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A new iteration method for solving a class of Hammerstein type integral equations system

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Abstract In this work, a new iterative method is proposed for obtaining the approximate solution of a class of Hammerstein type Integral Equations System. The main structure of this method is based on the Richardson iterative method for solving an algebraic linear system of equations. Some conditions for existence and unique solution of this type equations are imposed. Convergence analysis and error bound estimation of the new iterative method are also discussed. Finally, some numerical examples are given to compare the performance of the proposed method with the existing methods.

Keywords. Iterative method, Nonlinear integral equations system, Hammerstein integral equation, Fixed point iteration, Contraction operator.

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1. Introduction

Throughout this paper, we use the following notations. We use capital and lowercase letters to denote a vector and scalar functions, respectively. We consider the Banach space $\mathcal{X} = C(\Omega)$ and denote its associated norm by $||.||_\infty$. Also we denote $\mathcal{X}^m = \underbrace{\mathcal{X} \times \mathcal{X} \times \ldots \times \mathcal{X}}_{m-time}$. For a vector $\Phi = [\phi_1, \phi_2, ..., \phi_m]^T \in \mathcal{X}^m$, we define the following norm,

$$||\Phi||_\infty = \max_i \sup_{x\in\Omega} |\phi_i(x)| = \max_i ||\phi_i||_\infty.$$

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Systems of integral equations appear in a wide variety of physical applications. They are encountered in various fields of science and numerous applications such as elasticity, plasticity, game theory, control, queuing theory, electrical engineering, oscillation theory and fluid dynamics. The problem of finding numerical solution for the integral equations is one of the oldest problems in the applied mathematics literature and many computational methods are introduced in this field. Many researches are performed on the solution of linear integral equations system, while finding an approximate solution for the nonlinear kind is mostly difficult. Many scientific real world problems can be modeled by nonlinear differential and integral equations. In particular, the integral equations arise in fluid mechanics, biological models, solid state physics, kinetics in chemistry etc.

The aim of this work is to present a numerical method for approximating the solution of the following system of nonlinear integral equations

$$\phi_i(x) = f_i(x) + \lambda \sum_{j=1}^{m} \int_{\Omega} k_{ij}(x,y) g_{ij}(\phi_j(y)) dy, \; i = 1, \ldots, m; \; \Omega = [0,1], \quad (1.1)$$

where f_i and k_{ij}, $i,j = 1,2,...,m$ are known continuous functions. Also g_{ij}, $i,j = 1,2,...,m$ are in general known nonlinear continuous functions and ϕ_i, $i = 1,2,...,m$ are unknown functions to be determined. In equation (1.1), λ is a constant parameter. Note that the system of nonlinear integral equations (1.1) reduces to the linear ones, when $g_{ij}(x), i,j = 1,2,...,m$ are the linear functions.
By setting

$$F(x) = [f_1(x), ..., f_m(x)]^T, \; K(x,y) = [k_{ij}(x,y)]_{ij}, \quad i,j = 1,2,...,m,$$
$$\Phi(x) = [\phi_1(x), \phi_2(x), ..., \phi_m(x)]^T,$$

and

$$G(\Phi(x)) = [g_{ij}(\phi_j(x))]_{ij}, \; i,j = 1,2,...,m,$$

the system of nonlinear integral equations (1.1), can be rewritten as

$$\Phi(x) = F(x) + \lambda \int_{\Omega} [K(x,y) \circ G(\Phi(y))] \mathbf{1} dy, \quad (1.2)$$

where $\circ$ denotes the Hadamard product of two matrices $A, \; B \in \mathbb{R}^{m\times m}$, i.e., $A \circ B = (a_{ij}b_{ij})_{mm}, \; i,j = 1,...,m$, and $\mathbf{1} = [1,1,...,1]^T \in \mathbb{R}^m$. Equation (1.2) in the operator form is as follows

$$\begin{aligned} \Phi &= \tilde{T}(\Phi), \\ \tilde{T}(\Phi) &= F + \lambda \mathcal{K}\mathcal{G}(\Phi), \end{aligned} \quad (1.3)$$

where $\mathcal{K}\mathcal{G}(\Phi)(x) = \int_{\Omega} [K(x,y) \circ G(\Phi(y))] \mathbf{1} dy$.

There have been considerable interest in solving linear system of integral equations. Babolian et al. [3] applied an Adomian decomposition method for solving system of linear Fredholm integral equations of second kind. The Homotopy perturbation method [7] and its modification [8] were proposed by javidi and Golbabai. Maleknejad et al. presented a Taylor expansion method for a second kind Fredholm integral equations system with smooth or weakly singular kernel [13]. Their proposed method reduces the system of integral equations to the linear system of ordinary differential equations. Golbabai and Keramati presented a simple method to approximate the solution of system of linear integral equations of the second kind based on Adomian's decomposition method [9]. Convergence analysis of Sinc-collocation method for approximating the solution of the integral equations system was proposed by Rashidinia and Zarebnia [15]. Triangular functions method for the solution of Fredholm integral equations system has been proposed by Almasieh and Roodaki in [1]. Moreover, Jafarian and Measoomynia used Feed-back neurol networks(NNs) approach for finding the approximate solution of system of linear integral equations [11]. They substituted the Nth truncation of the Taylor expansion for unknown function in the origin system. As we know, a standard method can be improved to solve a system of integral equations. Recently, Karimi and Jozi proposed an iterative method for solving linear Fredholm integral equation [12] which can be improved easily to solve system of linear Fredholm integral equations.

Also there are many studies which focus on the solution of nonlinear integral equations system. These system of equations can be handled by some distinct methods such as, direct computation method, the modified Adomian method, the successive approximations method, and the series solution method, see Waswas [17]. Golbabai et al. [10] applied RBF network for solving nonlinear system equations (1.1). Moret et al. [14], applied an iterative quasi-Newton method for solving the vector and multidimensional integral equations of Urison type. Biazar and et al. [6], applied homotopy perturbation method (HPM) to solve the nonlinear integral equation of Fredholm type.

As we know, the nonlinear system (1.1) can be reduced to a system of nonlinear algebraic equations by using Nyström method which have solutions that are near to that of the continuous problem. There are many iterative methods to solve the reduced nonlinear system of equations, such as Newton method. Although Newton method is a good method for solving nonlinear equations, but it has some disadvantages from the point of view of practical calculation. The first of these is the difficulty of computing the Jacobian matrix and second is the conditions for convergence that depend on the initial estimate of the solution being sufficiently good, a requirement that is often impossible to realize in practice.

These disadvantages motivated us to present a new approach, namely Rich-type method, for obtaining the approximate solution of the system of nonlinear integral equations (1.1). The structure of this method is derived from Richardson iteration method for solving an algebraic linear system of equations $Ax = b$ [16]. We impose some conditions for existence and uniqueness of solution of equation (1.1) and we also discuss about the convergence analysis and error bound estimation of the new method. In implementation of the new method for solving system (1.1), at each iteration, some definite integrals involve which can be carried out by an appropriate quadrature rule.

The paper is organized as follows. In Section 2, we state several essential definitions, propositions and theorems to establish uniqueness and existence of solution (1.1). Section 3 is devoted to main idea of our method for solving equation (1.1). Also the convergence analysis and error bound estimation are discussed in this section. In Section 4 some numerical examples are given to compare the new method with some existing methods and illustrate the efficiency of the new method. Finally, we make some concluding remarks in Section 5.

2. Preliminaries

We review some necessary principles and definitions which are utilized throughout this paper.

Definition 2.1. $T : \mathcal{X} \to \mathcal{X}$ is said to be a contraction operator if there exists a constant $\eta \in [0, 1)$ such that

$$||T(\phi) - T(\psi)||_\infty \le \eta||\phi - \varphi||_\infty \qquad for\ all\ \phi,\ \varphi\ in\ \mathcal{X}.$$

The following theorem is known as Banach fixed point theorem and plays an important role in guaranteeing the existence and uniqueness of the solution of nonlinear equations.

Theorem 2.2. *[1] Let T be a contraction operator on $\mathcal{X}$. Then equation*

$$T(\phi) = \phi,$$

has a unique solution ϕ in $\mathcal{X}$. Such a solution is said to be a fixed point of T.

Definition 2.3. Let $a_{ij} \in C(\Omega \times \Omega),\ A = [a_{ij}(x, y)]_{i,j}\ i, j = 1, 2, ..., m$. Then

$$||A||_\infty = \max_i \sup_{(x,y)\in\Omega\times\Omega} \sum_{j=1}^{m} |a_{ij}(x, y)|.$$

Definition 2.4. Let $\mathcal{K} : \mathcal{X}^m \to \mathcal{X}^m$ be a bounded operator. Then

$$||\mathcal{K}|| = \sup_{\Phi\neq 0} \frac{||\mathcal{K}\Phi||_\infty}{||\Phi||_\infty}.$$

Theorem 2.5. *Let $k_{ij} \in C(\Omega \times \Omega),\ i, j = 1, 2, ...m$ and $g_{ij} \in C^1(\Omega)$, i, $j = 1, 2, ...m$. Then equation (1.3) has a unique solution in $\mathcal{X}^m$ for all F and sufficiently small $|\lambda|$, provided $\mathcal{KG}$ is a bounded operator.*

Proof: We assert that $\tilde{T}$ is a contraction operator. Let $\Phi = [\phi_1, \ldots, \phi_m]^T, \Psi = [\psi_1, \ldots, \psi_m]^T \in \mathcal{X}^m$. We have

$$\begin{aligned} ||\tilde{T}(\Phi) - \tilde{T}(\Psi)||_\infty &= |\lambda|||\mathcal{KG}(\Phi) - \mathcal{KG}(\Psi)||_\infty \\ &= |\lambda| \max_i ||\mathcal{KG}^{(i)}(\Phi) - \mathcal{KG}^{(i)}(\Psi)||_\infty, \end{aligned}$$

where $\mathcal{KG}^{(s)}(\Phi(x)) = \sum_{j=1}^{m} \int_\Omega k_{sj}(x, y)g_{sj}(\phi_j(y))dy$. Suppose that

$$\begin{aligned} \max_i ||\mathcal{KG}^{(i)}(\Phi) - \mathcal{KG}^{(i)}(\Psi)||_\infty &= ||\mathcal{KG}^{(l)}(\Phi) - \mathcal{KG}^{(l)}(\Psi)||_\infty \\ &= \max_{x\in\Omega} |\sum_{j=1}^{m} \int_\Omega k_{lj}(x, y)[g_{lj}(\phi_j(y)) - g_{lj}(\psi_j(y))]dy|. \end{aligned}$$

By applying the Role theorem, we have

$$||\mathcal{KG}^{(l)}(\Phi) - \mathcal{KG}^{(l)}(\Psi)||_\infty$$
$$= \max_{x\in\Omega} |\sum_{j=1}^{m} \int_\Omega k_{lj}(x,y)g'_{lj}(\xi_j(y))(\phi_j(y) - \psi_j(y))dy|,$$

for some $\xi_j(y)$ between $\phi_j(y)$ and $\psi_j(y)$ and where g'_{lj} denotes the derivative of g_{lj}. We assume that $M_k = \max_j ||k_{lj}||_\infty$ and $M_g = \max_j ||g'_{lj}||_\infty$. Therefore,

$$||\mathcal{KG}^{(l)}(\Phi) - \mathcal{KG}^{(l)}(\Psi)||_\infty \leq M_k M_g ||\Phi - \Psi||_\infty. \tag{2.1}$$

From (2.1) and (2.1), we have

$$||\tilde{T}(\Phi) - \tilde{T}(\Psi)||_\infty \leq |\lambda| M_k M_g ||\Phi - \Psi||_\infty.$$

It follows that for some small $|\lambda|$, $\tilde{T}$ is a contraction operator. The contractility of $\tilde{T}$ and Theorem 2.2 complete the proof. □

3. Iterative method

In this section, we present an iterative method, namely Rich-type method, for solving equation (1.1). For this purpose, first we present an iterative method to solve nonlinear Fredholm Hammerestein integral equation of second kind and then we generalize it for solving nonlinear integral equations system (1.1).

3.1. **Rich-type method.** Consider the following nonlinear Fredholm Hammerestein integral equation

$$\phi(x) = f(x) + \lambda \int_\Omega k(x,y)g(\phi(y))dy, \tag{3.1}$$

which is a special case of (1.1). The operator form of (3.1) is as

$$\phi = f + \lambda K g(\phi), \tag{3.2}$$

where $Kg(\phi(x)) = \int_\Omega k(x,y)g(\phi(y))dy$.
Since equation (3.1) is a special case of nonlinear integral equations system (1.1), we have the following remark.

Remark 3.1. Equation (3.2) has a unique solution for all f and sufficiently small λ, provided $|\lambda|||k||_\infty||g'||_\infty < 1$.

The wellknown Richardson iteration method is an iterative method for solving algebraic linear equation system $Ax = b$ with imposing some condition on the coefficient matrix A (for more elaborate details see [16]). Analogous Richardson iteration method, we present the following Rich-type method for solving (3.1).

Algorithm 3.2. Rich-type algorithm

(1) Choose $\phi_0 \in \mathcal{X}$.
(2) Compute $\phi_{n+1} = \check{T}\phi_n$ until convergent, where
$\check{T}\phi = \phi + \alpha(f - \phi + \lambda K g(\phi))$ and α is a constant parameter.

The following proposition shows that the sequence generated by Rich-type method converges to the solution of (3.1).

Proposition 3.3. *Let* $k \in C(\Omega \times \Omega)$, $g \in C^1(\Omega)$ *and suppose that assumption of Remark 3.1 is satisfied . Then the sequence* $\{\phi_n\}$ *generated by Rich-type method converges to the unique fixed point of the operator* $\check{T}$*, when*

$$0 < \alpha < \frac{2}{1 + |\lambda| ||k||_\infty ||g'||_\infty}.$$

Proof: We shall first show that under this assumption $\check{T}$ will be a contraction operator and then by Theorem 2.2 it has unique solution.

Note that

$$\begin{aligned} ||\check{T}(\phi) - \check{T}(\psi)||_\infty &= ||(1-\alpha)(\phi - \psi) + \alpha\lambda(Kg(\phi) - Kg(\psi))||_\infty \\ &\leq |1-\alpha| ||\phi - \psi||_\infty + |\alpha\lambda| ||Kg(\phi) - Kg(\psi)||_\infty \\ &= |1-\alpha| ||\phi - \psi||_\infty \\ &+ |\alpha\lambda| || \int_\Omega k(x,y)(g(\phi(y)) - g(\psi(y)))dy||_\infty. \end{aligned} \tag{3.3}$$

By applying the Role theorem, we have
$||\check{T}(\phi) - \check{T}(\psi)||_\infty = |1-\alpha| ||\phi - \psi||_\infty + |\alpha\lambda| || \int_\Omega k(x,y) g'(\zeta(y))(\phi(y) - \psi(y))dy||_\infty$, for some $\zeta(y)$ between $\phi(y)$ and $\psi(y)$ and where g' is the derivative of g. So

$$\begin{aligned} ||\check{T}(\phi) - \check{T}(\psi)||_\infty &\leq |1-\alpha| ||\phi - \psi||_\infty + |\alpha\lambda| \int_\Omega ||k||_\infty ||g'||_\infty ||\phi - \psi||_\infty dy \\ &= (|1-\alpha| + |\alpha\lambda| ||k||_\infty ||g'||_\infty) ||\phi - \psi||_\infty. \end{aligned} \tag{3.4}$$

A sufficient condition for the contractility of operator $\check{T}$ is that

$$|1-\alpha| + |\alpha\lambda| ||k||_\infty ||g'||_\infty < 1. \tag{3.5}$$

So we have the following cases.

- Case 1. If $\alpha \leq 0$ or $\alpha \geq \frac{2}{1+|\lambda| ||k||_\infty ||g'||_\infty}$ then (3.5) is not satisfied.
- Case 2. If $0 < \alpha \leq 1$, forasmuch as $|\lambda| ||k||_\infty ||g'||_\infty < 1$, we have
  $$\alpha|\lambda| ||k||_\infty ||g'||_\infty < \alpha \Rightarrow 1 - \alpha + \alpha|\lambda| ||k||_\infty ||g'||_\infty < 1,$$
  which holds inequality (3.5).
- Case 3. If $1 < \alpha < \frac{2}{1+|\lambda| ||k||_\infty ||g'||_\infty}$, then inequality (3.5) is easily satisfied.

Therefore, if $0 < \alpha < \frac{2}{1+|\lambda| ||k||_\infty ||g'||_\infty}$ and the nonlinear equation (3.1) be uniquely solvable, then the iteration operator $\{\phi_n\}$ converges to the solution of equation (3.1). □

3.2. **Implementation of Rich-type method to equation (1.3).** Consider the nonlinear operator equation (1.3). Rich-type method can be easily applied to solve (1.3). For this purpose, by choosing the initial guess $\Phi_0 \in \mathcal{X}^m$ we have

$$\Phi_{n+1} = \alpha F + (1-\alpha)\Phi_n + \alpha\lambda\mathcal{K}\mathcal{G}(\Phi_n)), \tag{3.6}$$

where α is a constant parameter which should be chosen such that the sequence $\{\Phi_n\}$ converges.
According to Theorem 2.5 and similar to Proposition 3.3, the following proposition is easily to prove.

Proposition 3.4. *Under the assumptions of Theorem 2.5, the sequence $\{\Phi_n\}$ generated by (3.6) converges to the solution of nonlinear system (1.3), when*

$$0 < \alpha < \frac{2}{1 + |\lambda| M_g M_k}, \tag{3.7}$$

where M_k and M_g are those of in the proof of Theorem 2.5.

3.3. **Error bound estimation.** As previously mentioned, some definite integrals have to be computed at each step of Rich-type algorithm. For computing these integrals, one can apply a proper quadrature rule. To this end, suppose that $\mathcal{KG}_N$ be approximate operator of $\mathcal{KG}$ obtained from an appropriate quadrature rule. Therefore for initial guess $\tilde{\Phi}_0 \in \mathcal{X}^m$ the recursive relation (3.6) reduces as follows

$$\tilde{\Phi}_{n+1} = \alpha F + (1-\alpha)\tilde{\Phi}_n + \alpha\lambda\mathcal{KG}_N(\tilde{\Phi}_n). \tag{3.8}$$

In the following theorem an upper bound for the error of exact solution with respect to the approximate operator $\mathcal{KG}_N$ is established.

Theorem 3.5. *Let $\{\tilde{\Phi}_n\}$ be the sequence generated by the recursive relation (3.8) and the inequality (3.7) satisfies. Also suppose that Φ^* is the exact solution of nonlinear equation (1.3). Then for some large n*

$$||\tilde{\mathcal{E}}_n||_\infty \leq \frac{\beta}{1-\theta},$$

where

$$\theta = |1-\alpha| + |\alpha\lambda| M_k M_g, \;\; \beta = |\alpha\lambda| ||\mathcal{KG}_N(\Phi^*) - \mathcal{KG}(\Phi^*)||_\infty,$$

and

$$\tilde{\mathcal{E}}_n = \tilde{\Phi}_n - \Phi^*.$$

Proof: Since Φ^* is the exact solution of (1.3), hence we have

$$\Phi^* = \alpha F + (1-\alpha)\Phi^* + \alpha\lambda\mathcal{KG}(\Phi^*). \tag{3.9}$$

By subtracting equation (3.9) from equation (3.8), we obtain

$$\begin{aligned} ||\tilde{\mathcal{E}}_n||_\infty &\leq |1-\alpha| ||\tilde{\mathcal{E}}_{n-1}||_\infty + |\alpha\lambda| ||\mathcal{KG}_N(\tilde{\Phi}_{n-1}) - \mathcal{KG}(\Phi^*)||_\infty \\ &\leq |1-\alpha| ||\tilde{\mathcal{E}}_{n-1}||_\infty + |\alpha\lambda| ||\mathcal{KG}_N(\tilde{\Phi}_{n-1}) - \mathcal{KG}_N(\Phi^*)||_\infty \\ &+ |\alpha\lambda| ||\mathcal{KG}_N(\Phi^*) - \mathcal{KG}(\Phi^*)||_\infty. \end{aligned} \tag{3.10}$$

Similar to equations (2.1) and (2.1) of Theorem 2.5, we obtain

$$\begin{aligned} ||\tilde{\mathcal{E}}_n||_\infty &\leq (|1-\alpha| + |\alpha\lambda| M_k M_g) ||\tilde{\mathcal{E}}_{n-1}||_\infty + \beta \\ &= \theta ||\tilde{\mathcal{E}}_{n-1}||_\infty + \beta \\ &\leq \theta^n ||\tilde{\mathcal{E}}_0||_\infty + \sum_{j=0}^{\infty} \theta^j \beta. \end{aligned} \tag{3.11}$$

For as much as $\theta < 1$, the following relation can be concluded

$$||\tilde{\mathcal{E}}_n||_\infty \quad \leq \quad \theta^n ||\tilde{\mathcal{E}}_0||_\infty + \tfrac{\beta}{1-\theta}. \tag{3.12}$$

So

$$\lim_{n\to\infty} ||\tilde{\mathcal{E}}_n||_\infty \leq \frac{\beta}{1-\theta},$$

which completes the proof of theorem. □

Note that the value of β depends on approximation $\mathcal{KG}_N$ of $\mathcal{KG}$. The more accurate approximation is the beta is closer to zero.

4. Numerical experiments

In this section, some numerical examples are presented to illustrate the effectiveness of the new method. All examples presented in this section were computed in double precision with a MATLAB code. As previously mentioned, in implementation of Rich-type method to solve (1.1), some definite integrals are involved. For computing these integrals numerically, we have used the 8-point Gauss-Legendre quadrature. Let $\Phi_n = [\phi_{1,n}, \ldots, \phi_{m,n}]^T$ be the sequence generated by (3.6), then we denote $e_{i,n}(x)$ the absolute solution error of ith component at nth iteration of $\Phi_n(x)$, as follows

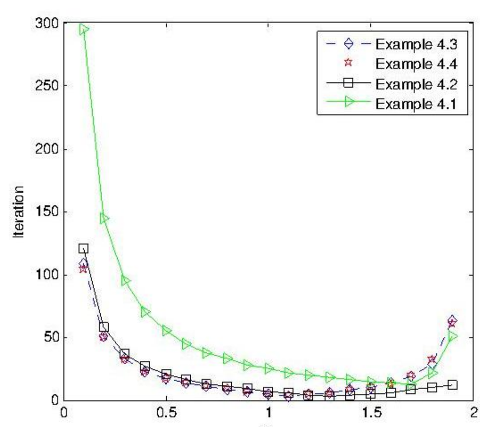
$$e_{i,n}(x) = |\phi_i^*(x) - \phi_{i,n}(x)|,$$

where $\phi_i^*(x)$ is ith component of the exact solution of system (1.1).

For the tests reported in this section we adopt the experimentally optimal parameter α (denoted by α_{eopt}) for Rich-type method which yields the least number of iterations.

The experimental optimal values of α for the examples 4.1, 4.2, 4.3 and 4.4 are depicted in Figure 1.

Figure 1. Pictures of iterations versus α for Examples 4.1,4.2,4.3 and 4.4.



Example 4.1 [10] Consider the system of nonlinear Fredholm equations

$$\begin{cases} \phi_1(x) - \lambda \int_0^1 \phi_1(y)dy - \lambda \int_0^1 \phi_2(y)dy &= x - 5/18, \\ \\ \phi_2(x) - \lambda \int_0^1 \phi_1^2(y)dy - \lambda \int_0^1 \phi_2(y)dy &= x^2 - 2/9, \end{cases}$$

where $\lambda = \frac{1}{3}$ and the exact solution is $(\phi_1^*(x), \phi_2^*(x)) = (x, x^2)$. We applied Rich-type method for solving this example with $\alpha_{eopt} = 1.6$. The numerical results are shown in Table 1. From the results presented in this table, we observe the good performance of Rich-type method.

Table 1. Numerical results of Rich-type method for Example 4.1.

x	$e_{1,21}(x)$	$e_{2,21}(x)$
0.0	$1.33e-10$	$1.95e-08$
0.1	$1.33e-10$	$1.95e-08$
0.2	$1.33e-10$	$1.95e-08$
0.3	$1.33e-10$	$1.95e-08$
0.4	$1.33e-10$	$1.95e-08$
0.5	$1.33e-10$	$1.95e-08$
0.6	$1.33e-10$	$1.95e-08$
0.7	$1.33e-10$	$1.95e-08$
0.8	$1.33e-10$	$1.95e-08$
0.9	$1.33e-10$	$1.95e-08$
1.0	$1.33e-10$	$1.95e-08$

Example 4.2 Consider the nonlinear Fredholm integral equation

$$\phi(x) - \frac{1}{2}\int_0^1 xy\phi^2(y)dy = \frac{7x}{8}, \tag{4.1}$$

with the exact solutions $\phi^*(x) = x, 7x$. We applied Rich-type method to solve this example with $\alpha = 0.9,\ 1.1$. The numerical results are shown in Table 2. From Figure 1, we have $\alpha_{eopt} = 1.3$ for this example. We get the exact solutions $\phi^*(x) = x$ with α_{eopt} at the one iteration.

TABLE 2. The values of $\phi_n(x)$ in different points with different values of α for Example 4.2

	$\alpha = 0.9$					$\alpha = 1.1$	
x\n	1	2	3	4	5	1	2
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.0980	0.0997	0.0999	0.1	0.1	0.0999	0.1
0.2	0.1961	0.1993	0.1999	0.2	0.2	0.1999	0.2
0.3	0.2941	0.2990	0.2998	0.3	0.3	0.2998	0.3
0.4	0.3921	0.3968	0.3998	0.4	0.4	0.3998	0.4
0.5	0.5882	0.5980	0.5996	0.5999	0.6	0.4997	0.5
0.6	0.5882	0.5980	0.5996	0.5999	0.6	0.5996	0.6
0.7	0.6862	0.6976	0.6996	0.6999	0.7	0.6996	0.7
0.8	0.7842	0.7973	0.7995	0.7999	0.8	0.7995	0.8
0.9	0.8822	0.8969	0.8995	0.8999	0.9	0.8995	0.9
1.0	09803	0.9966	0.9994	0.9999	1.0	0.9994	1.0

We reduced equation of Example 4.2 to a system of nonlinear algebraic equations by using Nyström method with 8-point Gauss-Legedre quadrature. Then we applied Rich-type and Newton methods to solve the corresponding nonlinear system. The numerical results are listed in Table 3. The stopping criterion is considered as $||r||_2 \leq 10^{-5}$, where $||r||_2$ denotes the residual norm. In this table, the initial guess has chosen zero or random vector for Newton method and right hand side for Rich-type method. From this table, Rich-type method is relatively better than Newton method in terms of CPU time. Of course, as mentioned, Newton method has own special difficulties.

TABLE 3. Numerical results for Example 4.2.

Method	Iteration	CPU time	initial guess
Richardson	3	2.1403	right hand side
Newton	3	4.2639	0
Newton	3	4.4795	random

Example 4.3 Consider the following system of nonlinear Fredholm integral equations

$$\begin{cases} \phi_1(x) - \lambda \int_0^1 e^x y \phi_1(y)dy - \lambda \int_0^1 (x-y)\phi_2^2(y)dy &= f_1(x), \\ \phi_2(x) - \lambda \int_0^1 e^{-x} y \phi_1(y)dy - \lambda \int_0^1 x^2 y \phi_2^2(y)dy &= f_2(x), \end{cases}$$

where $\lambda = 0.1$, the exact solution is $(\phi_1^*(x), \phi_2^*(x)) = (e^{x^2}, x)$ and

$$f_1 = e^{x^2} - \lambda \frac{e^x}{2}(e-1) - \lambda \frac{x}{3} + \frac{\lambda}{4},$$

$$f_2 = x - \lambda \frac{e^{-x}}{2}(e-1) - \lambda \frac{x^2}{4}.$$

We applied Rich-type method for solving this example with $\alpha_{eopt} = 1.1$. The numerical results are shown in Table 4. In this table $||r_{i,n}||_\infty$, $i = 1, 2$, denote the residual norm of ith component at nth iteration. As is shown from this table the convergence of Rich-type method is very fast.

Table 4. Error and residual norms for Example 4.3.

n	$\|\|e_{1,n}\|\|_\infty$	$\|\|e_{2,n}\|\|_\infty$	$\|\|r_{1,n}\|\|_\infty$	$\|\|r_{2,n}\|\|_\infty$
1	$3.10e-03$	$1.70e-03$	$5.60e-03$	$5.70e-03$
3	$7.68e-06$	$7.57e-06$	$7.57e-06$	$7.68e-03$
5	$4.57e-08$	$2.25e-08$	$4.89e-08$	$4.80e-08$
7	$4.52e-10$	$2.51e-10$	$4.49e-10$	$4.40e-10$
9	$4.76e-12$	$2.71e-12$	$4.65e-12$	$4.57e-12$

Example 4.4 Consider the following system of nonlinear Fredholm integral equations

$$\begin{cases} \phi_1(x) - \lambda \int_0^1 (x+y)\phi_1^2(y)dy - \lambda \int_0^1 xye^{-\phi_2(y)}dy = f_1(x), \\ \phi_2(x) - \lambda \int_0^1 e^x \sin(\phi_1(y))dy - \lambda \int_0^1 (x^2+y^2)\cos(\phi_2^2(y))dy = f_2(x), \end{cases}$$

where $f_1(x) = x - \lambda(\frac{x}{3}+\frac{1}{4}) + \lambda x(2e^{-1}-1)$, $f_2(x) = -x + \lambda e^x(\cos(1)-1) - \lambda(\sin(1)x^2 + 2\cos(1) - \sin(1))$ with $\lambda = 0.1$ and the exact solution is $(\phi_1^*(x), \phi_2^*(x)) = (x, -x)$. We applied Rich-type method to solve this example with $\alpha_{eopt} = 1.1$ and the numerical results are given in Table 5. From this table, the convergence of Rich-type method is quite impressive.

Table 5. Error norm for Example 4.4.

n	$\|\|e_{1,n}\|\|_\infty$	$\|\|e_{2,n}\|\|_\infty$
1	$7.10e-03$	$3.60e-03$
2	$4.20e-04$	$1.00e-03$
3	$4.10e-05$	$2.53e-05$
4	$2.23e-06$	$4.89e-06$
5	$2.34e-07$	$1.54e-07$
6	$1.40e-08$	$1.80e-08$
7	$1.28e-09$	$2.51e-09$
8	$1.09e-10$	$1.68e-10$
9	$6.27e-12$	$2.49e-11$

Example 4.5 Consider Example 1 of [6] as follows

$$\phi(x) - \frac{1}{5}\int_0^1 \cos(\pi x)\sin(\pi y)\phi^3(y)dy = \sin(\pi x).$$

The exact solution is $\phi^*(x) = \sin(\pi x) + \frac{20-\sqrt{391}}{3}\cos(\pi x)$. We applied Rich-type method for this example with $\alpha_{eopt} = 1$. We compared Rich-type method with HPM (homotopy perturbation method) of [6]. The numerical results are given in Table 6 for absolute solution errors in different points. From this table we see that our method converges faster than HPM.

TABLE 6. Numerical results for Example 4.5.

x	HPM(n=5)	Rich-type method(n=3)
0.0000	1.190e-6	9.432e-8
0.0625	1.167e-6	9.250e-8
0.1250	1.099e-6	8.714e-8
0.1875	9.896e-7	7.842e-8
0.2500	8.413e-7	6.669e-8
0.3125	6.610e-7	5.240e-8
0.3750	4.553e-7	3.609e-8
0.4375	2.321e-7	1.840e-8
0.5000	0.0	0.0
0.5625	2.321e-7	1.840e-8
0.6250	4.553e-7	3.609e-8
0.6875	6.610e-7	5.240e-8
0.7500	8.413e-7	6.669e-8
0.8125	9.893e-7	7.842e-8
0.8750	1.010e-6	8.714e-8
0.9375	1.167e-6	9.250e-8

Example 4.6 Consider Example 1 of [4] as follows

$$\phi(x) + \int_0^1 e^{x-2y}\phi^3(y)dy = e^{x+1},$$

with the exact solution $\phi^*(x) = e^x$. We applied Rich-type method for this example with $\alpha_{eopt} = 0.1$. The numerical results are given in Table 7.

TABLE 7. Numerical results for Example 4.6.

x	Exact solution	Rich-type method(n=29)
0.1	1.105170918	1.105170918
0.2	1.221402757	1.221402758
0.3	1.349858806	1.349858807
0.4	1.491824696	1.491824697
0.5	1.648721268	1.648721270
0.6	1.822118797	1.822118800
0.7	2.013752703	2.013752707
0.8	2.225540923	2.225540928
0.9	2.459603104	2.459603111

Example 4.7 [5] Consider the mathematical model for an adiabatic tubular chemical reactor which in the case of steady state solutions, can be stated as the ordinary

differential equation

$$u'' - \lambda u'(x) + F(\lambda, \mu, \beta, u(x)) = 0, \; x \in [0, 1],$$
$$u'(0) = \lambda u(0), \quad u'(1) = 0,$$

where $F(\lambda, \mu, \beta, u(x)) = \lambda\mu(\beta - u(x))e^{u(x)}$.
The problem can be converted into a Hammerstein integral equation of the form

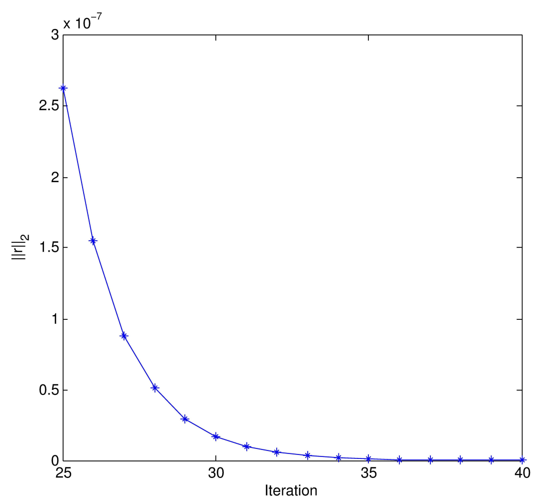
$$u(x) = \int_0^1 \mu k(x,t)(\beta - u(t))e^{u(t)}dt, \; x \in [0, 1],$$

where

$$k(x,t) = \begin{cases} 1, & t \leq x, \\ e^{\lambda(x-t)}, & t > x. \end{cases}$$

Similar to Example 4.2, we discretized equation of this example and obtained a system of nonlinear algebraic equations. Then we applied Rich-type method for the reduced algebraic system with $\alpha_{eopt} = 1$ and parameters $\lambda = 10$, $\mu = 0.02$ and $\beta = 3$. The convergence history of this method is depicted in Figure 2. The stopping criterion is considered as $||r||_2 \leq 10^{-10}$, where $||r||_2$ denotes the residual norm of the reduced algebraic system. This figure denotes the fast convergence of Rich-type method.

FIGURE 2. Convergence history of Rich-type method for Example 4.7.



Example 4.8 [5] One of the great interest in hydrodynamics is the physical problem

$$\begin{cases} u'' - e^{u(x)} = 0, & x \in [0, 1], \\ u(1) = u(0) = 0. \end{cases}$$

This equation can be reformulated as the nonlinear Fredholm Hammerstein integral equation

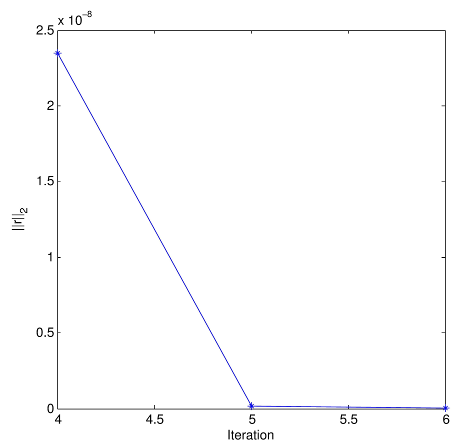
$$u(x) = \int_0^1 k(x,t)e^{u(t)}dt, \quad x \in [0,1],$$

where

$$k(x,t) = \begin{cases} -t(1-x), & t \leq x, \\ -x(1-t), & t > x. \end{cases}$$

We reduced equation of this example to a system of nonlinear algebraic equations by using Nyström method with 8-point Gauss-Legedre quadrature. Then we applied Rich-type method for the reduced algebraic system with $\alpha_{eopt} = 1.5$. The convergence history of this method is depicted in Figure 3. For this example, the stopping criterion is considered as $||r||_2 \leq 10^{-10}$, where $||r||_2$ denotes the residual norm of the reduced algebraic system.

FIGURE 3. Convergence history of Rich-type method for Example 4.8.

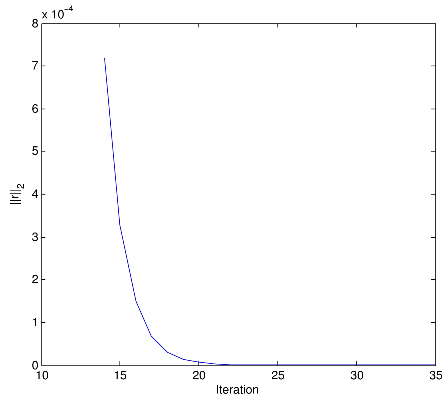


Example 4.9 Consider the following nonlinear integral equation

$$u(x) = f(x) + \lambda \int_0^1 e^{x-t} cos(u(t))dt, \quad x \in [0,1],$$

where $\lambda = 100$ and $f(x)$ is selected so that $u(x) = x$. We applied Rich-type method to solve the reduced algebraic system of this problem with $\alpha_{eopt} = 0.1$. The stopping criterion is considered as $||r||_2 \leq 10^{-10}$, where $||r||_2$ denotes the residual norm of reduced system. The results obtained for this example is shown in Figure 4.

Figure 4. Convergent history of Rich-type method for Example 4.9.



5. Conclusions

In this paper, we considered a class of Hammerstein type integral equations system. We verified existence and uniqueness of solutions of these equations. We also proposed the Rich-type method to solve this type system. Convergence of the Rich-type method was analyzed and an upper bound for the error of exact solution with respect to the approximate operator was established. Some numerical experiments was presented to illustrate the efficiency of the new method.

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References

[1] H. Almasieh and M. roodaki, *Triangular function method for the solution of Fredholm integral equations system*, *3* (2012), 411-416.

[2] K.E. Atkinson, *the numerical solution of integral equations of the second kind*, Cambridge Univ. Press, Cambridge, 1997.

[3] E. Babolian, J. Biazar and A.R. Vahidi, *The decomposition method applied to systems of Fredholm integral equations of the second Kind*, Appl. Math. Comput., *148* (2004), 443-452.

[4] E. Babolian and A. Shahsavaran, *Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets*, J. Comput. Appl. Math, *225* (2009), 87-95.

[5] S. Bazm, *Bernoulli polynomials for the numerical solution of some classes of linear and nonlinear integral equations*, J. Comput. Appl. Math., *275* (2015), 44-60.

[6] J. Biazar and H. Ghazvini, *Numerical solution for special non-linear Fredholm integral equation by HPM*, Appl. Math. Comput., *195* (2008), 681-687.

[7] A. Golbabai and M. Javidi, *A numerical solution for solving system of Fredholm integral equations by using Homotopy Perturbation method*, Appl. Math. Comput. *189* (2007), 1921-1928.

[8] A. Golbabai and M. Javidi, *Modified Homotopy Perturbation method for solving non-linear Fredholm integral equations*, Chaos. Solitons and Fractals, *40* (2009), 1408-1412.

[9] A. Golbabai and B. Keramati, *Easy computational approach to solution of system of linear Fredholm integral equations*, Chaos. Solitons and Fractals., *38* (2008), 568-574.

[10] A. Golbabai, M. Mammadov and S. Seifollahi, *Solving a system of nonlinear integral equations by an RBF network*, Comput. Math. Appl., *57* (2009), 1651-1658.

[11] A. Jafarian and S. Measoomy Nia, *Utilizing Feed-back Neural Network approach for solving linear Fredholm integral equations system*, Appl. Math. Model., *37* (2013), 5027-5038.
[12] S. Karimi and M. Jozi, *A new iterative method for solving linear Fredholm integral equations using the least squares method*, Appl. Math. and Comput., *250* (2015), 744-758.
[13] K. Maleknejad, N. Aghazadeh and M. Rabbani, *Numerical solution of second kind Fredholm integral equations system by using a Taylor-series expansion method*, Appl. Math. comput., *175* (2006), 1229-1234.
[14] I. Moret and P. Omari, *Iterative solution of integral equations by a quasi-Newton method*, J. Comput. Appl. Math., *20* (1987), 333-340.
[15] J. Rashidinia and M. Zarebnia, *Convergence of approximate solution of system of Fredholm integral equations*, J. Math. Anal. Appl., *333* (2007), 1216-1227.
[16] Y. Saad, *Iterative methods for sparse linear systems*, 2nd ed, SIAM; 2003.
[17] A.M. Waswas, *Linear and Nonlinear Integral Equations*, Spriger; 2011.