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Valuation of installment option by penalty method

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Abstract In this paper, installment options on the underlying asset which evolves according to Black-Scholes model and pays constant dividend to its owner will be considered. Applying arbitrage pricing theory, the non-homogeneous parabolic partial differential equation governing the value of installment option is derived. Then, penalty method is used to value the European continuous installment call option.

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1. Introduction

Installment option is a type of security in which instead of paying lump sum up front, the premium is paid over the life of the option. Thus the holder has the right to terminate the contract at any time prior to maturity. Based on the net present value of contract the holder makes decision to continue or stop the payments. Hence the owner will continue to pay future installments if the option worth the net present value of the remained payments. Otherwise the holder allows the contract to laps. After the last installment, the contract will become a vanilla option.

Installment options are traded in foreign currency market between banks and corporations. Several contracts can be considered as installment options such as: some life insurance contracts, capital investment projects, installment warrant, some contracts in pharmacy and employee stock options [4, 10, 12, 17, 18, 20], respectively.

A few paper exists in the case of continuous installment options. European and American continuous-installment options are investigated using Laplace-Carson transform by Kimura in [15] and [16], respectively. Ciurlia and Roko applied the multi piece exponential function (MEF) to solve the free boundary problem arised from the American continuous installment option [9]. Alobidi et al. used the integral transform to price European continuous installment options [2]. Alobaidi and Mallier also analyzed the behavior of the price of European installment options near expiry [1].

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In [8] Ciurlia have priced European continuous installment options using Monte Carlo approach. Perpetual American continuous installment option studied by Ciurlia1 and Caperdoni [7].

In the context of option pricing problems, the penalty method was applied by Zvan et al. [21]. Nielsen et al. used penalty method to solve American put option [19]. In this paper, for the fist time, we will apply penalty method to value installment options. In Section 2, we presents the modeling of European continuous installment option under Black-Scholes model. We describe penalty method using an ordinary differential equation in Section 3. In Section 4 the penalty method is used to solve the nonlinear parabolic partial differential equation governing the European continuous installment call option. In Section 5, numerical results for the price of European continuous installment call option are given.

2. The Model

Setting up a portfolio Π_t consisting of a European continuous installment option and Δ units of underlying asset, we get

$$\Pi_t = V(S_t, t; q) - \Delta S_t, \tag{2.1}$$

where $V(S_t, t; q)$ is the value of the European continuous installment option and q is the rate of installment that must be paid per unit time continuously and S_t is the value of underlying asset evolving according to the following stochastic differential equation, called Black-Scholes model [5],

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t, \tag{2.2}$$

in which r is the interest rate, δ is the dividend yield, σ is a positive constant called volatility and W_t is a one dimensional Wienner process.

The dynamic of the portfolio Π_t is given by

$$d\Pi_t = dV(S_t, t; q) - \Delta dS_t - \Delta(S_t \delta dt), \tag{2.3}$$

Applying Ito's lemma to $V(S_t, t; q$ yields

$$dV = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r-\delta)S_t \frac{\partial V}{\partial S_t} - q)dt + \sigma S_t \frac{\partial V}{\partial S_t} dW_t, \tag{2.4}$$

Substituting from (2.2) and (2.4) into (2.3), one can get

$$\begin{aligned} d\Pi_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r-\delta)S_t \frac{\partial V}{\partial S_t} + (r-\delta)S_t(\frac{\partial V}{\partial S_t} - \Delta) \\ - \delta \Delta S_t - q)dt + \sigma S_t(\frac{\partial V}{\partial S_t} - \Delta)dW_t. \end{aligned} \tag{2.5}$$

To avoid arbitrage opportunities the portfolio must satisfy $d\Pi_t = r\Pi_t dt$. On the other hand the portfolio must be riskless $\frac{\partial V}{\partial S_t} = \Delta$. Substituting from these relations into (2.5), we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r-\delta)S_t \frac{\partial V}{\partial S_t} - rV = q. \tag{2.6}$$

The only difference between the above partial differential equation and the one arising from the modeling of European vanilla option is the nonhomogeneous term q which is the rate of installment.

3. Penalty method

To describe the penalty method, an initial value problem, which satisfies an additional condition, is considered. Let us find the solution of the following problem with penalty method

$$y' = -y, \tag{3.1}$$
$$y(0) = 2, \tag{3.2}$$
$$y(t) \geq 1. \tag{3.3}$$

The only difference of this problem with a common initial value is that the solution of this problem must satisfy the inequality condition (3.3). Clearly the analytic solution of this problem is given by

$$y(t) = \begin{cases} 2e^{-t}, & t \leq \ln 2, \\ 1, & t > \ln 2. \end{cases} \tag{3.4}$$

Using the Brennan-Schwarz method [6], one can compute the solution of the problem (3.1)-(3.3) numerically as follows

$$\begin{aligned} &y_{n+1} = \max((1-\Delta t)y_n, 1), \;\; n \geq 0, \\ &y_0 = 2. \end{aligned} \tag{3.5}$$

By adding an extra term to (3.1), called penalty term, we apply a method to solve the problem (3.1)-(3.3) which automatically satisfy (3.3). This method is called penalty method. Now consider the new initial value problem

$$Y' = -Y + \frac{\epsilon}{Y+\epsilon-1}, \tag{3.6}$$
$$Y(0) = 2, \tag{3.7}$$

where $\frac{\epsilon}{Y+\epsilon-1}$ is called penalty term and $\epsilon > 0$ is a small parameter. The claim is that the following conditions are satisfied by $Y(t)$, for all $t \geq 0$,

$$1 \leq Y(t) \leq 2, \tag{3.8}$$
$$Y'(t) \leq 0, \tag{3.9}$$
$$Y''(t) \geq 0. \tag{3.10}$$

For $1 \leq Y(t) \leq 2$, one can get

$$Y(Y+\epsilon-1) \geq \epsilon, \tag{3.11}$$

which using (3.6) yields

$$Y' = \frac{\epsilon - Y(Y+\epsilon-1)}{Y+\epsilon-1} \leq 0. \tag{3.12}$$

This proves that $Y'(t) \leq 0$, for $t \geq 0$. On the other hand, by differentiating (3.6) with respect to t, we have

$$Y'' = -Y'(1 + \frac{\epsilon}{(Y + \epsilon - 1)^2}). \tag{3.13}$$

Therefore, using the fact that $Y'(t) \leq 0$, we have $Y''(t) \geq 0$, for all $t \geq 0$. Applying explicit finite difference method to the penalty problem (3.6)-(3.7), one can get

$$Y_{n+1} = (1 - \Delta t)Y_n + \frac{\Delta t \epsilon}{Y_n + \epsilon - 1}, \ n \geq 0, \tag{3.14}$$

$$Y_0 = 2. \tag{3.15}$$

For Δt sufficiently small, the following relations hold,

$$1 \leq Y_n \leq 2, \ n \geq 0, \tag{3.16}$$

$$Y_{n+1} \leq Y_n, \tag{3.17}$$

$$Y_n \to 1, \ as \ n \to \infty. \tag{3.18}$$

First, we will prove the relations (3.16) and (3.17). The relation (3.14) can be written as

$$Y_{n+1} = f(Y_n), \ n \geq 0, \tag{3.19}$$

where

$$f(Y) = (1 - \Delta t)Y + \frac{\Delta t \epsilon}{Y + \epsilon - 1}. \tag{3.20}$$

To prove the relation (3.16), the mathematical induction will be used. Clearly, Y_0 satisfies (3.16). Let us assume that $1 \leq Y_n \leq 2$. This, in turn, yields

$$Y_n(Y_n + \epsilon - 1) \geq \epsilon, \tag{3.21}$$

Using (3.14), this gives

$$\begin{aligned} Y_{n+1} =& Y_n(1 - \Delta t + \frac{\Delta t \epsilon}{Y_n(Y_n + \epsilon - 1)}) \\ &\leq Y_n(1 - \Delta t + \frac{\Delta t \epsilon}{\epsilon}) = Y_n. \end{aligned} \tag{3.22}$$

On the other hand

$$f'(Y) = 1 - \Delta t - \frac{\Delta t \epsilon}{(Y + \epsilon - 1)^2} \geq 1 - \Delta t - \frac{\Delta t \epsilon}{\epsilon^2}. \tag{3.23}$$

This, assuming $\Delta t \leq \frac{\epsilon}{1+\epsilon}$, yields $f'(Y) \geq 0$. Therefore, f is a nondecreasing function. Bearing this in mind and considering $Y_n \geq 1$, one can get

$$Y_{n+1} = f(Y_n) \geq f(1) = 1. \tag{3.24}$$

Thus, we have proved, by induction, that

$$1 \leq Y_{n+1} \leq Y_n \leq 2, \ n \geq 0. \tag{3.25}$$

In the next step, we will show that Y_n satisfies (3.18). Note that

$$\begin{aligned} Y_{n+1} - 1 =& Y_n - 1 - \Delta t Y_n + \frac{\Delta t \epsilon}{Y_n + \epsilon - 1} \\ =& (Y_n - 1)(1 - \Delta t) + \Delta t (\frac{\epsilon}{Y_n + \epsilon - 1} - 1). \end{aligned} \tag{3.26}$$

Using the fact that $Y_n + \epsilon - 1 \geq \epsilon$, we have

$$Y_{n+1} - 1 \leq (Y_n - 1)(1 - \Delta t). \tag{3.27}$$

This, in turn, yields

$$0 \leq Y_n - 1 \leq (1 - \Delta t)^n (Y_0 - 1) = (1 - \Delta t)^n. \tag{3.28}$$

Taking the limit as $n \to \infty$, we have $\lim_{n\to\infty} Y_n = 1$.

4. Applying to installment option problem

In this section, we will introduce European continuous installment call option problem and its penalty formulation. Then, a numerical method for solving the penalty problem will be presented.

4.1. **Penalty formulation.** Let $c(S_t, t; q)$ be the value of the European installment call option with the maturity T, the exercise price K and the payoff function $\max(S_T - K, 0)$. The valuation of European call can be done through the solution of the following inhomogeneous partial differential equation (PDE) [15]

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial S^2} + (r - \delta) S \frac{\partial c}{\partial S} - rc = q, \tag{4.1}$$

subject to the terminal condition

$$c(S_T, T; q) = \max(S_T - K, 0), \tag{4.2}$$

and along with boundary conditions

$$\lim_{S_t \to S_f(t)} c(S_t, t; q) = 0, \lim_{S_t \to S_f(t)} \frac{\partial c}{\partial S} = 0, \lim_{S_t \to \infty} \frac{\partial c}{\partial S} < \infty. \tag{4.3}$$

In this problem there are two unknowns $c(S_t, t; q)$ and $S_f(t)$. This problem is called free boundary problem. Solving this type of problem is a challenging work. In this paper, for the first time, we will apply penalty method to solve the mentioned free boundary problem. As seen in formulation (4.1)-(4.3), the free or stopping boundary belongs to the domain of the definition of the problem. Therefore difficulty arises in solving such a problem. The basic idea of penalty method is to remove the free boundary from the domain by adding a penalty term to the PDE (4.1). The resulted problem is a nonlinear PDE with known boundary conditions.

Now, we want to reformulate the above installment call option as a penalty problem. By adding the penalty term $\frac{\epsilon C}{u_\epsilon + \epsilon}$ to the PDE (4.1), the resulting problem, which is a nonlinear partial differential equation, can be written as

$$\frac{\partial u_\epsilon}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u_\epsilon}{\partial x^2} + (r-\delta)x\frac{\partial u_\epsilon}{\partial x} - ru_\epsilon + \frac{\epsilon C}{u_\epsilon + \epsilon} = q, \tag{4.4}$$

$$u_\epsilon(x,T) = \max(x-K,0), \tag{4.5}$$

$$u_\epsilon(0,t) = 0, \tag{4.6}$$

$$\lim_{x\to\infty} \frac{\partial u_\epsilon}{\partial x} < \infty, \tag{4.7}$$

where C is a positive constant, $0 < \epsilon \ll 1$ is a regularization parameter and $u_\epsilon(x,t)$ used in the place of $c(S,t)$ for $x = S$. This problem is called penalty formulation of the European installment call option. Note that for $u_\epsilon \gg 0$ the penalty term $\frac{\epsilon C}{u_\epsilon+\epsilon}$ is of order ϵ and it tends to C as u_ϵ approaches 0. In the next step, a numerical method for solving the penalty problem will be given and it will be proved that the approximate option values satisfy the discrete version of $u_\epsilon(x,t) \geq 0$.

4.2. **Upwind finite difference scheme.** In continuation, the finite difference method will be applied to solve the above problem. In this method, the derivatives in the PDE (4.4) are approximated by difference schemes. To apply finite difference method to the above problem, we need to bound the domain of the mentioned problem. Let x_∞ be sufficiently large number. This value plays the role of ∞ in the problem (4.4)-(4.7). For the boundary condition (4.7), the Neumann boundary conditions at x_∞ is defined as

$$\frac{\partial u}{\partial x}(x_\infty,t) = \frac{\partial}{\partial x}\max(x-K,0)|_{x=x_\infty} = 1. \tag{4.8}$$

Let $M, N > 0$ be integer numbers and define

$$h = \frac{x_\infty}{M+1},\ k = \frac{T}{N+1}, \tag{4.9}$$

$$x_i = ih,\ i = 0,1,\cdots,M+1, \tag{4.10}$$

$$t_j = jk,\ j = 0,1,\cdots,N+1. \tag{4.11}$$

Also assume that

$$u_i^j \approx u_\epsilon(x_i,t_j),\ 0 \leq i \leq M+1,\ 0 \leq j \leq N+1. \tag{4.12}$$

At this moment, the partial derivatives in (4.4) will be approximated by difference schemes. We will use upwind differencing for transport term, central differencing for diffusion term and backward Euler time for time derivative. Therefore, we apply the schemes

$$\frac{\partial u}{\partial t}(x_i,t_j) \approx \frac{u_i^{j+1} - u_i^j}{k}, \tag{4.13}$$

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) \approx \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2}, \tag{4.14}$$

$$\frac{\partial u}{\partial x}(x_i,t_j) \approx \frac{u_{i+1}^{j+1} - u_i^{j+1}}{h}, \tag{4.15}$$

for $0 \leq i \leq M+1,\ 0 \leq j \leq N$. Substituting these relations in (4.4) yields

$$\begin{aligned}\frac{u_i^{j+1}-u_i^j}{k}+&\frac{1}{2}\sigma^2 x_i^2\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{h^2}\\ &+(r-\delta)x_i\frac{u_{i+1}^{j+1}-u_i^{j+1}}{h}-ru_i^{j+1}+\frac{\epsilon C}{u_i^{j+1}+\epsilon}=q.\end{aligned} \tag{4.16}$$

This scheme is called upwind explicit finite difference method for equation (4.4). This can be written as

$$u_i^j=f(u_{i-1}^{j+1},u_i^{j+1},u_{i+1}^{j+1}),\ j=N,N-1,\cdots,0, \tag{4.17}$$

for $0 \leq i \leq M+1$, where

$$\begin{aligned}f(u_-,u,u_+)=&\frac{1}{2}\sigma^2x_i^2\frac{k}{h^2}u_-+(1-\sigma^2x_i^2\frac{k}{h^2}-\mu x_i\frac{k}{h}-kr)u\\ &+(\frac{1}{2}\sigma^2x_i^2\frac{k}{h^2}+\mu x_i\frac{k}{h})u_++\frac{\epsilon kC}{u+\epsilon}-kq,\end{aligned} \tag{4.18}$$

where $\mu=r-\delta$. The discretization of the terminal condition (4.5) is given by

$$u_i^{N+1}=\max(x_i-K,0),\ i=0,1,\cdots,M+1. \tag{4.19}$$

Moreover discretization of boundary conditions (4.6) and (4.8), for $0 \leq j \leq N+1$, yields

$$u_0^j=0,\ \frac{u_{M+2}^j-u_M^j}{2h}=1. \tag{4.20}$$

Therefore we have

$$u_0^j=0,\ u_{M+2}^j=2h+u_M^j,\ j=0,1,\cdots,N+1. \tag{4.21}$$

Note that, f depends on the values computed from time step t_{j+1}. Solving the problem will be done backward, then when we are at time step t_j the values at time step t_{j+1} have been computed. Plugging $i=M+1$ in (4.17), for $j=N,N-1,\cdots,0$, we have

$$u_{M+1}^j=f(u_M^{j+1},u_{M+1}^{j+1},u_{M+2}^{j+1}). \tag{4.22}$$

Now, substituting u_{M+2}^{j+1} from (4.21), for $j=N,N-1,\cdots,0$, yields

$$\begin{aligned}u_{M+1}^j=f(u_M^{j+1},u_{M+1}^{j+1})=&(\sigma^2x_{M+1}^2\frac{k}{h^2}+\mu x_{M+1}\frac{k}{h})u_M^{j+1}\\ &+(1-\sigma^2x_{M+1}^2\frac{k}{h^2}-\mu x_{M+1}\frac{k}{h}-kr)u_{M+1}^{j+1}\\ &+\sigma^2x_{M+1}^2\frac{k}{h}+2\mu x_{M+1}k+\frac{\epsilon kC}{u_{M+1}^{j+1}+\epsilon}-kq.\end{aligned} \tag{4.23}$$

In the next step, we will prove that f is a nondecreasing function in the variables u_-,u,u_+ and u_i^j satisfies $u_i^j \geq 0$ for all i,j.

Lemma 4.1. *Let $u \geq 0$ and k satisfies*

$$k \leq \frac{h^2}{\sigma^2 x_i^2 + \mu h x_i + (r + \frac{C}{\epsilon})h^2}. \tag{4.24}$$

Then, all of the partial derivatives of f,that is $\dfrac{\partial f}{\partial u_-}, \dfrac{\partial f}{\partial u}, \dfrac{\partial f}{\partial u_+}$, are nonnegative. In the case of $\mu < 0$, the following relations must also be satisfied

$$|\mu| < \sqrt{r + \frac{C}{\epsilon}} 2\sigma, \tag{4.25}$$

$$h \leq \frac{\sigma^2 x_i}{2\nu}, \tag{4.26}$$

where $\nu = -\mu$.

Proof. Taking the partial derivatives of f yields

$$\frac{\partial f}{\partial u_-} = \frac{1}{2}\sigma^2 x_i^2 \frac{k}{h^2}, \tag{4.27}$$

$$\frac{\partial f}{\partial u} = 1 - \sigma^2 x_i^2 \frac{k}{h^2} - \mu x_i \frac{k}{h} - kr - \frac{\epsilon k C}{(u+\epsilon)^2}, \tag{4.28}$$

$$\frac{\partial f}{\partial u_+} = \frac{1}{2}\sigma^2 x_i^2 \frac{k}{h^2} + \mu x_i \frac{k}{h}. \tag{4.29}$$

Clearly, $\frac{\partial f}{\partial u_-} \geq 0$. First we consider the case $\mu \geq 0$. Then, we have $\frac{\partial f}{\partial u_+} \geq 0$. Moreover, for $u \geq 0$, one can get

$$\frac{\partial f}{\partial u} \geq 1 - \sigma^2 x_i^2 \frac{k}{h^2} - \mu x_i \frac{k}{h} - kr - \frac{\epsilon k C}{\epsilon^2}. \tag{4.30}$$

For k satisfies $k \leq \dfrac{h^2}{\sigma^2 x_i^2 + \mu h x_i + (r + \frac{C}{\epsilon})h^2}$, we obtain

$$\frac{\partial f}{\partial u} \geq 1 - k\left(\frac{\sigma^2 x_i^2}{h^2} + \frac{\mu x_i}{h} + r + \frac{C}{\epsilon}\right) \geq 0. \tag{4.31}$$

In the next step, we consider the case $\mu < 0$. Setting $\nu = -\mu > 0$ yields

$$\frac{\partial f}{\partial u_+} = \frac{1}{2}\sigma^2 x_i^2 \frac{k}{h^2} - \nu x_i \frac{k}{h}. \tag{4.32}$$

If h satisfies $h \leq \frac{\sigma^2 x_i}{2\nu}$, we have $\frac{\partial f}{\partial u_+} \geq 0$. Once again for $u \geq 0$, one can obtain

$$\frac{\partial f}{\partial u} \geq 1 - \sigma^2 x_i^2 \frac{k}{h^2} - \mu x_i \frac{k}{h} - kr - \frac{\epsilon k C}{\epsilon^2}. \tag{4.33}$$

For $\frac{\partial f}{\partial u}$ to be non-negative, it is sufficient that the following conditions hold

$$k \leq \frac{h^2}{\sigma^2 x_i^2 + \mu h x_i + (r + \frac{C}{\epsilon})h^2}, \tag{4.34}$$

$$\sigma^2 x_i^2 + \mu h x_i + (r + \frac{C}{\epsilon})h^2 \geq 0. \tag{4.35}$$

Since $r + \frac{C}{\epsilon} > 0$, the sufficient condition for (4.35) to be non-negative is

$$\Delta = \mu^2 x_i^2 - 4(r + \frac{C}{\epsilon})\sigma^2 x_i^2 < 0. \tag{4.36}$$

This is equivalent to

$$|\mu| < \sqrt{r + \frac{C}{\epsilon}} 2\sigma, \tag{4.37}$$

and the proof is complete. □

Theorem 4.2. *Assume that k, h satisfy the assumptions of the Lemma 4.1. In addition, let $C \geq q$. Therefore, approximate installment option values u_i^j's computed by* (4.17) *satisfy*

$$u_i^j \geq 0, \tag{4.38}$$

for $i = 0, 1, \cdots, M+1$ and $j = N+1, N, \cdots, 0$.

Proof. Note that for $j = N+1$, by (4.19), we have

$$u_i^{N+1} = \max(x_i - K, 0) \geq 0, \ i = 0, 1, \cdots, M+1. \tag{4.39}$$

Thus (4.38) holds for $j = N+1$. Moreover, $u_0^j = 0$. This proves that u_i^j satisfies (4.38) for $i = 0$. Also for $i = M+1$, by (4.23), we have

$$u_{M+1}^j = f(u_M^{j+1}, u_{M+1}^{j+1}). \tag{4.40}$$

We claim that if (4.38) holds for $j = n$, it is also valid for $j = n-1$. Now, let u_i^n satisfies (4.38). Thus, the values $u_{i-1}^n, u_i^n, u_{i+1}^n$ are non-negative. On the other hand, by lemma 4.1, the function f is nondecreasing in all of its arguments. Therefore,

$$u_i^{n-1} = f(u_{i-1}^n, u_i^n, u_{i+1}^n) \geq f(0,0,0) = k(C-q). \tag{4.41}$$

Then for $C \geq q$, we have $u_i^{n-1} \geq 0$ and this proves the claim. Thus the proof is complete. □

5. Numerical results

In this section, we want to implement the penalty method for computing the solutions of the ODE and PDE penalty problems (3.6)-(3.7) and (4.4)-(4.7). First, we consider ODE penalty problem. Assume that the time step Δt and L^∞ error be defined as

$$\Delta t = \frac{\epsilon}{1+\epsilon}, \tag{5.1}$$

$$\|e\|_\infty = \max_n |y(t_n) - Y_n|, \tag{5.2}$$

where ϵ is defined as in table 1. The time interval $[0, 2]$ is divided by time step Δt. The L^∞ error estimates, in the third column of table 1, show that the numerical solutions computed by the penalty method converge to the exact solutions as ϵ tends to zero.

Now, we consider the PDE penalty problem (4.4)-(4.7). To compute the price of European installment call option at (x_i, t_j), one can develop the following algorithm:

Table 1. Numerical result for ODE penalty problem

ϵ	Δt	$\|e\|_\infty$
10^{-1}	9.0909×10^{-2}	6.3584×10^{-2}
10^{-2}	9.9009×10^{-3}	2.1746×10^{-2}
10^{-3}	9.9900×10^{-4}	3.7937×10^{-3}
10^{-4}	9.9990×10^{-5}	5.8148×10^{-4}

(1) Choose h, k, ϵ, C.
(2) Set $u_i^{N+1} = \max(x_i - K, 0)$ for $0 \leq i \leq M+1$.
(3) Compute $u_0^j = 0$ for $0 \leq j \leq N+1$.
(4) For $j = N, N-1, \cdots, 0$ compute:
- For $i = 1, 2, \cdots, M$, using (4.17), compute:

$$u_i^j = f(u_{i-1}^{j+1}, u_i^{j+1}, u_{i+1}^{j+1}).$$

- For $i = M+1$, using (4.23), compute:

$$u_{M+1}^j = f(u_M^{j+1}, u_{M+1}^{j+1}).$$

Now, we determine the parameters of the problem. Let's these parameters be given in table 2.

Table 2. Values of parameters

Parameter	Value
Maturity	$T = 1$
Risk-free interest rate	$r = 0.03$
Constant dividend yield	$\delta = 0.05$
Volatility	$\sigma = 0.2$
Strike	$K = 100$

Assume also that $x_\infty = 200$. In discretizing the domain $[1, 200] \times [0, 1]$ by step length h and time step k, one must check that the assumptions of lemma 4.1 are held. For this purpose, it is sufficient that h, k satisfy

$$h \leq \frac{\sigma^2 x_1}{2\nu}, \tag{5.3}$$

$$k \leq \frac{h^2}{\sigma^2 x_\infty^2 + \mu h x_\infty + (r + \frac{C}{\epsilon})h^2}. \tag{5.4}$$

Also we need to check that for any given set of parameters the constraint (4.25) is satisfied. The parameters C and ϵ are chosen as $C = 3$ and $\epsilon = 10^{-4}$. Thus the required discretization parameters are $h = 0.1$ and $k = 0.0002$.

Based on the values of ϵ_i for $i = 1, 2, 3, 4$, we construct 4 tables and set $\epsilon_1 = \frac{1}{10}$ and $\epsilon_i = \frac{\epsilon_{i-1}}{10}$ for $i = 2, 3, 4$. Running the mentioned algorithm, the prices of European installment call option are computed and these values are reported in tables 3, 4, 5, 6.

TABLE 3. Call option values for $\epsilon = 10^{-1}$

q	S_0	Value
1	95	3.7065
	105	8.3984
	115	14.8539
3	95	2.2256
	105	6.6395
	115	12.9671
6	95	0.6746
	105	4.2719
	115	10.2505

TABLE 4. Call option values for $\epsilon = 10^{-2}$

q	S_0	Value
1	95	3.7033
	105	8.3947
	115	14.8578
3	95	2.2289
	105	6.6348
	115	12.9657
6	95	0.6773
	105	4.2744
	115	10.2548

TABLE 5. Call option values for $\epsilon = 10^{-3}$

q	S_0	Value
1	95	3.7036
	105	8.3959
	115	14.8571
3	95	2.2285
	105	6.6346
	115	12.9665
6	95	0.6772
	105	4.2748
	115	10.2559

TABLE 6. Call option values for $\epsilon = 10^{-4}$

q	S_0	Value
1	95	3.7039
	105	8.3959
	115	14.8574
3	95	2.2288
	105	6.6340
	115	12.9665
6	95	0.6772
	105	4.2746
	115	10.2556

In these tables, for underlying asset price, three values $S_0 \in \{95, 105, 115\}$ are chosen. Then, for each value of underlying asset, three values for installment rate $q \in \{1, 3, 6\}$ are given. In the next step, for different values of underlying asset and installment rate, the price of European installment call option is computed. As it is clear from tables $3, 4, 5, 6$, an increase in values of installment rate causes a decrease in the prices of call options. On the other hand, when $q = 0$ European installment option becomes European vanilla option. Therefore, one can deduce that the premium of European vanilla call option is larger than the premium of European installment call option. But as a whole, the sum of the premium and the installments paid for European installment option is larger than the premium of the European vanilla option.

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