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An application of differential transform method for solving nonlinear optimal control problems

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Abstract In this paper, we present a capable algorithm for solving a class of nonlinear optimal control problems (OCP's). The approach rest mainly on the differential transform method (DTM) which is one of the approximate methods. The DTM is a powerful and efficient technique for finding solutions of nonlinear equations without the need of a linearization process. Utilizing this approach, the optimal control and the corresponding trajectory of the OCP's are found in the form of rapidly convergent series with easily computed components. Numerical results are also given for several test examples to demonstrate the applicability and the efficiency of the method.

Keywords. Optimal control problems; Differential transform method; Hamiltonian system.

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1. Introduction

Optimal control problems arise in a wide variety of scientific and engineering applications including physics, aerospace engineering, robotics, chemical engineering, economy, etc. [14, 28, 30, 34]. In practice, many optimal control problems are subject to constraints on the state and/or control variables. It is well known that constrained optimal control problems are very difficult to solve. In particular, their analytical solutions are in many cases out of the question. Thus, numerical methods are needed for solving many of these real world problems. There are now many

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numerical methods available in the literature for various optimal control problems [1,3,6,8,9,11,15,21–24,26,31–33,36]. Among them, Bellman [7] proposed an approach using dynamic programming. This approach leads to the Hamilton- Jacobi-Bellman (HJB) equation which is hard to solve in most cases. Due to the use of embedding method, Rubio [32] proposed a numerical approach for solving optimal control problems of ODE's and PDE's. The advantages of the proposed method lies on the fact that the method is not iterative, it is self-starting, and it does not need to solve corresponding boundary value problems. However, this method has high computational complexity. Huang and Lin [21] and Abu-khalaf et al. [1] suggested a numerical approach which finds the Taylor series solution of the Hamilton-Jacobi-Isaacs (HJI) equation associated with the nonlinear H^{∞} control problem. The coefficients of the Taylor series are generated by solving one Riccati algebraic equation and a sequence of linear algebraic equations. However, deriving each linear equation in the sequence requires a number of matrix computations, which may introduce computational complexity. An excellent literature review on the methods for solving the HJB equation is provided by Beard et al. [6] where a Successive Galerkin Approximation (SGA) approach is also considered. In the SGA, a sequence of generalized HJB equations is solved iteratively to obtain a sequence of approximations leading eventually to the solution of the HJB equation. However, the above-mentioned sequence may converge very slowly or even diverge. Banks and Dinesh [4] proposed the Approximating Sequence of Riccati Equations (ASRE). From a practical point of view, the ASRE is attractive. But its shortcoming is that it suffers from computational complexity, since it needs solving a sequence of linear quadratic time-varying matrix Riccati differential equations. Cimen [9] presented the State-Dependent Riccati Equation (SDRE) method which has been widely used in various applications. However, its major limitation is that it requires solving a sequence of matrix Riccati algebraic equations. This property may use up a lot of computing time and memory space.

One well known method for solving optimal control problems can also be derived using the Pontryagin's maximum principle (PMP) [31]. For the nonlinear OCP's, this approach leads to a nonlinear Two-Point Boundary Value Problem (2PBVP) that unfortunately in general cannot be analytically solved. Therefore, many researchers have tried to find an approximate solution for the nonlinear 2PBVPs [3]. In the recent years, some better results have been obtained. For instance, a new Successive Approximation Approach (SAA) has been proposed by Tang in [33], where instead of directly solving the nonlinear 2PBVP, derived from the maximum principle, a sequence of nonhomogeneous linear time-varying 2PBVPs is solved iteratively. It should be noted that solving time-varying equations are much more difficult than solving time-invariant ones. Effati and Saberi Nik [11] obtained an analytical approximate solution for non-linear OCP's using the homotopy perturbation method. As a modification, Jajarmi et al. [23] applied the optimal homotopy perturbation method for OCP's. Yousefi et al. [36] utilized another approximate analytical scheme called the variational iteration method to find optimal control of linear systems.

Recently, a growing interest has been appeared toward the application of DTM in the nonlinear problems. The concept of the differential transform was first proposed by Zhou [37], and its main applications therein is solved both linear and non-linear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of polynomials. It is different from the high-order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally expansive for large orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. In recent years the application of differential transform theory has been appeared in many researches [2, 10, 12, 17, 19, 20, 25, 27, 35, 37]. Especially, the differential transform method has been successfully used for solving 2PBVP [5, 13, 18, 29].

Motivated by the above discussions, the aim of this paper is to employ the DTM for solving a class of linear and nonlinear OCP's. Applying the DTM to the 2PBVP derived from the PMP, the optimal control and the corresponding trajectory are obtained in the form of rapid convergent series. Moreover, the convergence of the obtained series is controlled by an absolute tolerance. Thus, just a few iterations yield to find a suboptimal trajectory-control pair for the nonlinear OCP's. Illustrative examples are provided to demonstrate the applicability and efficiency of the technique.

The paper is organized as follows. Section 2 describes the mathematical modelling of a linear-quadratic OCP's which using PMP leads to a linear 2PBVP. Section 3 is similar to section 2, where instead of linear-quadratic OCP's, a nonlinear OCP's with the corresponding 2PBVP is studied. The basic idea of DTM is explained in Section 4. In Section 5, the DTM is employed to propose a new optimal control design strategy. In Section 6, effectiveness of the proposed approach is verified by solving several numerical examples. Conclusions are given in Section 7.

2. Linear-quadratic optimal control system

Consider the linear system

$$\dot{x} = Ax(t) + Bu(t),\ t_0 \leq t \leq t_f, \tag{2.1}$$

with the initial condition

$$x(t_0) = x_0, \tag{2.2}$$

where $x_0 \in \mathbb{R}^n$ is a given vector and the matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$. we shall consider $u \in L_2^m[t_0, t_f]$ and $x : [t_0, t_f] \to \mathbb{R}^n$ an absolutely continuous function on $[t_0, t_f]$ such that $\dot{x} \in L_2^n[t_0, t_f]$ where $L_2^n[t_0, t_f]$ is the Hilbert space of measurable square integrable functions on $[t_0, t_f]$ with values in $\mathbb{R}^n$. The problem may now be stated as follows:

Given the dynamical system (2.1), find the optimal control $u \in L_2^m[t_0, t_f]$ and the corresponding state vector $x(t)$ satisfying (2.1) and (2.2) while minimizing the quadratic cost functional

$$J = \frac{1}{2}x(t_f)^T Sx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (x^T Px + 2x^T Qu + u^T Ru)dt, \tag{2.3}$$

where the final time t_f is fixed, $S \in \mathbb{R}^{n\times n}, P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times m}$ are real symmetric positive semi-definite matrices and $R \in \mathbb{R}^{m\times m}$ is a positive definite matrix. It is assumed that the states and controls are not bounded and $x(t_f)$ is free.

The Hamiltonian is

$$H(x,u,p,t) = \frac{1}{2}(x^T Px + 2x^T Qu + u^T Ru) + p^T(Ax + Bu), \tag{2.4}$$

where $p \in \mathbb{R}^n$ is a co-state vector. According to the PMP, the necessary conditions for optimality are

$$\dot{x} = Ax(t) + Bu(t), \tag{2.5}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -Px(t) - Qu(t) - A^T p(t), \tag{2.6}$$

$$0 = \frac{\partial H}{\partial u} = Q^T x(t) + Ru(t) + B^T p(t). \tag{2.7}$$

Equation (2.7) can be solved for $u(t)$ to give

$$u(t) = -R^{-1}Q^T x(t) - R^{-1}B^T p(t),\ t \in [t_0, t_f], \tag{2.8}$$

the existence of R^{-1} is assured, since R is a positive definite matrix. Substituting (2.8) into (2.5) yields,

$$\dot{x}(t) = [A - BR^{-1}Q^T]x(t) - BR^{-1}B^T p(t),$$

thus, we have the set of $2n$ linear homogenous differential equations,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A - BR^{-1}Q^T & -BR^{-1}B^T P \\ -P + QR^{-1}Q^T & QR^{-1}B^T - A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}. \tag{2.9}$$

The solution for these equations has the form

$$\begin{bmatrix} x(t_f) \\ p(t_f) \end{bmatrix} = \phi(t_f, t) \begin{bmatrix} x(t) \\ p(t) \end{bmatrix},$$

where ϕ is the transition matrix of the system (2.9). Partitioning the transition matrix, we have

$$\begin{bmatrix} x(t_f) \\ p(t_f) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t_f,t) & \phi_{12}(t_f,t) \\ \phi_{21}(t_f,t) & \phi_{22}(t_f,t) \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}, \tag{2.10}$$

where $\phi_{11}, \phi_{12}, \phi_{21}$ and ϕ_{22} are $n \times n$ matrices. Since $x(t_f)$ is free, from the boundary condition equations, (see [26] page 200, entry 2 of Table 5.1), we find that

$$p(t_f) = Sx(t_f). \tag{2.11}$$

Substituting this for $p(t_f)$ in (2.10) gives

$$x(t_f) = \phi_{11}(t_f,t)x(t) + \phi_{12}(t_f,t)p(t), \tag{2.12}$$

$$Sx(t_f) = \phi_{21}(t_f,t)x(t) + \phi_{22}(t_f,t)p(t), \tag{2.13}$$

which when solved for $p(t)$ yields

$$p(t) = [\phi_{22}(t_f,t) - S\phi_{12}(t_f,t)]^{-1}[S\phi_{11}(t_f,t) - \phi_{21}(t_f,t)]x(t). \tag{2.14}$$

Kalman [24] has shown that the required inverse exists for all $t \in [t_0, t_f]$. It is easy to verify that (2.14) can be written as

$$p(t) = k(t)x(t), \tag{2.15}$$

which means that $p(t)$ is a linear function of the states of the systems; k is a $n \times n$ matrix. Actually k depends on t_f also, but t_f is specified. Substituting in (2.8) we obtain,

$$u(t) = -R^{-1}Q^T x - R^{-1}B^T k(t)x(t),\ t \in [t_0, t_f], \tag{2.16}$$

which indicates that the optimal control law for problem is a linear, albeit time varying, combination of the system states.

3. Nonlinear quadratic optimal control problems

Consider a nonlinear control system described by:

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t),\ t \in [t_0, t_f], \\ x(t_0) = x_0,\ x(t_f) = x_f, \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are respectively the state and control vectors, $f(t, x(t)) \in \mathbb{R}^n$ and $g(t, x(t)) \in \mathbb{R}^{n\times m}$ are two continuously differentiable functions in all arguments, $x_0 \in \mathbb{R}^n$ and $x_f \in \mathbb{R}^n$ are the initial and final state vectors, respectively. Our goal is to find the optimal control law $u(t)$, which minimizes the following quadratic performance as

$$J[x, u] = \frac{1}{2}\int_{t_0}^{t_f} (x(t)^T \mathcal{Q} x(t) + u(t)^T R u(t))dt, \tag{3.2}$$

subject to the dynamic system (3.1) for $\mathcal{Q} \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$, positive semi-definite and positive definite matrices, respectively.

According to the PMP, the necessary optimality conditions are obtained as following [11]:

$$\dot{x} = f(t, x) + g(t, x)u, \tag{3.3}$$
$$\dot{p} = -H_x(x, u, p), \tag{3.4}$$
$$u = \text{argmin}_u H(x, u, p), \tag{3.5}$$
$$x(t_0) = x_0,\ x(t_f) = x_f, \tag{3.6}$$

where $H(x, u, p) = \frac{1}{2}[x^T \mathcal{Q} x + u^T R u] + p^T[f(t, x) + g(t, x)u]$ is referred to as the Hamiltonian and $p(t) \in \mathbb{R}^n$ is the co-state vector. Equivalently, (3.3)-(3.6) can be written as

$$\begin{cases} \dot{x} = f(t, x(t)) - g(t, x(t))R^{-1}g^T(t, x(t))p(t), \\ \dot{p} = -\left(\mathcal{Q}x(t) + (\frac{\partial f(t,x(t))}{\partial x})^T p(t) + \sum_{i=1}^n p_i(t)[-R^{-1}g^T(t, x(t))p(t)]^T \frac{\partial g_i(t,x(t))}{\partial x}\right), \\ x(t_0) = x_0,\ x(t_f) = x_f. \end{cases} \tag{3.7}$$

The optimal control law is also given by

$$u(t) = -R^{-1}g^T(t, x(t))p(t),\ t \in [t_0, t_f]. \tag{3.8}$$

Unfortunately, system (3.7) contains a nonlinear 2PBVP that in general cannot be solved analytically except in a few simple cases. In order to overcome this difficulty, we will introduce the differential transform scheme in the next section.

4. Analysis of the differential transform method

For convenience of the reader, we will present a review of the differential transform procedure.

As stated in [19], the differential transform of the derivative of a function is defined as follows

Definition 4.1. The one-dimensional differential transform of function $w(x)$ is defined as follows:

$$W(k) = \frac{1}{k!}[\frac{d^k w(x)}{dx^k}]_{x=0}, \tag{4.1}$$

where $w(x)$ is the original function and $W(k)$ is the transformed function, which is called the T -function.

Definition 4.2. The differential inverse transform of $W(k)$ is defined as follows:

$$w(x) = \sum_{k=0}^{\infty} W(k)x^k. \tag{4.2}$$

Substituting (4.1) into (4.2) we have

$$w(x) = \sum_{k=0}^{\infty} \frac{1}{k!}[\frac{d^k w(x)}{dx^k}]_{x=0}\ x^k. \tag{4.3}$$

In real applications, the function $w(x)$ by a finite series of (4.2) can be written as

$$w(x) = \sum_{k=0}^{n} W(k)x^k, \tag{4.4}$$

and (4.2) implies that

$$w(x) = \sum_{k=n+1}^{\infty} W(k)x^k, \tag{4.5}$$

is neglected as it is small. Usually, the values of n are decided by a convergency of the series coefficients.

The fundamental mathematical operations performed by one dimensional differential transform method can readily be obtained and are listed in Table 1.

Here, we propose a new idea in order to use the DTM to solve optimization problem (3.2) with constrain (3.1). We consider the the following initial value problem (IVP)

$$\begin{cases} \dot{x} = f(t,x(t)) - g(t,x(t))R^{-1}g^T(t,x(t))p(t), \\ \dot{p} = -\Big(\mathcal{Q}x(t) + (\frac{\partial f(t,x(t))}{\partial x})^T p(t) + \sum_{i=1}^{n} p_i(t)[-R^{-1}g^T(t,x(t))p(t)]^T \frac{\partial g_i(t,x(t))}{\partial x}\Big), \\ x(t_0) = x_0,\ p(t_0) = \alpha, \end{cases} \tag{4.6}$$

where $\alpha \in \mathbb{R}$ is an unknown parameter. Using the DTM, we find the series solutions of $x(t)$ and $p(t)$ consist of an unknown constant α. To find this constant, we impose the boundary condition $x(t_f) = x_f$ to the obtained approximate solution (4.4) which results in an equation in α. By solving this equation that usually is nonlinear, we find

α and then the optimal pair $(x(.), u(.))^T$ is immediately given. A similar procedure is done to solve problem (2.3) with respect to (2.1) and (2.2), where the imposed boundary condition is given by (2.11).

TABLE 1. The operations for the one-dimensional differential transform method.

Original function	Transformed function
$w(x) = u(x) \mp v(x),$	$W(k) = U(k) \mp V(k)$
$w(x) = \alpha u(x)$	$W(k) = \alpha U(k)$
$w(x) = \frac{\partial u(x)}{\partial t}$	$W(k) = (k+1)U(k+1)$
$w(x) = \frac{\partial^n u(x)}{\partial x^n}$	$W(k) = (k+1)(k+2)...(k+n)U(k+n)$
$w(x) = u(x)v(x)$	$W(k) = \sum_{r=0}^{k} U(r)V(k-r)$
$w(x) = e^{\lambda t}$	$W(k) = \frac{\lambda^k}{k!}$
$w(x) = x^n$	$W(k) = \delta(k-n)$, where $\delta(k-n) = \begin{cases} 1, & k=n \\ 0, & \text{otherwise} \end{cases}$

According to the above discussions, the following theorem can be stated:

Theorem 4.3. *Consider the OCP of the nonlinear system in (3.1) with performance index in (3.2). Employing the DTM, the optimal pair* $(x(.), u(.))^T$ *is given as follows*

$$\begin{cases} x^*(t) = \sum_{k=0}^{\infty} X(k)t^k, \ t \in [t_0, t_f], \\ u^*(t) = -R^{-1}g^T(t, x^*(t)) \sum_{k=0}^{\infty} P(k), \ t \in [t_0, t_f]. \end{cases} \tag{4.7}$$

A similar theorem can be concluded for linear system (2.1)-(2.2) with the quadratic performance index (2.3).

5. Application of the DTM for non-linear OCP's

In this section, we propose a practical implementation to design the suboptimal control to system (3.1) with cost function (3.2). For the linear-quadratic problem (2.3) with constrains (2.1) and (2.2) the strategy is similar.

It is clearly impossible to obtain the optimal trajectory and the optimal control law as in (4.7), since it contains infinite series. In practice, the Nth order suboptimal trajectory-control pair is obtained by replacing ∞ with a finite positive integer N in (4.7) as follows:

$$\begin{cases} x^N(t) = \sum_{k=0}^{N} X(k)t^k, \\ u^N(t) = -R^{-1}g^T(t, x^N(t)) \sum_{k=0}^{N} P(k)t^k. \end{cases} \tag{5.1}$$

The integer N is generally determined according to a concrete control precision. For example, the Nth order suboptimal trajectory-control pair in (5.1) has the desired accuracy if for a given positive constant $\epsilon > 0$, the following condition holds:

$$|\frac{J^{(N)} - J^{(N-1)}}{J^{(N)}}| < \epsilon, \tag{5.2}$$

where

$$J^{(N)} = \frac{1}{2}\int_{t_0}^{t_f} \left((x^{(N)}(t))^T Q x^{(N)}(t) + (u^{(N)}(t))^T R u^{(N)}(t) \right) dt. \tag{5.3}$$

If the tolerance error bound $\epsilon > 0$ be chosen small enough, the Nth-order suboptimal trajectory-control law will be very close to the optimal pair $(x^*(t), u^*(t))^T$, the value of performance $J^{(N)}$ in (5.3) will be very close to its optimal value J^* and the boundary conditions will be tightly satisfied.

6. Simulation results

In this section we present some numerical examples to illustrate the proposed method.

Example 6.1: [11]

$$\min\ J = \int_0^1 u^2(t)dt$$

$$\text{subject to} \begin{cases} \dot{x} = \frac{1}{2}x^2(t)\sin x(t) + u(t),\ t \in [0,1], \\ x(0) = 0,\ x(1) = 0.5\ . \end{cases}$$

According to the PMP, the following nonlinear 2PBVP is obtained

$$\begin{cases} \dot{x} = \frac{1}{2}x^2(t)\sin x(t) - \frac{1}{2}p(t),\ t \in [0,1], \\ \dot{p} = -p(t)x(t)\sin x(t) - \frac{1}{2}p(t)x^2(t)\cos x(t),\ t \in [0,1], \\ x(0) = 0,\ x(1) = 0.5, \end{cases} \tag{6.1}$$

and the optimal control law is given by

$$u(t) = -\frac{1}{2}p(t). \tag{6.2}$$

We consider the following IVP

$$\begin{cases} \dot{x} = \frac{1}{2}x^2(t)\sin x(t) - \frac{1}{2}p(t),\ t \in [0,1], \\ \dot{p} = -p(t)x(t)\sin x(t) - \frac{1}{2}p(t)x^2(t)\cos x(t),\ t \in [0,1], \\ x(0) = 0,\ p(0) = \alpha. \end{cases} \tag{6.3}$$

According to the DTM for (6.3) by an iterative procedure, we obtain the following components

$$\begin{cases} (k+1)X(k+1) = \frac{1}{2}\left(\sum_{r=0}^{k}\sum_{s=0}^{r} X(s)X(r-s)U(k-r) - P(k)\right), \\ (k+1)P(k+1) = \sum_{r=0}^{k}\sum_{s=0}^{r} P(s)X(r-s)U(k-r) - \\ \frac{1}{2}\left(\sum_{r=0}^{k}\sum_{s=0}^{r}\sum_{p=0}^{s} P(p)X(s-p)X(r-s)V(k-r)\right), \\ X(0) = 0,\ P(0) = \alpha, \end{cases} \tag{6.4}$$

where U and V are the DTM of $\sin(x)$ and $\cos(x)$ as follows

$$\begin{cases} U(0) = \sin(0); V(0) = \cos(0), \\ U(k) = \sum_{i=0}^{k-1} \left[\frac{k-i}{k} V(i) X(k-i)\right], \\ V(k) = -\sum_{i=0}^{k-1} \left[\frac{k-i}{k} U(i) X(k-i)\right]. \end{cases} \tag{6.5}$$

Table 2. Simulation results of the proposed method in different iteration times for Example 6.1.

k	$X(k)$	$P(k)$	$J^{(k)}$	$\lvert\frac{J^{(k)}-J^{(k-1)}}{J^{(k)}}\rvert$
0.	0.	-1.	0.25	–
1.	0.5	0.	0.25	0.
2.	0.	0.	0.25	0.
3.	0.	0.125	0.234933	0.064133
4.	0.	0.	0.234933	0.
5.	0.	-0.00520833	0.235332	0.00169325
6.	0.	-0.0078125	0.235844	0.00217126
7.	0.000558036	0.0000651042	0.23584	0.0000158168
8.	0.	0.000651042	0.235807	0.000140205
9.	-0.000036169	0.000418139	0.235788	0.0000808384
10.	0.	-0.0000217014	0.235789	$3.80609\ \times 10^{-6}$

In order to obtain a suboptimal trajectory-control pair with remarkable accuracy, we use the proposed guideline in Section 5 with the tolerance error bound $\epsilon = 5\times 10^{-5}$. Substituting (6.5) into (6.4) and after some manipulations, we acquire $\lvert\frac{J^{(10)}-J^{(9)}}{J^{(10)}}\rvert = 3.80609 \times 10^{-6} < \epsilon$; where by imposing the boundary condition $x^{10}(1) = 0.5$, we have $\sum_{k=0}^{10} X(k) = 0.5$ i.e. $\alpha = -0.998958 \simeq -1$. Thus we obtain

$$x(t) = \frac{1}{2}t,$$

and from (6.2)

$$u(t) = -\frac{1}{2}\sum_{k=0}^{10} p(k) = \frac{1}{2} - \frac{t^3}{16} + \frac{t^5}{384} + \frac{t^6}{256} - \frac{t^7}{30720} - \frac{t^8}{3072} - \frac{t^9}{4783} + \frac{t^{10}}{92160}.$$

The results are summarized in Table 2. The above problem has also been solved in [32] via the measure theory in which to find an acceptable solution, a linear programming problem with 1000 variables and 20 constraints should be solved. The performance index value is gotten $J = 0.2425$. Comparison of the results of the above example with that obtained in the corresponding reference, shows the efficiency of this algorithm more clearly.

Example 6.2: [11]

$$\min J = \frac{1}{2}x^2(1) + \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt,$$

$$\text{subject to} \begin{cases} \dot{x} = -2x(t) + u(t) \\ x(0) = 1, \ x(1) \text{ is free.} \end{cases}$$

The exact solution of $k(t)$ is

$$k(t) = \frac{\sqrt{5}\cosh\sqrt{5}(1-t) - \sinh\sqrt{5}(1-t)}{\sqrt{5}\cosh\sqrt{5}(1-t) + 3\sinh\sqrt{5}(1-t)}. \tag{6.6}$$

We solve the following IVP

$$\begin{cases} \dot{x} = -2x(t) - p(t), \\ \dot{p} = -x(t) + 2p(t), \\ x(0) = 1, \ p(0) = \alpha. \end{cases} \tag{6.7}$$

Implementing the DTM for (6.7) we have

$$\begin{cases} (k+1)X(k+1) = -2X(k) - P(k), \\ (k+1)P(k+1) = -X(k) + 2P(k), \\ X(0) = 1, \ P(0) = \alpha. \end{cases} \tag{6.8}$$

We select the tolerance error bound $\epsilon = 5\times 10^{-5}$ and get $|\frac{J^{(12)} - J^{(11)}}{J^{(12)}}| = 0.0000272704 < \epsilon$; where by employing the boundary condition $x^{12}(1) - p^{12}(1) = 0$, we have $\alpha = 0.243534$. Therefore using (2.15) we have

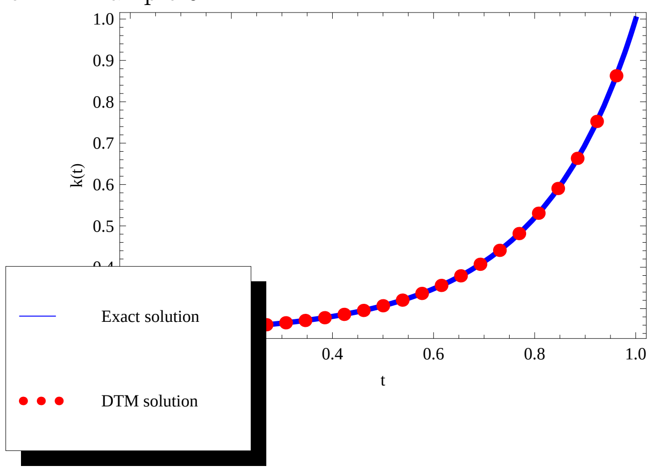
$$k(t) = \frac{\sum_{k=0}^{12} P(k)t^k}{\sum_{k=0}^{12} X(k)t^k}, \tag{6.9}$$

which is the approximate solution of $k(t)$. In Figure 1, the approximate value for $k(t)$ obtained from DTM with $N = 12$ and the following exact value of $k(t)$ are plotted. Table 3 gives the results of the DTM and the exact solution of this example and illustrates the relative errors.

TABLE 3. Simulation results of the proposed method in different iteration times for Example 6.2.

k	$X(k)$	$P(k)$	$J^{(k)}$	$\lvert \frac{J^{(k)}-J^{(k-1)}}{J^{(k)}} \rvert$
0.	1.	0.243534	1.02965	–
1.	-2.24353	-0.512933	1.00137	0.0282411
2.	2.5	0.608834	1.08208	0.0745874
3.	-1.86961	-0.427444	0.310381	2.48631
4.	1.04167	0.253681	0.224674	0.381468
5.	-0.467403	-0.106861	0.116038	0.936216
6.	0.173611	0.0422801	0.127557	0.0903023
7.	-0.0556432	-0.0127215	0.120514	0.0584406
8.	0.015501	0.00377501	0.122105	0.0130347
9.	-0.00386411	-0.000883441	0.121693	0.00339046

FIGURE 1. Comparison of the exact solution of $k(t)$ with the DTM solution in Example 6.2.



Example 6.3: [11]

$$\min\ J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt$$

$$\text{subject to} \begin{cases} \dot{x} = -x(t) + u(t), \\ x(0) = 1,\ x(1) \text{ is free .} \end{cases}$$

The exact solution of $k(t)$ is

$$k(t) = -\frac{(1+\sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2}+\beta)\sinh(\sqrt{2}t)}{\cosh(\sqrt{2}t) + \beta\sinh(\sqrt{2}t)}, \tag{6.10}$$

where

$$\beta = -\frac{\cosh(\sqrt{2}) + (\sqrt{2})\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})}. \tag{6.11}$$

We consider

$$\begin{cases} \dot{x} = -x(t) - p(t), \\ \dot{p} = -x(t) + p(t), \\ x(0) = 1, \; p(0) = \alpha. \end{cases} \tag{6.12}$$

Implementing the DTM for (6.12) we have

$$\begin{cases} (k+1)X(k+1) = -X(k) - P(k) \\ (k+1)P(k+1) = -X(k) + P(k), \\ X(0) = 1, \; P(0) = \alpha. \end{cases} \tag{6.13}$$

Table 4. Simulation results of the proposed method in different iteration times for Example 6.3.

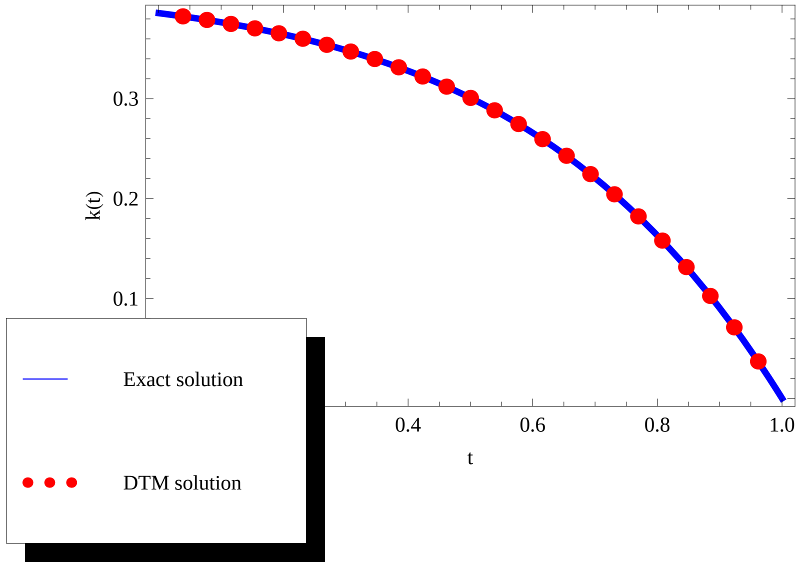
k	$X(k)$	$P(k)$	$J^{(k)}$	$\lvert\frac{J^{(k)}-J^{(k-1)}}{J^{(k)}}\rvert$
0.	1.	0.385819	0.574428	–
1.	-1.38582	-0.614181	0.145989	2.93473
2.	1.	0.385819	0.238132	0.386938
3.	-0.46194	-0.204727	0.184162	0.293056
4.	0.166667	0.0643032	0.195237	0.0567272
5.	-0.046194	-0.0204727	0.192453	0.0144678
6.	0.0111111	0.00428688	0.192989	0.00277637
7.	-0.00219971	-0.00097489	0.192897	0.000477353
8.	0.000396825	0.000153103	0.192911	0.0000746814
9.	-0.0000611031	-0.0000270803	0.192909	0.0000102555

We select the tolerance error bound $\epsilon = 5 \times 10^{-5}$ and acquire $\lvert\frac{J^{(9)}-J^{(8)}}{J^{(9)}}\rvert = 0.0000102555 < \epsilon$; where by employing $x^9(1) - p^9(1) = 0$, we get $\alpha = 0.385819$. Therefore we have

$$k(t) = \frac{\sum_{k=0}^{9} P(k)t^k}{\sum_{k=0}^{9} X(k)t^k}, \tag{6.14}$$

that is the approximate value of $k(t)$. In Figure 2, the approximate value for $k(t)$ obtained from DTM with $N = 9$ and the following exact value are shown. Table 4 gives the results of this example.

FIGURE 2. Comparison of the exact solution of $k(t)$ with the DTM solution in Example 6.3.



Example 6.4: [26]

$$\min\ J = \frac{1}{2}Sx^2(15) + \frac{1}{4}\int_0^{15} u^2(t)dt$$

$$\text{subject to} \begin{cases} \dot{x} = -0.2x(t) + u(t), \\ x(0) = 5,\ x(15) \text{ is free},\ S > 0. \end{cases}$$

The optimal control low is $u(t) = -2K(t)x(t)$ where

$$K(t) = [e^{0.2(t_f - t)} + \frac{S}{0.2}[e^{0.2(t_f-t)} - e^{-0.2(t_f-t)}]]^{-1}[Se^{-0.2(t_f-t)}].$$

From equation (9), we consider

$$\begin{cases} \dot{x}(t) = -0.2\ x(t) - 2\ p(t), \\ \dot{p}(t) = 0.2\ p(t), \\ x(0) = 1,\ p(0) = \alpha. \end{cases} \tag{6.15}$$

Employing the DTM for (6.15) we have

$$\begin{cases} X(k) = \frac{1}{k}(-0.2X(k-1) - 2P(k-1)), \\ P(k) = \frac{1}{k}(0.2P(k-1)), \\ X(0) = 1,\ P(0) = \alpha. \end{cases} \tag{6.16}$$

TABLE 5. Simulation results of the proposed method in different iteration times for Example 6.4.

k	$X(k)$	$P(k)$	$J^{(k)}$	$\vert\frac{J^{(k)}-J^{(k-1)}}{J^{(k)}}\vert$
0.	5.	0.00238911	12.5	–
1.	-1.00478	0.000477822	-25.179	1.49645
2.	0.1	0.0000477822	31.0713	1.81036
3.	-0.00669852	3.18548×10^{-6}	-25.4471	2.22101
4.	0.000333333	1.59274×10^{-7}	16.7407	2.52008
5.	-0.000013397	6.37096×10^{-9}	-8.69261	2.92585
6.	4.44444×10^{-7}	2.12365×10^{-10}	3.96373	3.19304
7.	-1.27591 10-8	6.06758×10^{-12}	-1.48626	3.66691
8.	3.1746×10^{-10}	1.5169×10^{-13}	0.547789	3.7132
9.	-7.08838×10^{-12}	3.37088×10^{-15}	-0.13346	5.10451
10.	1.41093×10^{-13}	6.74176×10^{-17}	0.0699449	2.90808
11.	-2.57759×10^{-15}	1.22577×10^{-18}	0.0142062	3.92353
12.	4.27556×10^{-17}	2.04296×10^{-20}	0.0280747	0.493985
13.	-6.60921×10^{-19}	3.14301×10^{-22}	0.0248591	0.129357
14.	9.39683×10^{-21}	4.49001×10^{-24}	0.0255449	0.0268471
15.	-1.2589×10^{-22}	5.98669×10^{-26}	0.025407	0.0054243
16.	1.56614×10^{-24}	7.48336×10^{-28}	0.0254328	0.0010112
17.	-1.85132×10^{-26}	8.80395×10^{-30}	0.0254282	0.000179331
18.	2.04724×10^{-28}	9.78217×10^{-32}	0.025429	0.0000297457
19.	-2.16529×10^{-30}	1.0297×10^{-33}	0.0254288	4.71913×10^{-6}
20.	2.15499×10^{-32}	1.0297×10^{-35}	0.0254289	7.04506×10^{-7}

Setting $\epsilon = 5 \times 10^{-5}$ we obtain $|\frac{J^{(20)}-J^{(19)}}{J^{(19)}}| = 7.04506 \times 10^{-7} < \epsilon$; where by using the boundary condition $Sx(t) - p(t) = 0$, we have

$$\begin{cases} \alpha = 0.00238911, & \text{for } S = 5, \\ \alpha = 0.00177368, & \text{for } S = 0.5, \\ \alpha = 0.000495996, & \text{for } S = 0.05. \end{cases} \tag{6.17}$$

Therefore

$$k(t) = \frac{\sum_{k=0}^{20} P(k)t^k}{\sum_{k=0}^{20} X(k)t^k}, \tag{6.18}$$

is the approximate solution of $k(t)$. Figure 3 shows the approximate value of $k(t)$ with $N = 20$ and $S = 5, 0.5$ and 0.05. The corresponding approximate optimal pair

$(x(.), u(.))^T$ are also shown in Figures 4 and 5. One can compare these approximate results with the exact results in [26] page 213.

To end this section, we answer a natural question: Are there advantages of the proposed collocation method compared to the existing ones? To answer this, we summarize what we have observed from numerical experiments and theoretical results as below.

• The technique that we used, which is based on Taylor series expansion enables us to obtain a series solution by means of an iterative procedure.

• The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms.

• A specific advantage of this method over any purely numerical method is that it offers a smooth, functional form of the solution over a time step.

• The other advantages of this method, compared to other analytic methods are controllable accuracy, and high efficiency which is exhibited by the rapid convergence of the solution.

• This method can be applied directly to differential equations without requiring linearization, discretization or perturbation.

• Large computational work and roundoff errors are avoided.

• All the calculations in the method are very easy.

FIGURE 3. The approximate solution of $k(t)$ with various values of S in Example 6.4.

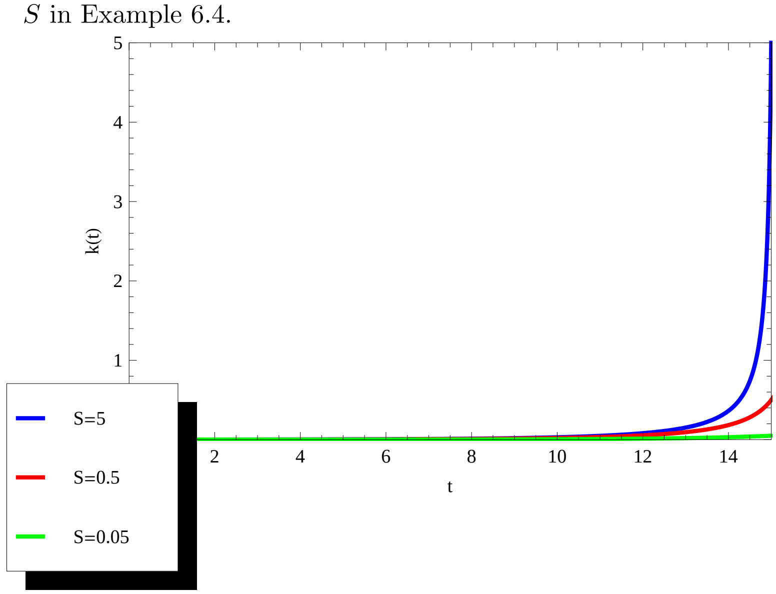


FIGURE 4. The approximate solution of $x(t)$ with various values of S in Example 6.4.

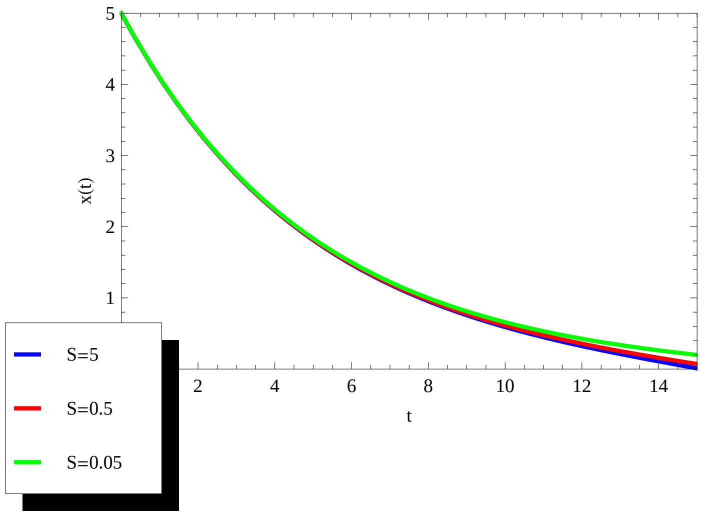
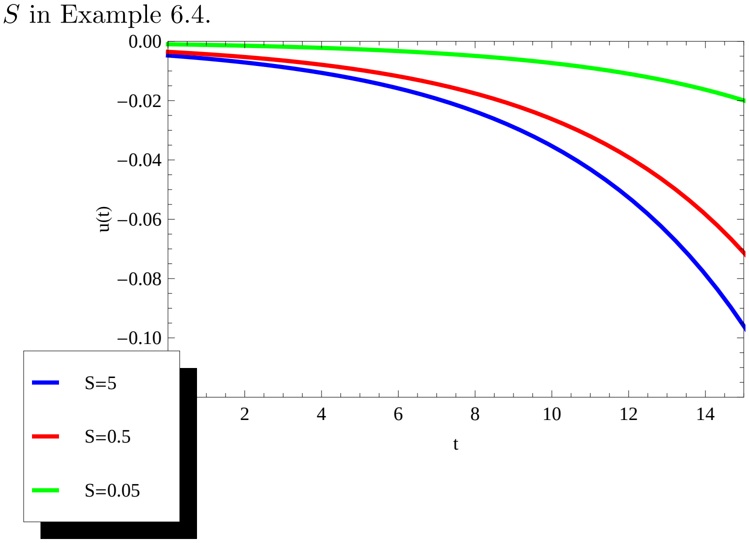


FIGURE 5. The approximate solution of $u(t)$ with various values of S in Example 6.4.



7. CONCLUSION

This paper presented a new analytical technique, called the DTM, for solving a class of linear and nonlinear OCPs. Despite the other approximate approaches such as SAA [33], ASRE [4], SDRE [9] and SGA [6], the suggested technique keeps away from solving a sequence of linear time-varying 2PBVPs or a sequence of matrix Riccati differential (or algebraic) equations or a sequence of generalized HJB equations.

Applying the DTM, the optimal control law and the corresponding optimal trajectory are determined in the form of rapid convergent series with easy computable terms. The present study has confirmed that the DTM offers great advantages of straightforward applicability, computational efficiency and high accuracy. The Differential transform method needs less work in comparison with the traditional methods. Therefore, this method can be applied to many complicated linear and non-linear problems and does not require linearization, discretization or perturbation. Moreover, in view of computational complexity, the proposed method is more practical than the other approximate approaches. Mathematica and Matlab have been used for computations and simulations in this paper.

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