t_0 < t_1 < ... < t_n < ... < T, \qquad t_n = n\Delta t, \qquad n = 0,1,.... \;.$$

Through this section, linear combination between cubic B-splines (LCCBS) with different basis functions is used to solve (2.1). The approximate solution, $U^N(x,t)$, $V^N(x,t)$ to the analytical solution $u(x,t), v(x,t)$, in the form:

$$U^N(x,t) = \sum_{j=-1}^{N+1} c_j(t)\; L_j(x), V^N(x,t) = \sum_{j=-1}^{N+1} \delta_j(t)\; L_j(x), \tag{3.1}$$

where $c_j(t)$ and $\delta_j(t)$ are time-dependent unknowns to be determined and $L_j(x)$ is (LCCBS) basis functions as

$$L_j(x) = \gamma\, CT\, B_j(x) + (1-\gamma)\; B_j(x),$$

where $CTB_j(x)$ the cubic trigonometric B-spline basis function at knots is given by

$$CTB_j(x) = \frac{1}{\theta}\begin{cases} \omega^3(x_{j-2}), & x_{j-2} \le x \le x_{j-1}\,, \\ \omega\,(x_{j-2})\ \ (\omega\,(x_{j-2})\,\phi(x_j) + \omega\,(x_{j-1})\,\phi(x_{j+1})\,) \\ +\omega^{\,2}(x_{j-1})\,\phi(x_{j+2}), \quad x_{j-1} \le x \le x_j \\ \omega\,(x_{j-2})\,\phi^2(x_{j+1}) + \phi(x_{j+2})\ (\omega\,(x_{j-1})\,\phi(x_{j+1}) \\ +\omega\,(x_j)\,\phi(x_{j+2})), \quad x_{j-1} \le x \le x_j \\ \phi^3(x_{j+2}), & x_{j+1} \le x \le x_{j+2}\,, \\ 0 & otherwise\,, \end{cases} \tag{3.2}$$

where $\omega(x_j) = \sin\left(\frac{x-x_j}{2}\right)$, $\phi(x_j) = \sin\left(\frac{x_j-x}{2}\right)$, $\theta = \sin\left(\frac{h}{2}\right)\ \sin(h)\ \sin\left(\frac{3h}{2}\right)$ and $B_j(x)$ is the exponential cubic B-spline basis functions at knots is are given by

$$B_j(x) = \begin{cases} b_2\left((x_{j-2}-x) - \frac{1}{p}\,(\sinh\,(p\,(x_{j-2}-x)))\right), & x_{j-2} \le x \le x_{j-1} \\ a_1\ +\ b_1\,(x_j - x)\ + c_1\ \exp\ \ (p\ (x_j - x)) \\ +d_1\,\exp\ \ (-p\,(x_j - x))\ , & x_{j-1} \le x \le x_j, \\ a_1\ +\ b_1\,(x - x_j)\ \ + c_1\ \exp\ \ (p\,(x - x_j)) \\ +d_1\,\exp\ \ (-p\,(x - x_j))\,, & x_j \le x \le x_{j+1}\,, \\ b_2\left((x - x_{j+2}) - \frac{1}{p}\,(\sinh\,(p\,(x - x_{j+2})))\right), & x_{j+1} \le x \le x_{j+2}\,, \\ 0 & otherwise\,, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} &a_1 = \frac{p\,h\,c}{p\,h\,c-s}, \qquad b_1 = \frac{p}{2}\left[\frac{c\,(c-1)+s^2}{(p\,h\,c-s)(1-c)}\right], \\ &b_2 = \frac{p}{2(p\,h\,c-s)}, \qquad d_1 = \frac{1}{4}\left[\frac{\exp\,(-p\,h)\,(1-c)+s(\exp\,(-p\,h)-1)}{(p\,h\,c-s)(1-c)}\right], \\ &d_2 = \frac{1}{4}\left[\frac{\exp\,(p\,h)\,(1-c)+s(\exp\,(p\,h)-1)}{(p\,h\,c-s)(1-c)}\right], \\ &s = \sinh\,(p\,h)\,, \quad c = cosh\ (p\,h)\,, \quad h = \frac{b-a}{N}. \end{aligned}$$

The value of γ plays an important role in the (LCCBS) basis function. If $\gamma = 0$, the basis function is equal to cubic trigonometric B-spline basis function and if $\gamma = 1$, the basis function is equal to exponential cubic B-spline basis functions. Hence, this work just considers the value of $0 < \gamma < 1$.

Due to local support properties of B-spline basis function, there are only three nonzero basis functions; namely, L_{j-1}, L_j and L_{j+1}. Thus, the approximate values of $U^N(x)$, $V^N(x)$ and its two derivatives at the knots/nodes are determined in terms of the time parameters c_j, δ_j as follows:

$$\begin{aligned}
&(U_1)_{\mathrm{j}} = (\mathrm{U}_1)(x_j) = a_1 c_{j-1} + a_2 c_j + a_1 c_{j+1},\\
&(\mathrm{U}')_j = (\mathrm{U}')(x_j) = a_3 c_{j-1} - a_3 c_{j+1},\\
&(\mathrm{U}'')_j = (\mathrm{U}'')(x_j) = a_4 c_{j-1} + a_5 c_j + a_4 c_{j+1}\\
&(\mathrm{V})_{\mathrm{j}} = (\mathrm{V})(x_j) = a_1\,\delta_{j-1} + a_2\,\delta_j + a_1\,\delta_{j+1},\\
&(\mathrm{V}')_j = (\mathrm{V}')(x_j) = a_3\,\delta_{j-1} - a_3\,\delta_{j+1},\\
&(\mathrm{V}'')_j = (\mathrm{V}'')(x_j) = a_4\,\delta_{j-1} + a_5\,\delta_j + a_4\,\delta_{j+1},
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
&a_1 = \gamma\,\alpha_1 + (1-\gamma)m_1, \quad a_2 = \gamma\,\alpha_2 + (1-\gamma),\\
&a_3 = (1-\gamma)m_2 - \gamma\,\alpha_3, \quad a_4 = \gamma\,\alpha_4 + (1-\gamma)m_3,\\
&a_5 = \gamma\,\alpha_5 - 2\,(1-\gamma)m_3,\\
&m_1 = \tfrac{(s-p\,h)}{2(p\,h\,c-s)}, \;\; m_2 = \tfrac{p\,(1-c)}{2(p\,h\,c-s)}, m_3 = \tfrac{p^2 s}{2(p\,h\,c-s)},\\
&\alpha_1 = \sin^2\left(\tfrac{h}{2}\right)\;\csc(h)\;\csc\;\left(\tfrac{3h}{2}\right), \quad \alpha_2 = \tfrac{2}{1+2\cos\,(h)}, \;\; \alpha_3 = \tfrac{3}{4}\csc\;\left(\tfrac{3h}{2}\right),\\
&\quad \alpha_4 = \tfrac{3\left((1+3\cos\,(h))\csc^2\,\left(\tfrac{h}{2}\right)\right)}{16\left(2\cos\,\left(\tfrac{h}{2}\right)+\cos\,\left(\tfrac{3h}{2}\right)\right)}, \qquad\qquad \alpha_5 = -\tfrac{3\cot^2\left(\tfrac{h}{2}\right)}{2+4\cos\,(h)},
\end{aligned}$$

4. Solution of KdV equation

To apply the proposed method, we rewrite (2.1) as

$$\begin{aligned}
&\tfrac{\partial u(x,t)}{\partial t} + \varepsilon\,u\,(x,t)\tfrac{\partial u(x,t)}{\partial x} + \mu\tfrac{\partial^2 v(x,t)}{\partial x^2} = 0,\\
&\tfrac{\partial u\,(x,t)}{\partial x} = v(x,t).
\end{aligned} \tag{4.1}$$

We take the approximations $u(x,t) = U_j^n$ and $v(x,t) = V_j^n$, then from famous Cranck-Nicolson scheme and forward finite difference approximation for the derivative t [5], we get

$$\begin{aligned}
&\tfrac{U_j^{n+1}-U_j^n}{k} + \varepsilon\left[\tfrac{(UU_x)_j^{n+1}+(UU_x)_j^n}{2}\right] + \mu\left[\tfrac{(V)_{xx}{}_j^{n+1}+(V)_{xx}{}_j^n}{2}\right] = 0,\\
&\left[\tfrac{U_x{}_j^{n+1}+U_x{}_j^n}{2}\right] = \tfrac{V_j^{n+1}+V_j^n}{2},
\end{aligned} \tag{4.2}$$

where $k = \Delta t$ is the time step.

The nonlinear terms in (4.2) is linearized using the form given by Rubin and Graves [14] as: we take linearization of the nonlinear term as follows:

$$(UU_x)_j^{n+1} = U_j^n U_x{}_j^{n+1} + U_j^{n+1} U_x{}_j^n - U_j^n U_x{}_j^n. \tag{4.3}$$

Using (3.4) and (4.3) in (4.2), we get a system of ordinary differential equations of the form:

$$\begin{aligned}
A_1 c_{j-1}^{n+1} + A_2 c_j^{n+1} + A_3 c_{j+1}^{n+1} + A_4 \delta_{j-1}^{n+1} + A_5 \delta_j^{n+1} + A_4 \delta_{j+1}^{n+1} &=\\
a_1 c_{j-1}^n + a_2 c_j^n + a_3 c_{j+1}^n - A_4 \delta_{j-1}^n - A_5 \delta_j^n - A_4 \delta_{j+1}^n&,
\end{aligned}$$

$$\tag{4.4}$$

$$\begin{aligned} a_3c_{j-1}^{n+1} - a_3c_{j+1}^{n+1} - a_1\delta_{j-1}^{n+1} - a_2\delta_j^{n+1} - a_1\delta_{j+1}^{n+1} = \\ - a_3c_{j-1}^{n} + a_3c_{j+1}^{n} + a_1\delta_{j-1}^{n} + a_2\delta_j^{n} + a_1\delta_{j+1}^{n}, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} A_1 &= a_1(1 + \tfrac{\varepsilon\Delta t}{2} z_2) + a_3 z_1, & A_2 &= a_2(1 + \tfrac{\varepsilon\Delta t}{2} z_2), \\ A_3 &= a_1(1 + \tfrac{\varepsilon\Delta t}{2} z_2) - a_3 z_1, & A_4 &= \tfrac{\mu\,\Delta t}{2} a_4, \quad A_5 = \tfrac{\mu\,\Delta t}{2} a_5. \end{aligned}$$

The system thus obtained on simplifying (4.4) and (4.5) consists of $(2N+2)$ linear equations in the $(2N+2)$ unknowns $(c_{0,}......., c_N)^T$,$(\delta_{0,}......., \delta_N)^T$, which is the tridiagonal system that can be solved by any algorithm.

4.1. **Initial values.** The initial vectors c_j^0, δ_j^0 can be obtained from the initial condition and boundary values of the derivatives of the initial condition as the following expressions:

$$\begin{array}{lr} U^N(x_0,0) = f_1(0), & for\ j=0, \\ U^N(x_j,0) = f(x_j), & for\ j=1,2,\dots,N-1, \\ U^N(x_N,0) = f_2(0), & for\ j=N, \\ V^N(x_0,0) = g_1(0), & for\ j=0, \\ V^N(x_j,0) = g(x_j), & for\ j=1,2,\dots,N-1, \\ V^N(x_N,0) = g_2(0), & for\ j=N. \end{array} \tag{4.6}$$

This yields a $(2N+2)\times(2N+2)$ system equations of the form

$$\begin{gathered} \begin{pmatrix} a_2 & 2a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2a_1 & a_2 \end{pmatrix} \begin{pmatrix} c_0^0 \\ c_1^0 \\ \vdots \\ c_{N-1}^0 \\ c_N^0 \end{pmatrix} = \begin{pmatrix} f_1(x_0) \\ f(x_0) \\ \vdots \\ \vdots \\ f(x_N) \\ f_N(x_0) \end{pmatrix}, \\ \begin{pmatrix} a_2 & 2a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2a_1 & a_2 \end{pmatrix} \begin{pmatrix} \delta_0^0 \\ \delta_1^0 \\ \vdots \\ \delta_{N-1}^0 \\ \delta_N^0 \end{pmatrix} = \begin{pmatrix} g_1(x_0) \\ g(x_0) \\ \vdots \\ \vdots \\ g(x_N) \\ g_N(x_0) \end{pmatrix}, \end{gathered} \tag{4.7}$$

The solution of (4.7) can be solved by any algorithm.

4.2. **Stability analysis of the method.** The stability analysis of nonlinear partial differential equations is not easy task to undertake. Most researchers copy with the problem by linearizing the partial differential equation. Our stability analysis will be based on the Von-Neumann concept in which the growth factor of a typical Fourier mode defined as

$$\begin{aligned} c_j^n &= A\zeta^n \exp(ij\phi), \\ \delta_j^n &= B\zeta^n \exp(ij\phi), \end{aligned} \tag{4.8}$$

$$g = \frac{\zeta^{n+1}}{\zeta^n},$$

where A and B are the harmonics amplitude, $\phi = k\,h$, k is the mode number, $i = \sqrt{-1}$ and g is the amplification factor of the schemes. We will be applied the stability of the schemes by assuming the nonlinear term as a constants λ. System (4.4) and (4.5) can be written as

$$\begin{aligned} & a_1 c_{j-1}^{n+1} + a_2 c_j^{n+1} + a_1 c_{j+1}^{n+1} + \tfrac{\lambda\,k\,\varepsilon}{2}\left(a_3 c_{j-1}^{n+1} - a_3 c_{j+1}^{n+1}\right) + \\ & \quad \tfrac{k\mu}{2}\left(a_4 \delta_{j-1}^{n+1} + a_5 \delta_j^{n+1} + a_4 \delta_{j+1}^{n+1}\right) = a_1 c_{j-1}^n + a_2 c_j^n + a_1 c_{j+1}^n - \\ & \quad\quad \tfrac{\lambda\,k\,\varepsilon}{2}\left(a_3 c_{j-1}^n - a_3 c_{j+1}^n\right) - \tfrac{k\mu}{2}\left(a_4 \delta_{j-1}^n + a_5 \delta_j^n + a_4 \delta_{j+1}^n\right)\,, \end{aligned} \tag{4.9}$$

$$\begin{aligned} & a_3 c_{j-1}^{n+1} - a_3 c_{j+1}^{n+1} - a_1 \delta_{j-1}^{n+1} - a_2 \delta_j^{n+1} - a_1 \delta_{j+1}^{n+1} = \\ & \qquad - a_3 c_{j-1}^n + a_3 c_{j+1}^n + a_1 \delta_{j-1}^n + a_2 \delta_j^n + a_1 \delta_{j+1}^n. \end{aligned} \tag{4.10}$$

Substituting (4.8) into the difference (4.9), we get

$$\begin{aligned} & \zeta^{n+1}\left[A\,(2a_1\cos(\phi) + a_2) + \tfrac{B\,k\mu}{2}\,(2a_4\cos(\phi) + a_5) - i\lambda\,k\,\varepsilon\,A\,a_3 \sin(\phi)\right] \\ & = \zeta^n\left[A\,(2a_1\cos(\phi) + a_2) - \tfrac{B\,k\mu}{2}\,(2a_4\cos(\phi) + a_5) + i\lambda\,k\,\varepsilon\,A\,a_3 \sin(\phi)\right]\,, \end{aligned} \tag{4.11}$$

$$\frac{\zeta^{n+1}}{\zeta^n} = \frac{\left[A\,(2a_1\cos(\phi) + a_2) - \tfrac{B\,k\mu}{2}\,(2a_4\cos(\phi) + a_5) + i\lambda\,k\,\varepsilon\,A\,a_3 \sin(\phi)\right]}{\left[A\,(2a_1\cos(\phi) + a_2) + \tfrac{B\,k\mu}{2}\,(2a_4\cos(\phi) + a_5) - i\lambda\,k\,\varepsilon\,A\,a_3 \sin(\phi)\right]}\,, \tag{4.12}$$

$$g = \frac{\zeta^{n+1}}{\zeta^n} = \frac{X_1 + iY}{X_2 - iY}\,, \tag{4.13}$$

where

$$\begin{aligned} X_1 &= A\,(2a_1\cos(\phi) + a_2) - \tfrac{B\,k\mu}{2}\,(2a_4\cos(\phi) + a_5)\,, \\ X_2 &= A\,(2a_1\cos(\phi) + a_2) + \tfrac{B\,k\mu}{2}\,(2a_4\cos(\phi) + a_5)\,, \\ Y &= \lambda\,k\,\varepsilon\,A\,a_3 \sin(\phi). \end{aligned}$$

Substituting (4.8) into the difference (4.10), we get

$$\begin{aligned} &\zeta^{n+1}\left[-B\ \left(2a_1\cos(\phi)+a_2\right)-2i\,A\,a_3\sin\left(\phi\right)\right] \\ &\qquad=\zeta^{n}\left[B\ \left(2a_1\cos(\phi)+a_2\right)+2i\,A\,a_3\sin\left(\phi\right)\right]\ , \end{aligned} \tag{4.14}$$

$$\frac{\zeta^{n+1}}{\zeta^{n}}=\frac{\left[B\ \left(2a_1\cos(\phi)+a_2\right)+2i\,A\,a_3\sin\left(\phi\right)\right]}{\left[-B\ \left(2a_1\cos(\phi)+a_2\right)-2i\,A\,a_3\sin\left(\phi\right)\right]}\,, \tag{4.15}$$

$$g=\frac{\zeta^{n+1}}{\zeta^{n}}=\frac{X_3+iZ}{X_4-iZ}\,, \tag{4.16}$$

where

$$\begin{aligned} &X_3=B\ \left(2a_1\cos(\phi)+a_2\right), \\ &X_4=-B\ \left(2a_1\cos(\phi)+a_2\right), \\ &Z=2A\,a_3\sin\left(\phi\right). \end{aligned}$$

From (4.13) and (4.16) we get $|g|\leq 1$, hence the schemes are unconditionally stable. It means that there is no restriction on the grid size, i.e. on h and Δt, but we should choose them in such a way that the accuracy of the scheme is not degraded.

5. Numerical testes and results of KdV equation

In this section, we present numerical example to test validity of our scheme for solving KdV equation.

The norms L_2-norm and L_∞-norm are used to compare the numerical solution with the analytical solution [6].

$$\begin{aligned} &L_2=\left\|u^E-u^N\right\|=\sqrt{h\sum_{i=0}^{N}(u_j^E-u_j^N)^2}, \\ &L_\infty=\max_j\left|u_j^E-u_j^N\right|,\ j=0,\,1,\cdots,N, \end{aligned} \tag{5.1}$$

where u^E is the exact solution u and u^N is the approximation solution U_N, and the quantities I_1, I_2 and I_3 are shown to measure conservation for the schemes.

$$\begin{aligned} &I_1=\int_{-\infty}^{\infty}u\left(x,t\right)dx\cong h\sum_{j=0}^{N}\left(U\right)_j^n, \\ &I_2=\int_{-\infty}^{\infty}\left(u\left(x,t\right)^2\right)\,dx\cong h\sum_{j=0}^{N}\left(U^2\right)_j^n, \\ &I_3=\int_{-\infty}^{\infty}\left[\left(u\left(x,t\right)^3-\tfrac{3\mu}{\varepsilon}u_x\left(x,t\right)^2\right)\right]dx\cong\ h\sum_{j=0}^{N}\left[\left(\left(U^3\right)_j^n-\tfrac{3\mu}{\varepsilon}\left(U_x^2\right)_j^n\right)\right], \end{aligned} \tag{5.2}$$

Now we consider this test problem.

Test problem

We assume that the solution of the KdV equation is negligible outside the interval $[a,b]$, together with all its x derivatives tend to zero at the boundaries. Therefore, in our numerical study we replace Eq. (1.1) as shown in section 2 by

$$\begin{aligned} &u_t + \varepsilon\, u u_x + \mu\, v_{xx} = 0, \\ &u_x = v, \end{aligned} \tag{5.3}$$

where ε, μ are positive parameters and the subscripts x and t denote differentiation.

The boundary conditions of (5.3) are given by

$$\begin{aligned} &u(a,t) = 0, \qquad u(b,t) = 0, \\ &v(a,t) = 0, \qquad v(b,t) = 0, \quad 0 \le t \le T. \end{aligned} \tag{5.4}$$

And initial conditions of (5.3) are given by

$$\begin{aligned} &u(x,0) = f(x), \\ &v(x,0) = g(x), \qquad a \le x \le b. \end{aligned} \tag{5.5}$$

Then the exact solutions of system (5.5) is

$$u(x,t) = 3c\ \mathrm{sech}^2\,(A\,x - B\,t + D\,), \tag{5.6}$$

where $A = \frac{1}{2}\sqrt{\frac{3c}{\mu}},\ B = \varepsilon\, c\, A.$

This solution represents propagation of single soliton, having velocity $\varepsilon\, c$ and amplitude $3c$.

To investigate the performance of the proposed schemes we consider solving the following problem.

5.1. **Single soliton.** In previous section, we have provided modified exponential cubic B-spline schemes for the KdV equation, and we can take the following initial condition.

$$u(x,0) = 3c\ \mathrm{sech}^2\,(A\,x + D\,), \tag{5.7}$$

where $A = \frac{1}{2}\sqrt{\frac{3c}{\mu}}$.

The norms L_2 and L_∞ are used to compare the numerical results with the analytical values and the quantities I_1, I_2 and I_3 are shown to measure conservation for the schemes.

Now, for comparison, we consider a test problem where, $k = 0.005,\ D = -6,\ c = 0.3, \varepsilon = 1,\ \mu = 4.84\times10^{-4},\ p = 1.64\times10^{-5},\ a = 0,\ b = 2$. The invariant $I_1,\ I_2$ and I_3 changed by less than 1.62×10^{-4}, 4×10^{-6} and 1.471×10^{-4} respectively. Errors, also, at time 2 are satisfactorily small L_2-error $=3\times10^{-3}$ and L_∞-error $= 9\ \times10^{-3}$ in the computer program for the scheme at $\gamma = 0.9$. The invariant $I_1,\ I_2$ and I_3 changed by less than 9.8×10^{-5}, 1.3×10^{-6} and 6.23×10^{-5} respectively. Errors, also, at time 2 are satisfactorily small L_2-error $=1\times10^{-3}$ and L_∞-error $= 5\ \times10^{-3}$ in the computer program for the scheme at $\gamma = 0.5$. The invariant I_1 changed by less than 5.8×10^{-5},I_2 approach to zero and I_3 changed by less than 2.05×10^{-5} in the computer program for the scheme at $\gamma = 0.001$. Our results are recorded in Table 1. The motion of solitary wave using our scheme is plotted at times $t = 0,\ 1,\ 1.5,\ 2,\ 2.5,\ 3$ in Figure 1. These results illustrate that the scheme has a highest accuracy and best conservation at $\gamma = 0.001$ than other scheme at $\gamma = 0.9,\ \gamma = 0.5$. So we use the scheme at$\gamma = 0.001$ to study the motion of single solitary waves and interaction between two and three solitons.

Table 1. Invariants and errors for single solitary wave $k = 0.005,\ D = -6, p = 1.64\times10^{-5},\ \varepsilon = 1,\ c = 0.3,\ \mu = 4.84\times10^{-4},\ a = 0, b = 2$.

γ	T	I_1	I_2	I_3	L_2**-norm**	L_∞**-norm**
$\gamma = 0.9$	0.0	0.144598	0.0867593	0.0624667	0.000	0.000
	0.5	0.144559	0.0867604	0.0623538	0.001	0.005
	1.0	0.144721	0.0867633	0.0623423	0.002	0.007
	1.5	0.144702	0.0867623	0.0623254	0.003	0.009
	2.0	0.144661	0.0867614	0.0623196	0.003	0.009
$\gamma = 0.5$	0.0	0.144598	0.0867593	0.0624667	0.000	0.000
	0.5	0.144567	0.0867599	0.0624142	0.001	0.002
	1.0	0.144665	0.0867606	0.0624148	0.001	0.003
	1.5	0.144656	0.0867602	0.0624055	0.001	0.004
	2.0	0.144644	0.0867600	0.0624044	0.001	0.005
$\gamma = 0.001$	0.0	0.144598	0.0867593	0.0624667	0.0000	0.0000
	0.5	0.144575	0.0867593	0.0624669	0.0004	0.0009
	1.0	0.144617	0.0867593	0.0624786	0.0004	0.0009
	1.5	0.144614	0.0867593	0.0624758	0.0004	0.0009
	2.0	0.144627	0.0867593	0.0624785	0.0004	0.0009
	2.5	0.144618	0.0867593	0.0624872	0.0004	0.0009
	3.0	0.144633	0.0867593	0.0624826	0.0004	0.0009

In the next table we make comparison between the results of our scheme and the results have been published in [15], [2], [3] and [7].

The results of our scheme are accurate than the results in [15], [3]b, [3]c and [3]d and related with the results in [3]a, [3]e and [7] and not better than the results in [2].

5.2. **Interaction of two solitary waves.** The interaction of two solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider KdV equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

$$u(x,0) = 3c_j\ \mathrm{sech}^2\,(A\,x + D_j\,), \tag{5.8}$$

FIGURE 1. Single solitary wave with $k = 0.005,\ D = -6,\ \varepsilon = 1,\ c = 0.3,\ \mu = 4.84 \times 10^{-4},\ \gamma = 0.001,\ 0 \le x \le 2,\ t = 0, 1, 1.5, 2, 2.5, 3$ respectively.

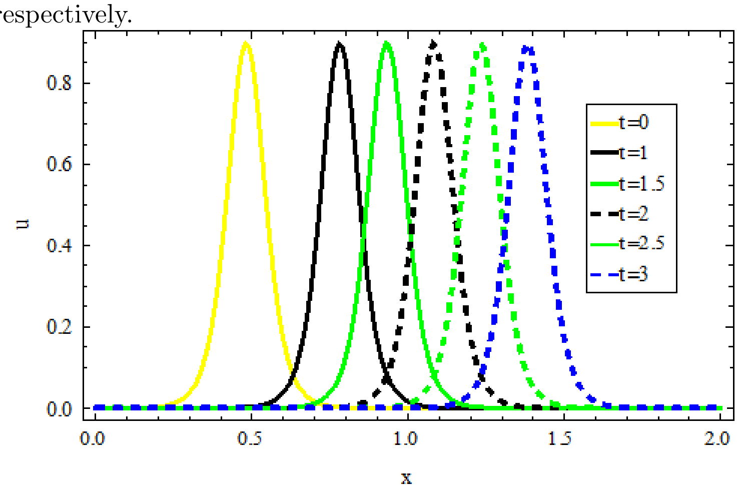


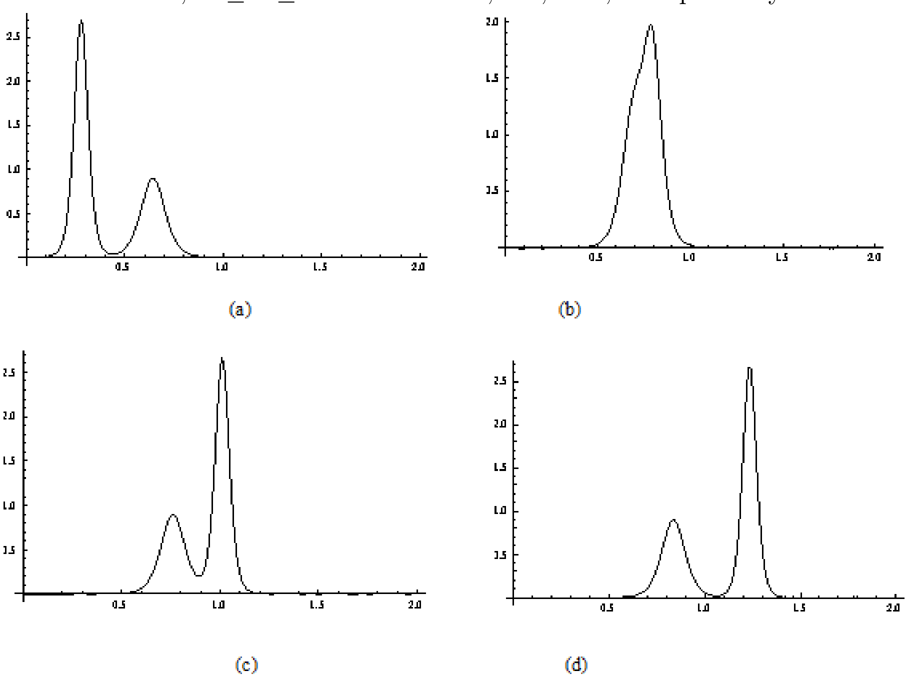
TABLE 2. Invariants and errors for single solitary wave $k = 0.005,\ D = -6,\ \varepsilon = 1,\ c = 0.3,\ p = 1.64 \times 10^{-5},\ \gamma = 0.001,\ \mu = 4.84 \times 10^{-4},\ 0 \le x \le 2,\ t = 3$.

Method	I_1	I_2	L_2**-norm**	L_∞**-norm**
Analytical	0.144598	0.0867593	0.0000	0.00000
Our scheme	0.144633	0.0867593	0.0004	0.0009
[15]	0.014460	0.08675	-	0.001
[2]	0.144597	0.086761	-	0.00004
[3]**a (G)**	0.144601	0.086760	0.00004	0.0001
[3]**b (TPS)**	0.144261	0.086762	0.002	0.006
[3]**c (IQ)**	0.144598	0.086759	0.001	0.002
[3]**d (IMQ)**	0.144623	0.086765	0.002	0.005
[3]**e (MQ)**	0.144606	0.086759	0.00006	0.0001
[7]	0.144597	0.0867593	-	0.0007

where $A = \frac{1}{2}\sqrt{\frac{3c_j}{\mu}},\ j = 1, 2,\ 3,\ \ c_j$ and D_j are arbitrary constants. In our computational work. Now, we choose $c_1 = 0.9,\ c_2 = 0.3,\ D_1 = -6,\ D_2 = -8,\ \mu = 4.84 \times 10^{-4},\ \varepsilon = 1,\ h = 0.01,\ k = 0.005$ with interval $[0,\ 2]$. In Figure 3, the interactions of these solitary waves are plotted at different time levels.

5.3. **Interaction of three solitary waves.** The interaction of three solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the KdV equation with initial conditions given by the linear sum of three

FIGURE 2. Interaction two solitary waves with $c_1 = 0.9$, $c_2 = 0.3$, $D_1 = -6$, $D_2 = -8$, $\mu = 4.84 \times 10^{-4}$, $\varepsilon = 1$, $h = 0.01$, $k = 0.005$, $0 \leq x \leq 2$ at time $t = 0,\ 0.5,\ 0.75,\ 1$ respectively.



well separated solitary waves of various amplitudes

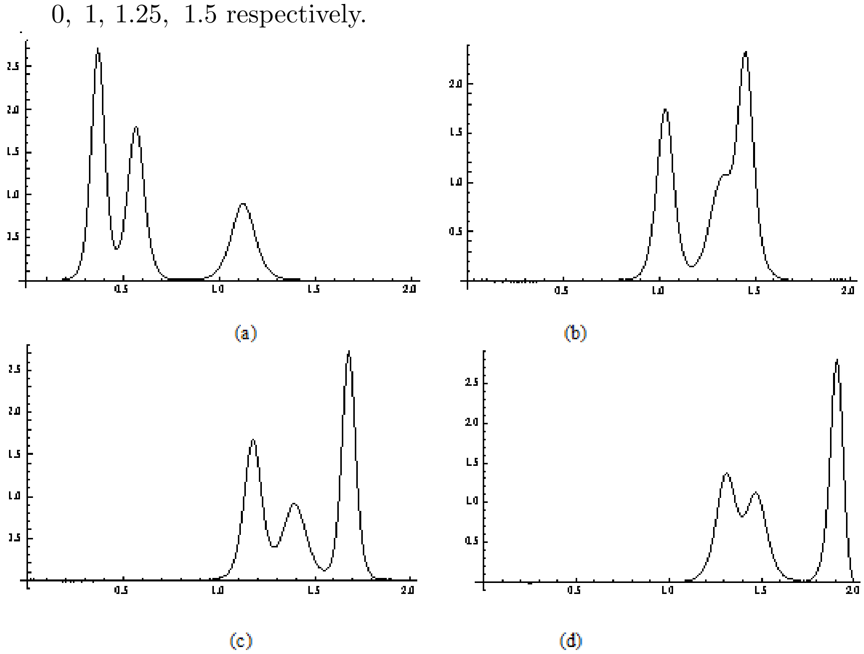
$$u(x, 0) = 3c_j \ \mathrm{sech}^2\,(A\,x + D_j\,), \tag{5.9}$$

where $A = \frac{1}{2}\sqrt{\frac{3c_j}{\mu}}$, $j = 1, 2, 3$, $-c_j$ and D_j are arbitrary constants. In our computational work. Now, we choose $c_1 = 0.9$, $c_2 = 0.6$, $c_3 = 0.3$, $D_1 = -8$, $D_2 = -10$, $D_3 = -14$, $\varepsilon = 1$, $h = 0.01$, $\mu = 4.84 \times 10^{-4}$, $k = 0.005$ with interval [0, 2]. In Figure 4 the interactions of these solitary waves are plotted at different time levels.

6. Conclusion

In this paper, we applied the linear combination between exponential cubic B-spline method and trigonometric cubic B-spline to develop a numerical method for solving KdV equation and shown that the scheme is unconditionally stable. We tested our schemes through a single solitary wave in which the analytic solution is known, then extend it to study the interaction of solitons where no analytic solution is known during the interaction and its accuracy was shown by calculating error norms L_2 and L_∞.

FIGURE 3. Interaction three solitary waves with $c_1 = 0.9$, $c_2 = 0.6$, $c_3 = 0.3$, $D_1 = -8$, $D_2 = -10$, $D_3 = -14$, $\varepsilon = 1$, $h = 0.01$, $\mu = 4.84 \times 10^{-4}$, $k = 0.005$, $0 \leq x \leq 2$ at times $t = 0,\ 1,\ 1.25,\ 1.5$ respectively.



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