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Stability analysis of a fractional order prey-predator system with nonmonotonic functional response

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Abstract In this paper, we introduce fractional order of a planar fractional prey-predator system with a nonmonotonic functional response and anti-predator behaviour such that the adult preys can attack vulnerable predators. We analyze the existence and stability of all possible equilibria. Numerical simulations reveal that anti-predator behaviour not only makes the coexistence of the prey and predator populations less likely, but also damps the predator-prey oscillations. Therefore, anti-predator behaviour helps the prey population to resist predator aggression.

Keywords. Bifurcation, Fractional Prey-predator model, Stability of equilibrium, Dynamical behavior, Limit cycle.

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1. Introduction

Fractional calculus is a generalization of classical differentiation and integration to arbitrary order. In recent years, fractional calculus has been a fruitful field of research in science and engineering. Meanwhile, applications of fractional differential equations (FDEs) to physics, biology and engineering are a recent focus of interests [4, 3, 6, 7, 8, 9, 11, 13]. In this paper we consider a differential equation introduced in [14, 15]. Biologists routinely label the animals as predator or prey, there is sometimes no obvious winner as prey can sometimes inflict harm on their predators, which indicates that cyclic dominance is also important for predator-prey interactions. Indeed, role reversals between predator and prey (i.e. anti-predator behaviour) often occur. Experiments show that anti-predator behaviour of prey populations is realized in terms of morphological changes or through changes in behaviour, or the prey attack their predators [1, 5, 12]. Anti-predator behaviour requires that adult prey are not just invulnerable to their predators, but they can even kill the juveniles of their predators.

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Based on the system given in [14], we consider the impact of anti-predator behaviour of the following model:

$$\begin{cases} \dfrac{dx}{dt} = rx(1-\dfrac{x}{k}) - \dfrac{\beta xy}{a+x^2}, \\ \dfrac{dy}{dt} = \dfrac{\mu\beta xy}{a+x^2} - dy - \eta xy, \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ are the densities of the prey and the predator , respectively. r is the intrinsic growth rate of the prey, k is the carrying capacity of the environment, β is the capture rate of the predator, μ is the conversion rate of prey into predator, d is the natural death rate of the predator population, a is the half-saturation constant, and η is the rate of anti-predator behaviour of prey to the predator population. The model is biologically feasible if all the parameters are positive.
Starting from the integer-order prey-predator model presented by (1.1), we introduce the fractional order derivatives by replacing the usual integer-order derivatives by fractional order Caputo-type derivatives to obtain the following fractional order system:

$$\begin{cases} \dfrac{d^\alpha x}{dt} = rx(1-\dfrac{x}{k}) - \dfrac{\beta xy}{a+x^2}, \\ \dfrac{d^\alpha y}{dt} = \dfrac{\mu\beta xy}{a+x^2} - dy - \eta xy, \end{cases} \tag{1.2}$$

with the initial conditions

$$x(0) > 0, y(0) > 0,$$

where $\alpha \in (0, 1)$.
The fractional order systems are more suitable than integer-order in biological modelling due to the memory effects.
The aim of this paper is to examine the stability properties of the equilibria of the system (1.2) and deriving conditions under which the system may exhibit a Hopf bifurcation. It turns out that dynamical behaviour of system (1.2) changes dramatically, comparing to that of the integer order system, when α decreases and crosses a threshold.

The rest of this paper is organized as follows: In Section 2, we present some basic materials on fractional calculus. A detailed analysis on the stability and Hopf bifurcation of the equilibria is carried out in Section 3. In Section 4, we perform numerical simulations of the system by computing different orbits of the fractional system. In Section 5, we conclude the paper.

2. Preliminaries

Two types of fractional derivatives of Riemann-Liouville and Caputo derivatives, have been often used in fractional differential systems. We briefly recall these two definitions.

Definition 2.1. The Riemann-Liouville integral $J_{t_0}^{\alpha}$ with fractional order $\alpha \in (0,\infty)$ of function $x(t)$ is defined as:

$$J_{t_0}^{\alpha}x(t) := \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}x(\tau)d\tau,$$

where $\Gamma(.)$ is the gamma function. For $\alpha = 0$ we set $J_{t_0}^{0} := Id$, the identity operator.

Definition 2.2. The Riemann-Liouville derivative with fractional order $\alpha \in \mathbb{R}_+ := [0,\infty)$ of function $x(t)$ is defined by:

$${}_{RL}D_{t_0}^{\alpha}x(t) := \frac{d^m}{dt^m}J_{t_0}^{m-\alpha}x(t),$$

where $m = \lceil\alpha\rceil := \min\{k \in \mathbb{Z} : k \geq \alpha\}$, is the ceiling of α. For $\alpha = 0$, we set ${}_{RL}D_{t_0}^{0} = I$ the identity operator..

Definition 2.3. The Caputo derivative with fractional order $\alpha \in \mathbb{R}_+$ of function $x(t)$ is defined by:

$$\frac{d^\alpha}{dt^\alpha}x(t) := J_{t_0}^{m-\alpha}\frac{d^m}{dt^m}x(t),$$

where $m = \lceil\alpha\rceil$.

Proposition 2.4. [2]
Consider the $N-$dimensional fractional differential equation system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t),\ x(0) = x_0 \in \mathbb{R}^N,\ \alpha \in (0,1),$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T \in \mathbb{R}^N$ and $A_{N\times N}$ is an arbitrary constant matrix.
(i) *The solution $x = 0$ is asymptotically stable, if and only if all eigenvalues λ_j $(j = 1, 2, \ldots, N)$ of A satisfy $|\arg(\lambda_j)| > \frac{\alpha\pi}{2}$.*
(ii) *The solution $x = 0$ is stable, if and only if all eigenvalues of A satisfy $|\arg(\lambda_j)| \geq \frac{\alpha\pi}{2}$ and eigenvalues with $|\arg(\lambda_j)| = \frac{\alpha\pi}{2}$ have the same geometric multiplicity and algebraic multiplicity.*

Proposition 2.5. [10]
Consider the following fractional-order system:

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(x(t)),\ x(0) = x_0 \in \mathbb{R}^N,\ \alpha \in (0,1),$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T \in \mathbb{R}^N$ and $f : [f_1, f_2, \ldots, f_N]^T : \mathbb{R}^N \longrightarrow \mathbb{R}^N$.The equilibrium points of the above system are solutions to the equation $f(x(t)) = 0$. An equilibrium is asymptotically stable if all eigenvalues λ_j of the Jacobian matrix $J = \frac{\partial f}{\partial x} = \frac{\partial(f_1, f_2, \ldots, f_N)}{\partial(x_1, x_2, \ldots, x_N)}$ evaluated at the equilibrium satisfy $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$.

Definition 2.6. An equilibrium point is of saddle type if the Jacobian matrix at this point has at least one eigenvalue λ_1 in the stable region i.e.,
$|\arg(\lambda_1)| > \frac{\alpha\pi}{2}$ and has at least one eigenvalue λ_2 in the unstable region i.e., $|\arg(\lambda_2)| < \frac{\alpha\pi}{2}$.

3. Equilibria of the fractional order system and their stability

Equilibria of the system (1.2) are solutions to the system:

$$\begin{cases} \frac{d^\alpha x(t)}{dt^\alpha} = 0, \\ \frac{d^\alpha y(t)}{dt^\alpha} = 0, \end{cases}$$

which are as follows:

1) The origin $E_0 = (0, 0)$.

2) The axial equilibrium point at $E_k = (k, 0)$ which exists irrespective of any parametric restriction.
3) The interior equilibrium satisfy the following equations

$$\begin{cases} rx(1 - \frac{x}{k}) - \frac{\beta xy}{a+x^2} = 0, \\ \frac{\mu\beta x}{a+x^2} - d - \eta x = 0, \end{cases}$$

Thus we must have $x < k$ and

$$\eta x^3 + dx^2 + (\eta a - \mu\beta)x + da = 0 \tag{3.1}$$

Therefore, existence of a positive equilibrium of model (1.2) is equivalent to the Eq. (3.1) admitting a positive root which is less than k. In this case, if we set

$$\Delta = M^3 + (9\eta^2 a - 9\mu\beta\eta - 3d^2)M + 2d^3 + 18\eta^2 ad + 9\mu\beta\eta d,$$
$$M = \sqrt{d^2 - 3\eta(\eta a - \mu\beta)},$$

then we deduce the following theorem:

Proposition 3.1. (i) *If $\eta a - \mu\beta \geq 0$ the system(1.2) does not have any positive equilibrium.*
(ii) *If $\eta a - \mu\beta < 0$ and $\Delta > 0$ then the system(1.2) does not have any positive equilibrium.*
(iii) *If $\eta a - \mu\beta < 0$ and $\Delta < 0$, there are two positive equilibria*

$$E_1 = (x_1, y_1) = \Big(\frac{-d + M(\cos(\frac{\theta}{3}) - \sqrt{3}\sin(\frac{\theta}{3}))}{3\eta}, \frac{r}{\beta}(1 - \frac{x_1}{k})(a + x_1^2)\Big),$$
$$E_2 = (x_2, y_2) = \Big(\frac{-d + M(\cos(\frac{\theta}{3}) + \sqrt{3}\sin(\frac{\theta}{3}))}{3\eta}, \frac{r}{\beta}(1 - \frac{x_2}{k})(a + x_2^2)\Big),$$

whenever $x_2 < k$, in which

$$\theta = \arccos(\frac{2M^2 d - 3\eta(d\eta a - d\mu\beta - 9\eta ad)}{2M^3})$$

(iv) *If* $\eta a - \mu\beta < 0$ *and* $\Delta = 0$ *then the positive equilibria* E_1 *and* E_2 *coincide into one positive equilibrium, which is denoted by*

$$E_* = (x_*, y_*) = (\frac{-d+M}{3\eta}, \frac{r}{\beta}(1 - \frac{x_*}{k})(a + x_*^2)),$$

whenever $x_* < k$.

Proof. We define $f(x) = \eta x^3 + dx^2 + (\eta a - \mu\beta)x + da$. It is easy to see that $f''(x) > 0$ for all $x > 0$, which indicates that $f'(x)$ is strictly monotonically increasing in the interval $(0, +\infty)$. If $\eta a - \mu\beta \geq 0$, then we can see that $f'(x) > 0$ for all $x > 0$, and consequently the function $f(x)$ is strictly monotonically increasing for all $x > 0$. This shows that $f(x) > f(0) = ad > 0$ when $x > 0$. Thus, there is no positive root for the equation $f(x) = 0$, and hence the system (1.2) does not have any positive equilibrium.

If $\eta a - \mu\beta < 0$, it is easy to see that $f'(x) = 0$ has a positive root, denoted by $x_* = \frac{-d+M}{3\eta}$. Thus, $f(x)$ is strictly monotonically increasing in the interval $(0, x_*)$ and is strictly monotonically decreasing in the interval $(x_*, +\infty)$. We notice that if $f(x_*) > 0$ i.e., $\Delta > 0$ then there is no positive root of the equation $f(x) = 0$.

If $f(x_*) < 0$, i.e., $\Delta < 0$, then $f(x) = 0$ has two positive roots denoted by x_1 and x_2.

If $f(x_*) = 0$, i.e. $\Delta = 0$, then $f(x) = 0$ has only one positive root x_*.
Now, by using this fact that $x_1 < x_* < x_2$ the proof becomes clear. □

Remark 3.2. If $\eta a - \mu\beta < 0$, $\Delta < 0$ and $x_1 < k \leq x_2$, then the system (1.2) has only one positive equilibrium E_1.

We now derive stability conditions for feasible equilibrium points of the system (1.2) by using Theorem 2.5. The Jacobian matrix of the system (1.2) at any point (x, y) is given by:

$$J = \begin{pmatrix} r(1 - \frac{x}{k}) - \frac{\beta y}{a+x^2} - \frac{rx}{k} + \frac{2\beta x^2 y}{(a+x^2)^2} & \frac{-\beta x}{a+x^2} \\ \frac{\mu\beta y(a-x^2)}{(a+x^2)^2} - \eta y & \frac{\mu\beta x}{a+x^2} - d - \eta x \end{pmatrix},$$

Proposition 3.3.
(i) E_0 *is always a saddle point.*
(ii) $E_k = (k, 0)$ *is asymptotically stable when* $\frac{\mu\beta k}{a+k^2} - d - \eta k < 0$.

Proof. (i) The Jacobian matrix of system (1.2) evaluated at E_0 is

$$J|_{E_0} = \begin{pmatrix} r & 0 \\ 0 & -d \end{pmatrix},$$

The eigenvalues of $J|_{E_0}$ are $\lambda_1 = r > 0$, $\lambda_2 = -d < 0$. Thus $|\arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$ and $|\arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$. Therefore origin is a saddle point.
(ii) The Jacobian matrix of system (1.2) evaluated at E_k is

$$J|_{E_k} = \begin{pmatrix} -r & \frac{-\beta k}{a+k^2} \\ 0 & \frac{\mu\beta k}{a+k^2} - d - \eta k \end{pmatrix},$$

The eigenvalues of $J|_{E_k}$ are $\lambda_1 = -r < 0$, $\lambda_2 = \frac{\mu\beta k}{a+k^2} - d - \eta k$. Thus $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ and by assumption we deduce $|\arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$. Therefore E_k is asymptotically stable.

□

Remark 3.4. E_k becomes a saddle point when $\frac{\mu\beta k}{a+k^2} - d - \eta k > 0$.

Proposition 3.5. *The equilibrium E_1 is locally asymptotically stable if one of the following mutually exclusive conditions holds:*
(a) *If $tr^2(J|_{E_1}) - 4\det(J|_{E_1}) \geq 0$ one must have $tr(J|_{E_1}) < 0$.*
(b) *If $tr^2(J|_{E_1}) - 4\det(J|_{E_1}) < 0$ one must have*

$$\alpha < \Big(\frac{2}{\pi}\arctan\sqrt{\frac{4\det(J|_{E_1})}{tr^2(J|_{E_1})} - 1}\Big)_{tr(J|_{E_1})\geq 0},$$

or

$$\alpha < \Big(2 - \frac{2}{\pi}\arctan\sqrt{\frac{4\det(J|_{E_1})}{tr^2(J|_{E_1})} - 1}\Big)_{tr(J|_{E_1})<0}.$$

Proof. The Jacobian of (1.2) evaluated at E_1 is given by

$$J|_{E_1} = \begin{pmatrix} -\frac{rx_1}{k} + \frac{2\beta x_1^2 y_1}{(a+x_1^2)^2} & \frac{-\beta x_1}{a+x_1^2} \\ \frac{\mu\beta y_1(a-x_1^2)}{(a+x_1^2)^2} - \eta y_1 & 0 \end{pmatrix}.$$

We conclude that

$$\det(J|_{E_1}) = \frac{\beta x_1 y_1}{a+x_1^2}\Big(\frac{\mu\beta(a-x_1^2)}{(a+x_1^2)^2} - \eta\Big),$$

also we have $a = \dfrac{\eta x_1^3 + dx_1^2 - \mu\beta x_1}{-\eta x_1 - d}$, thus

$$\det(J|_{E_1}) = \frac{y_1}{\mu(a+x_1^2)}\Big(\mu\beta d - 2\eta^2 x_1^3 - 4\eta d x_1^2 - 2d^2 x_1\Big).$$

Let $g(x) = \mu\beta d - 2\eta^2 x^3 - 4\eta d x^2 - 2d^2 x$. It is easy to check that the function $g(x)$ is strictly monotonically decreasing when $x > 0$. It follows from $f'(x_1) < 0$ that

$$g(x_1) = \mu\beta d - 2\eta^2 x_1^3 - 4\eta d x_1^2 - 2d^2 x_1 > \eta(-2\eta x_1^3 - dx_1^2 + da) > \eta(-2\eta x_*^3 - dx_*^2 + da) = 0.$$

Then we have $\det(J|_{E_1}) > 0$, So the stability of the equilibrium E_1 depends on the sign of $tr(J|_{E_1}) = -\frac{rx_1}{k} + \frac{2\beta x_1^2 y_1}{(a+x_1^2)^2}$. If $tr^2(J|_{E_1}) - 4\det(J|_{E_1}) \geq 0$ and $tr(J|_{E_1}) < 0$, since $\det(J|_{E_1}) > 0$, we deduce

$tr(J|_{E_1}) \pm \sqrt{tr^2(J|_{E_1}) - 4\det(J|_{E_1})} < 0$. This proves the part (a).
Now we suppose that $tr^2(J|_{E_1}) - 4\det(J|_{E_1}) < 0$, then the eigenvalues of $J|_{E_1}$ are

$$\lambda_{1,2} = \frac{1}{2}\Big[tr(J|_{E_1}) \pm i\sqrt{4\det(J|_{E_1}) - tr^2(J|_{E_1})}\Big],$$

if we set $\alpha_0 = \frac{2}{\pi}|\arg(\lambda_{1,2})|$, we deduce

$$|\arg(\lambda_{1,2})| = \begin{cases} \arctan(\frac{\sqrt{4\det(J|_{E_1}) - tr^2(J|_{E_1})}}{tr(J|_{E_1})}), \ tr(J|_{E_1}) \geq 0 \\ \pi - \arctan(\frac{\sqrt{4\det(J|_{E_1}) - tr^2(J|_{E_1})}}{|tr(J|_{E_1})|}), \ tr(J|_{E_1}) < 0. \end{cases}$$

Therefore,

$$\alpha_0 = \begin{cases} \frac{2}{\pi}\arctan\frac{\sqrt{4\det(J|_{E_1}) - tr^2(J|_{E_1})}}{tr(J|_{E_1})}, \ tr(J|_{E_1}) \geq 0 \\ \frac{2}{\pi}(\pi - \arctan\frac{\sqrt{4\det(J|_{E_1}) - tr^2(J|_{E_1})}}{|tr(J|_{E_1})|}), \ tr(J|_{E_1}) < 0. \end{cases}$$

By using condition of Theorem 2.5, we can see if $\alpha < \alpha_0$ then E_1 becomes asymptotically stable and E_1 becomes unstable whenever $\alpha > \alpha_0$. □

Remark 3.6. If

$$\alpha = \Big(\frac{2}{\pi}\arctan\sqrt{\frac{4\det(J|_{E_1})}{tr^2(J|_{E_1})} - 1}\Big)_{tr(J|_{E_1})\geq 0},$$

or

$$\alpha = \Big(2 - \frac{2}{\pi}\arctan\sqrt{\frac{4\det(J|_{E_1})}{tr^2(J|_{E_1})} - 1}\Big)_{tr(J|_{E_1})<0},$$

then E_1 undergoes a Hopf bifurcation .

Remark 3.7. When $0 < \alpha < 1$, one can see that if $tr^2(J|_{E_1}) - 4\det(J|_{E_1}) < 0$ and $tr(J|_{E_1}) < 0$. So, we have a pair of complex conjugate root $\lambda_{1,2}$ with negative real parts. In this case, we also can deduce that E_1 becomes asymptotically stable.

Proposition 3.8. *A feasible E_2 is of saddle type.*

Proof. Since $f'(x_*) = 0$ and $f(x_*) < 0$ then

$$\begin{aligned} g(x_*) &= \mu\beta d - 2\eta^2 x_*^3 - 4\eta d x_*^2 - 2d^2 x_*, \\ &= \eta(-2\eta x_*^3 - d x_*^2 + da), \\ &< \eta\Big(-2\eta x_*^3 - d x_*^2 - \eta x_*^3 - d x_*^2 - (\eta a - \mu\beta)x_*\Big), \\ &= \eta x_*\Big(-3\eta x_*^2 - 2d x_* - (\eta a - \mu\beta)\Big) = 0. \end{aligned}$$

This implies $g(x_2) < g(x_*) < 0$ for $x_* < x_2$, and consequently $\det(J|_{E_2}) < 0$, hence E_2 is a saddle point. □

4. **Numerical simulation**

We perform numerical simulations based on the fractional Adams-Bashforth-Moulton method [2]. It was shown that for a differential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = f(t, x(t)),$$

the fractional variant of the one-step Adams-Moulton method is given by (corrector formula)

$$x_{n+1} = \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^i}{i!} x_0^{(i)} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} f(t_i, x_i) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, x_{n+1}^p),$$

in which $t_i = ih$ with some fixed h, and

$$a_{i,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, \ i = 0 \\ (n-i+2)^{\alpha+1} + (n-i)^{\alpha+1} - 2(n-i+1)^{\alpha+1}, \ 1 \le i \le n. \end{cases}$$

We first need to compute the values x_{n+1}^p, given by the generalize one-step Adams-Bashforth method as a predictor formula

$$x_{n+1}^p = \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^i}{i!} x_0^{(i)} + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} f(t_i, x_i),$$

where $b_{i,n+1} = (n+1-i)^\alpha - (n-i)^\alpha$. This method is said to be of Predict, Evaluate, Correct, Evaluate ($PECE$) type, because in a concrete implementation, we would start to calculate the predictor x_{n+1}^p, then we evaluate $f(t_{n+1}, x_{n+1}^p)$. Next, we use these quantities to calculate the corrector in x_{n+1} , and finally evaluate $f(t_{n+1}, x_{n+1})$. This result is stored for future use in the next integration step.

To perform an error analysis of the presented method, we first assume that $t_i = ih = \frac{i\mathcal{T}}{N}$ with some $N \in \mathbb{N}$, and we have the following theorem:

Proposition 4.1. [2] *Let $\alpha > 0$ and $\frac{d^\alpha x(t)}{dt^\alpha} \in C^2[0, \mathcal{T}]$ for some suitable $\mathcal{T}$, then,*

$$\max_{0\le i\le N} |x(t_i) - x_i| = \begin{cases} O(h^2), \ \alpha \ge 1, \\ O(h^{1+\alpha}), \ \alpha < 1, \end{cases}$$

Applying the corrector formula improves the accuracy of its input (the predictor) by a factor of h^α up to order of $O(h^2)$ for which a saturation is reached. Thus by replacing the plain $PECE$ structure by a $P(EC)^\mu E$ method (additional corrector iterations) a corrector iteration is of the form (corrector formula)

$$x_{n+1}^{[l]} = \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^i}{i!} x_0^{(i)} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} f(t_i, x_i) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, x_{n+1}^{[l-1]}),$$

in which $x_{n+1}^{[l]}$ denotes the approximation after l corrector steps, $x_{n+1}^{[0]} = x_{n+1}^p$ is the predictor, and $x_{n+1} := x_{n+1}^{[\mu]}$ is the final approximation after μ corrector steps. The following theorem provides an error analysis of this method.

Proposition 4.2. [2] *Assume* $\frac{d^\alpha x(t)}{dt^\alpha} \in C^2[0, \mathcal{T}]$ *for some suitable* $\mathcal{T}$*, and* $\alpha > 0$*. Then, the approximation obtained by the* $P(EC)^\mu E$ *method described above satisfies*

$$\max_{0 \le i \le N} |x(t_i) - x_i| = O(h^q),$$

where $q = \min\{2, 1 + \mu\alpha\}$*.*

Now by employing this method for $0 < \alpha < 1$ to the following system

$$\frac{d^\alpha x(t)}{dt^\alpha} = F(t, x(t), y(t)),$$
$$\frac{d^\alpha y(t)}{dt^\alpha} = G(t, x(t), y(t)),$$

we deduce that

$$x_{n+1}^{[l]} = x_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} F(t_i, x_i, y_i, z_i) + \frac{h^\alpha}{\Gamma(\alpha+2)} F(t_{n+1}, x_{n+1}^{[l-1]}, y_{n+1}^{[l-1]}, z_{n+1}^{[l-1]}),$$
$$y_{n+1}^{[l]} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} G(t_i, x_i, y_i, z_i) + \frac{h^\alpha}{\Gamma(\alpha+2)} G(t_{n+1}, x_{n+1}^{[l-1]}, y_{n+1}^{[l-1]}, z_{n+1}^{[l-1]}),$$
$$x_{n+1}^{[0]} = x_{n+1}^{p} = x_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} F(t_i, x_i, y_i, z_i),$$
$$y_{n+1}^{[0]} = y_{n+1}^{p} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} G(t_i, x_i, y_i, z_i),$$

We now consider k (carrying capacity of the environment) as a bifurcation parameter and consider the fix parameter values $r = 0.05, a = 0.8, \mu = 0.8, \alpha = 0.98$.
Our numerical simulations are plotted in Figure 1 for $d = 0.24, \eta = 0.01, \beta = 0.6, k = 1.6$ and Figure 2 for $d = 0.015, \eta = 0.01, \beta = 0.4, k = 5$ show that α has an essential role on the stability behaviour of this system.

The analytical results can be exploited to examine the obtained numerical results. We now consider the set of fixed parameters specified for Figure 1. For $\alpha = 0.98$ we have $arg(\lambda|_{E_1}) = 1.5649$ and $\frac{\alpha\pi}{2} = 1.5394$. Hence, $|arg(\lambda|_{E_1})| > \frac{\alpha\pi}{2}$, which indicates the condition of asymptotic stability of E_1, based on the Theorem 3.5, holds. Further, for $\alpha = 1$ we have $arg(\lambda|_{E_1}) = 1.5649$ and $\frac{\alpha\pi}{2} = 1.5708$, then $|arg(\lambda|_{E_1})| < \frac{\alpha\pi}{2}$. This reveals that the condition of asymptotic stability of E_1 is violated and E_1 becomes unstable. Hence, a stable limit cycle emerges around E_1. These scenarios are depicted in Figure 1 (right) and Figure 1 (left), respectively.
We next consider the set of fixed parameters specified for Figure 2. For $\alpha = 0.98$ we have $arg(\lambda|_{E_1}) = 1.5695$ and $\frac{\alpha\pi}{2} = 1.5394$, then $|arg(\lambda|_{E_1})| > \frac{\alpha\pi}{2}$ Hence, the condition of asymptotic stability of E_1 holds. Also for $\alpha = 1$ we have $arg(\lambda|_{E_1}) = 1.5695$ and $\frac{\alpha\pi}{2} = 1.5708$, then $|arg(\lambda|_{E_1})| < \frac{\alpha\pi}{2}$. Hence, the condition of asymptotic stability of E_1 is violated and E_1 becomes unstable. Hence, a stable limit cycle emerges around

E_1. These scenarios are depicted in Figure 2 (right) and Figure 2 (left), respectively.

5. Conclusion

We consider a planar fractional order prey-predator system with a nonmonotonic functional response and anti-predator behaviour such that the adult prey can attack vulnerable predators. We derive conditions for existence and asymptotic stability of different equilibrium points. We also derive conditions under which a Hopf bifurcation occurs and confirm these conditions by numerical evidences. Numerical simulations reveal that α has an essential role on the stability and dynamics of this system.

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FIGURE 1. Phase space of system (1.2). (left) for $\alpha = 1$, (right) for $\alpha = 0.98$.

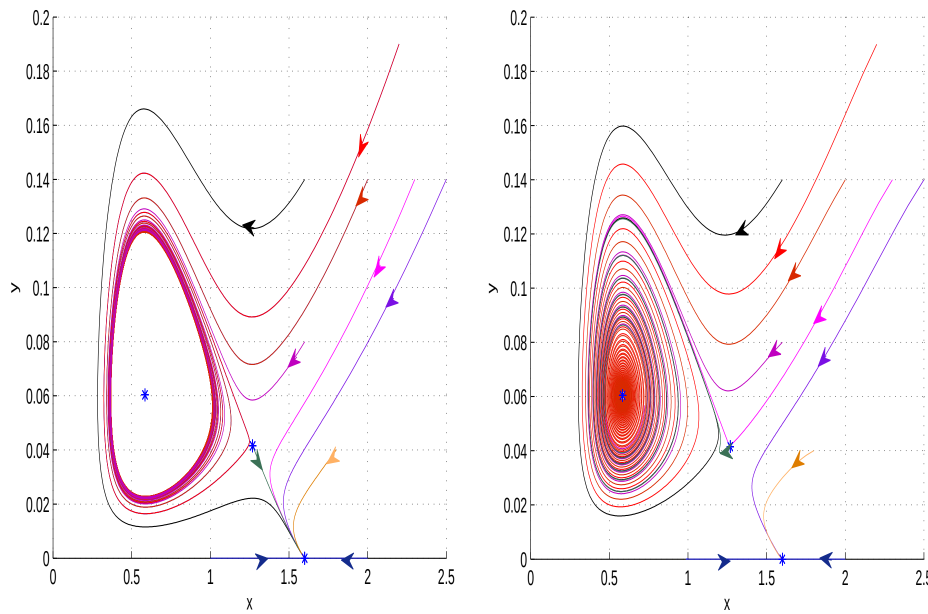

FIGURE 2. Phase space of system (1.2). (left) for $\alpha = 1$, (right) for $\alpha = 0.98$.