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Explicit exact solutions for variable coefficient Broer–Kaup equations

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Abstract Based on symbolic manipulation program Maple and using Riccati equation mapping method several explicit exact solutions including kink, soliton-like, periodic and rational solutions are obtained for (2+1)-dimensional variable coefficient Broer–Kaup system in quite a straightforward manner. The known solutions of Riccati equation are used to construct new solutions for variable coefficient Broer–Kaup system.

Keywords. Broer–Kaup equations, Riccati equation mapping method, Explicit exact solutions.

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1. Introduction

Last two decades have witnessed considerable number of new methods for solving nonlinear partial differential equations including those with constant and variable coefficients as well. With advent of tanh method [22, 23] many known physically relevant PDEs were investigated for explicit exact solutions and the introduction of several symbolic manipulation programs like "Maple" and "Mathematica" greatly helped the researcher to perform complex algebraic calculations. The Jacobi elliptic function expansion method [20] followed by its generalization like the F-expansion method [33], and Exp-function method [10]. The two new methods appear to have some advantages over the first one in term of its simplicity and generalization. Beside these algebraic manipulations for solving nonlinear PDEs, there are much more sophisticated methods like Lie group method [2], inverse scattering transform [24], Hirota's direct method [11] for solving nonlinear PDEs.

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In the present paper we investigate variable coefficient (2+1)-dimensional Broer–Kaup equations using Riccati equation mapping method

$$\begin{aligned} u_{t,y} =& b(t)\left[u_{x,x,y} - 2\left(uu_x\right)_y - 2v_{x,x}\right], \\ v_t =& b(t)\left[-v_{x,x} - 2\left(uv\right)_x\right], \end{aligned} \tag{1.1}$$

where we have introduced the time-dependent function $b(t)$ in order to incorporate more realistic physical phenomena. If $b(t) = 1$, then system (1.1) reduces to Broer–Kaup system [21] and for $y = x$ it reduces to (1+1)–dimensional model which can be used to describe long waves in shallow water [31]. Moreover, using some transformations [4] (2+1)–dimensional Broer–Kaup equation can be transformed into (2+1)–dimensional dispersive long wave equation [3]. Recently the system (1.1) has been investigated for non-traveling wave solutions and fractal solitons [18] and many soliton-like solutions have also been obtained [29]. Through simple transformation $v = u_y$, the Broer–Kaup system (1.1) reduced to

$$u_{t,y} + b(t)\left[u_{x,x,y} + 2\left(uu_x\right)_y\right] = 0. \tag{1.2}$$

The Broer–Kaup equation (1.2) has been rigourously investigated using Painlevé and Lie group analysis [17], wherein under one dimensional optimal sub-algebra several dimensional reductions of (1.2) and some interesting periodic, kink and soliton like solutions have also been obtained.

After inception of Broer–kaup equation in Ref. [21], this equation has been firstly studied for integrability and infinite symmetries by Hann–Yu and Yi–Xin [9], Bäcklund transformations related to truncated Painlevé analysis have been used to construct rich variety of soliton solutions [12, 19, 25] and variety of other techniques are used to investigate constant coefficient Broer–Kaup equation [1, 5, 7, 14, 26, 27, 30]. Beside this, the variable coefficient Broer–Kaup equation (1.1) has been investigated for soliton fission and fusion phenomena [6, 8] and exact solutions have been constructed using several other techniques [13, 15, 28, 32].

Nevertheless, in present work we do not intend to outcast existing methodologies for finding exact solutions but in wake of Ref. [16], a same meromorphic structure of solutions is used in each method, so by "Eremenko" theorem these methods can not give new solutions or results. However, we believe that the Riccati equation mapping method is simplest of all methodologies presented so far and this method has never been applied to variable coefficient Broer–Kaup equation (1.1) for traveling wave solutions.

2. Riccati equation mapping method

In this section we would like introduce some basic steps of Riccati equation mapping method. We assume a solution of equation (1.2) in the form

$$u(x,t) = \sum_{i=0}^{n} a_i(t)\phi^i(\xi), \quad \xi = kx + ly + \int \tau(t)dt, \tag{2.1}$$

where n is required to be positive integer which can be fixed by balancing highest order linear derivative term with that of nonlinear derivative term in (1.2).

However, it is possible that this n may not be positive in all cases of PDEs but still those cases can be handled by making appropriate transformations in the given PDE. It is easy to see that the order of nonlinear term $u_x u_y$ or $u u_{x,y}$ in (1.2) is $2n+2$ and of highest order linear term $u_{x,x,y}$ is $n+3$, therefore on equating both these orders we obtain $n=1$.

The key point of this method is to take full advantage of Riccati equation by introducing new function $\phi(\xi)$ as solution of Riccati equation

$$\phi'(\xi) = r + p\phi(\xi) + q\phi^2(\xi), \tag{2.2}$$

r, p and q are real constants. In contrast to Ref. [34] where (2.2) is wrongly interpreted as generalised Riccati equation whereas in this manuscript we prefer calling it as Riccati equation only. Depending on the values of p, q and r the Riccati equation (2.2) admits several types of solutions [34].

Differentiating (2.2) twice w.r.t. ξ we get

$$\phi''(\xi) = pr + p^2\phi(\xi) + 3\,pq\phi(\xi)^2 + 2\,q\phi(\xi)\,r + 2\,q^2\phi(\xi)^3, \tag{2.3}$$

$$\begin{aligned}\phi'''(\xi) = {} & p^2 r + p^3\phi(\xi) + 7\,p^2 q\phi(\xi)^2 + 8\,pq\phi(\xi)\,r + 12\,pq^2\phi(\xi)^3 + 2\,qr^2 \\ & + 8\,q^2 r\phi(\xi)^2 + 6\,q^3\phi(\xi)^4 .\end{aligned} \tag{2.4}$$

Using repeated derivatives (2.2), (2.3) and (2.4) in (2.1) and keeping in mind that $n=1$, we have

$$u_x = a_1(t)\,qk\phi(\xi)^2 + a_1(t)\,pk\phi(\xi) + a_1(t)\,rk, \tag{2.5a}$$

$$u_y = a_1(t)\,ql\phi(\xi)^2 + a_1(t)\,pl\phi(\xi) + a_1(t)\,rl, \tag{2.5b}$$

$$\begin{aligned}u_{t,y} = {} & 2\,a_1(t)\,q^2 l\tau(t)\,\phi(\xi)^3 + \left(3\,a_1(t)\,pql\tau(t) + a_1'(t)\,ql\right)\phi(\xi)^2 \\ & + \left(a_1'(t)\,pl + a_1(t)\left(p^2 + 2\,qr\right)l\tau(t)\right)\phi(\xi) \\ & + a_1(t)\,prl\tau(t) + a_1'(t)\,rl,\end{aligned} \tag{2.5c}$$

$$\begin{aligned}u_{x,y} = {} & 2\,a_1(t)\,q^2 lk\phi(\xi)^3 + 3\,a_1(t)\,pqlk\phi(\xi)^2 \\ & + a_1(t)\left(p^2 + 2\,qr\right)lk\phi(\xi) + a_1(t)\,prlk,\end{aligned} \tag{2.5d}$$

$$\begin{aligned}u_{x,x,y} = {} & 6\,a_1(t)\,q^3 lk^2\phi(\xi)^4 + 12\,a_1(t)\,pq^2 lk^2\phi(\xi)^3 \\ & + a_1(t)\left(3\,p^2 q + 4\,q\left(p^2 + 2\,qr\right)\right)lk^2\phi(\xi)^2, \\ & + pa_1(t)\left(p^2 + 8\,qr\right)lk^2\phi(\xi) + ra_1(t)\left(p^2 + 2\,qr\right)lk^2.\end{aligned} \tag{2.5e}$$

On substituting (2.1) and (2.5) in (1.2) and equating coefficients of powers of $\phi(\xi)$ to zero yields the following set of algebraic equations:

$$6\,b(t)\,a_1(t)\,k^2 lq^3 + 6\,b(t)\,a_1(t)^2\,klq^2 = 0, \tag{2.6a}$$

$$\begin{aligned}& 12\,b(t)\,a_1(t)\,k^2 lpq^2 + 10\,b(t)\,a_1(t)^2\,klpq + 4\,b(t)\,a_1(t)\,a_0(t)\,klq^2 \\ & - 2\,a_1(t)\,\tau(t)\,lq^2 = 0,\end{aligned} \tag{2.6b}$$

$$\begin{aligned}& 7\,b(t)\,a_1(t)\,k^2 lp^2 q + 8\,b(t)\,a_1(t)\,k^2 lq^2 r + 4\,b(t)\,a_1(t)^2\,klp^2 \\ & + 8\,b(t)\,a_1(t)^2\,klqr + 6\,b(t)\,a_1(t)\,a_0(t)\,klpq\end{aligned}$$

$$-3\,a_1\,(t)\,\tau\,(t)\,lpq - a_1'\,(t)\,lq = 0, \tag{2.6c}$$

$$b\,(t)\,a_1\,(t)\,k^2lp^3 + 8\,b\,(t)\,a_1\,(t)\,k^2lpqr + 6\,b\,(t)\,a_1\,(t)^2\,klpr$$
$$+\,2\,b\,(t)\,a_1\,(t)\,a_0\,(t)\,klp^2 + 4\,b\,(t)\,a_1\,(t)\,a_0\,(t)\,klqr$$
$$-\,a_1\,(t)\,\tau\,(t)\,lp^2 - 2\,a_1\,(t)\,\tau\,(t)\,lqr - a_1'\,(t)\,lp = 0, \tag{2.6d}$$

$$b\,(t)\,a_1\,(t)\,lk^2p^2r + 2\,b\,(t)\,a_1\,(t)\,lk^2qr^2 + 2\,b\,(t)\,a_1\,(t)^2\,klr^2$$
$$+\,2\,b\,(t)\,a_1\,(t)\,lka_0\,(t)\,pr$$
$$-\,a_1\,(t)\,l\tau\,(t)\,pr - a_1'\,(t)\,lr = 0. \tag{2.6e}$$

Solving algebraic equations (2.6), we obtain:

$$a_0(t) = \frac{-b\,(t)\,k^2p + \tau\,(t)}{2b\,(t)\,k}, \quad a_1(t) = -kq, \tag{2.7}$$

where $\tau(t)$ are arbitrary function.

Based on different values of p, q and r the Riccati equation (2.2) admits 27 kink, soliton like and periodic solutions as mentioned in Ref. [34], those solutions are rewritten hereinafter correcting several typing errors. Hence, via expression (2.1) the variable coefficient Broer-Kaup equation (1.2) admits abundant new exact solutions:

Case 2.1. When $\delta = p^2 - 4pq > 0$ and $pq \neq 0$, then (1.2) admits kink and soliton like solutions

$$u_1 = \frac{\tau\,(t)}{2kb\,(t)} + \frac{k}{2}\left[\sqrt{\delta}\tanh\left(\frac{\sqrt{\delta}}{2}\,\xi\right)\right],$$

$$u_2 = \frac{\tau\,(t)}{2kb\,(t)} + \frac{k}{2}\left[\sqrt{\delta}\coth\left(\frac{\sqrt{\delta}}{2}\,\xi\right)\right],$$

$$u_3 = \frac{\tau\,(t)}{2kb\,(t)} + \frac{\sqrt{\delta}k}{2}\left[\tanh\left(\sqrt{\delta}\,\xi\right) \pm \mathrm{i}\,\mathrm{sech}\left(\sqrt{\delta}\xi\right)\right],$$

$$u_4 = \frac{\tau\,(t)}{2kb\,(t)} + \frac{\sqrt{\delta}k}{2}\left[\coth\left(\sqrt{\delta}\,\xi\right) \pm \mathrm{i}\,\mathrm{csch}\left(\sqrt{\delta}\xi\right)\right],$$

$$u_5 = \frac{\tau\,(t)}{2kb\,(t)} + \frac{\sqrt{\delta}k}{4}\left[\tanh\left(\frac{\sqrt{\delta}}{4}\,\xi\right) + \mathrm{i}\,\coth\left(\frac{\sqrt{\delta}}{4}\xi\right)\right],$$

$$u_6 = \frac{\tau\,(t)}{2kb\,(t)} - \frac{k}{2}\left[\frac{\sqrt{(A^2+B^2)\,(\delta)} - A\sqrt{\delta}\cosh\left(\sqrt{\delta}\xi\right)}{A\sinh\left(\sqrt{\delta}\xi\right) + B}\right],$$

$$u_7 = \frac{\tau\,(t)}{2kb\,(t)} + \frac{k}{2}\left[\frac{\sqrt{(B^2-A^2)\,(\delta)} + A\sqrt{\delta}\sinh\left(\sqrt{\delta}\xi\right)}{A\cosh\left(\sqrt{\delta}\xi\right) + B}\right],$$

where constants A and B are such that $B^2 - A^2 > 0$.

$$u_8 = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} - \frac{2qrk\cosh\left(\frac{\sqrt{\delta}}{2}\xi\right)}{\sqrt{\delta}\sinh\left(\frac{\sqrt{\delta}}{2}\xi\right) - p\cosh\left(\frac{\sqrt{\delta}}{2}\xi\right)},$$

$$u_9 = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} + \frac{2qrk\sinh\left(\frac{\sqrt{\delta}}{2}\xi\right)}{p\sinh\left(\frac{\sqrt{\delta}}{2}\xi\right) - \sqrt{\delta}\cosh\left(\frac{\sqrt{\delta}}{2}\xi\right)},$$

$$u_{10} = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} - \frac{2qrk\cosh\left(\sqrt{\delta}\xi\right)}{\sqrt{\delta}\sinh\left(\sqrt{\delta}\xi\right) - p\cosh\left(\sqrt{\delta}\xi\right) \pm \mathrm{i}\sqrt{\delta}},$$

$$u_{11} = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} - \frac{2qrk\sinh\left(\sqrt{\delta}\xi\right)}{-p\sinh\left(\sqrt{\delta}\xi\right) + \sqrt{\delta}\cosh\left(\sqrt{\delta}\xi\right) \pm \sqrt{\delta}},$$

$$u_{12} = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} - \frac{4qrk\sinh\left(\frac{\sqrt{\delta}}{4}\xi\right)\cosh\left(\frac{\sqrt{\delta}}{4}\xi\right)}{-2p\sinh\left(\frac{\sqrt{\delta}}{4}\xi\right)\cosh\left(\frac{\sqrt{\delta}}{4}\xi\right) + 2\sqrt{\delta}\cosh^2\left(\frac{\sqrt{\delta}}{4}\xi\right) - \sqrt{\delta}},$$

where $\xi = kx + ly + \int \tau(t)\,dt$ and $\tau(t)$ is arbitrary function.

Case 2.2. When $p^2 - 4qr < 0$ and $pq \neq 0$, then (1.2) admits periodic solutions

$$u_{13} = \frac{\tau(t)}{2kb(t)} - \frac{k\sqrt{-\delta}}{2}\tan\left(\frac{\sqrt{-\delta}}{2}\xi\right),$$

$$u_{14} = \frac{\tau(t)}{2kb(t)} + \frac{k\sqrt{-\delta}}{2}\cot\left(\frac{\sqrt{-\delta}}{2}\xi\right),$$

$$u_{15} = \frac{\tau(t)}{2kb(t)} - \frac{k\sqrt{-\delta}}{2}\left[\tan\left(\sqrt{-\delta}\xi\right) \pm \sec\left(\sqrt{-\delta}\xi\right)\right],$$

$$u_{16} = \frac{\tau(t)}{2kb(t)} + \frac{k\sqrt{-\delta}}{2}\left[\cot\left(\sqrt{-\delta}\xi\right) \pm \csc\left(\sqrt{-\delta}\xi\right)\right],$$

$$u_{17} = \frac{\tau(t)}{2kb(t)} - \frac{k\sqrt{-\delta}}{4}\left[\tan\left(\frac{\sqrt{-\delta}}{4}\xi\right) - \cot\left(\frac{\sqrt{-\delta}}{4}\xi\right)\right],$$

$$u_{18} = \frac{\tau(t)}{2kb(t)} - \frac{k}{2}\left[\frac{\sqrt{(A^2 - B^2)(-\delta)} - A\sqrt{-\delta}\cos\left(\sqrt{-\delta}\xi\right)}{A\sin\left(\sqrt{-\delta}\xi\right) + B}\right],$$

$$u_{19} = \frac{\tau(t)}{2kb(t)} + \frac{k}{2}\left[\frac{\sqrt{(A^2 - B^2)(-\delta)} + A\sqrt{-\delta}\cos\left(\sqrt{-\delta}\xi\right)}{A\sin\left(\sqrt{-\delta}\xi\right) + B}\right],$$

where constants A and B are such that $A^2 - B^2 > 0$.

$$u_{20} = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} + \frac{2qrk\cos\left(\frac{\sqrt{-\delta}}{2}\xi\right)}{\sqrt{-\delta}\sin\left(\frac{\sqrt{-\delta}}{2}\xi\right) + p\cos\left(\frac{\sqrt{-\delta}}{2}\xi\right)},$$

$$u_{21} = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} - \frac{2qrk\sin\left(\frac{\sqrt{-\delta}}{2}\xi\right)}{-p\sin\left(\frac{\sqrt{-\delta}}{2}\xi\right) + \sqrt{-\delta}\cos\left(\frac{\sqrt{-\delta}}{2}\xi\right)},$$

$$u_{22} = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} + \frac{2qrk\cos\left(\sqrt{-\delta}\xi\right)}{\sqrt{-\delta}\sin\left(\sqrt{-\delta}\xi\right) + p\cos\left(\sqrt{-\delta}\xi\right) \pm \sqrt{-\delta}},$$

$$u_{23} = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} - \frac{2qrk\sin\left(\sqrt{-\delta}\xi\right)}{-p\sin\left(\sqrt{-\delta}\xi\right) + \sqrt{-\delta}\cos\left(\sqrt{-\delta}\xi\right) \pm \sqrt{-\delta}},$$

$$u_{24} = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} - \frac{4qrk\sin\left(\frac{\sqrt{-\delta}}{4}\xi\right)\cos\left(\frac{\sqrt{-\delta}}{4}\xi\right)}{-2p\sin\left(\frac{\sqrt{-\delta}}{4}\xi\right)\cos\left(\frac{\sqrt{-\delta}}{4}\xi\right) + 2\sqrt{-\delta}\cos^2\left(\frac{\sqrt{-\delta}}{4}\xi\right) - \sqrt{-\delta}},$$

where $\xi = kx + ly + \int \tau(t)\,dt$ and $\tau(t)$ is arbitrary function.

Case 2.3. When $r = 0$ and $pq \neq 0$, then equation (1.2) admits the following solutions

$$u_{25} = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} + \frac{pd_1k}{d_1 + \cosh(p\xi) - \sinh(p\xi)},$$

$$u_{26} = \frac{\tau(t)}{2kb(t)} - \frac{kp}{2} + \frac{pk(\cosh(p\xi) + \sinh(p\xi))}{d_1 + \cosh(p\xi) + \sinh(p\xi)},$$

where $\xi = kx + ly + \int \tau(t)\,dt$ and $\tau(t)$ is arbitrary function.

Case 2.4. When $q \neq 0$ and $r = p = 0$, then equation (1.2) admits a rational solution

$$u_{27} = \frac{\tau(t)}{2kb(t)} + \frac{qk}{q\xi + d_2}, \tag{2.8}$$

where $\xi = kx + ly + \int \tau(t)\,dt$ and $\tau(t)$ is arbitrary function. Further, we want to remark here that the rational solution (2.8) bears similarity with rational solution reported in [17].

By invoking the transformation $v = u_y$, we have been able to construct several new explicit exact solutions for variable coefficient (2+1)-dimensional Broer–Kaup equations (1.1). Besides enriching the solution structure of Broer–Kaup system (1.1) we have demonstrated the effectiveness of Riccati equation mapping method.

3. Conclusion

The variable coefficient Broer–Kaup system (1.1) which has been obtained as integrable system by virtue of symmetry constraints of Kadomtsev–Petviashvili equation has many physical applications, including those in nonlinear optics, plasma physics, statistical physics, we have consider its variable coefficient version in order to incorporate more realistic physical phenomena. The Riccati equation mapping method has been successfully applied to Broer–Kaup system to construct several explicit exact solutions. To best of our knowledge these solutions have never been reported before and we believe that these solutions might add new meanings to understandings of the underlying physical phenomena in a better way.

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