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Solving high-order partial differential equations in unbounded domains by means of double exponential second kind Chebyshev approximation

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Abstract In this paper, a collocation method for solving high-order linear partial differential equations (PDEs) with variable coefficients under more general form of conditions is presented. This method is based on the approximation of the truncated double exponential second kind Chebyshev (ESC) series. The definition of the partial derivative is presented and derived as new operational matrices of derivatives. All principles and properties of the ESC functions are derived and introduced by us as a new basis defined in the whole range. The method transforms the PDEs and conditions into block matrix equations, which correspond to system of linear algebraic equations with unknown ESC coefficients, by using ESC collocation points. Combining these matrix equations and then solving the system yield the ESC coefficients of the solution function. Numerical examples are included to test the validity and applicability of the method.

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1. Introduction

It is well known that the numerical methods have played an important role in solving PDEs. Some of the well-known numerical methods are finite differences and finite element methods [6,17]. Recently, various approximate methods were discussed, such as differential transform method, Adomian decomposition method and Homotopy analysis method [3,8,11,13,15,18]. Furthermore, spectral methods are one of the principal methods for solving differential equations. The main idea of spectral methods is to approximate the solutions of differential equations by means of truncated series of orthogonal polynomials. The most used versions of spectral methods are tau, collocation, and Galerkin methods [1,5,9]. One of the most important orthogonal polynomials is Chebyshev polynomials. The well-known Chebyshev polynomials of the second kind $U_n(x)$ [12] are orthogonal with respect to the weight-function $w(x) = \sqrt{1-x^2}$ on the interval $[-1, 1]$, and the recurrence relation is

$$U_0(x) = 1,\ U_1(x) = 2x,\ U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad n \geq 1.$$

Many studies of Chebyshev polynomials are considered on the first kind $T_n(x)$. On the other hand, in [10] and [14] a modified type of Chebyshev polynomials was proposed as new alternative technique to the solutions of ordinary and partial differential equations given in whole domain. In their studies, the basis functions called exponential Chebyshev functions, which are orthogonal on $(-\infty, \infty)$ and are defined by

$$E_n(x) = T_n\left(\frac{e^x - 1}{e^x + 1}\right),$$

where $T_n(x)$ is the first kind Chebyshev polynomials. This kind of extension tackles the problems over the whole real domain. In this paper we introduce a new type of Chebyshev polynomials in the whole real range, called exponential second kind Chebyshev (ESC) functions. The rest of the paper is organized as follows. In section 2, the definition and properties of ESC functions are listed, while in section 3 the form of high-order linear non-homogeneous partial differential equations is presented. In section 4, we formulated the fundamental matrix relation based on collocation points. In section 5, method of solution is presented. Finally, section 6 contains numerical illustrations and results that are compared with the exact solutions to demonstrate the applicability and accuracy of the present method.

2. Properties of double ESC functions

We introduce the definition of expotennial second kind Chebyshev ESC to be of the form

$$E_n^U(x) = U_n\left(\frac{e^x-1}{e^x+1}\right), \tag{2.1}$$

where the corresponding recurrencs relation is

$$E_0^U(x) = 1, \;\; E_1^U(x) = 2\left(\frac{e^x-1}{e^x+1}\right),$$

$$E_{n+1}^U(x) = 2\left(\frac{e^x-1}{e^x+1}\right)E_n^U(x) - E_{n-1}^U(x).$$

In Basu [4], the expression $T_{r,s}(x,y) = T_r(x)\cdot T_s(y)$ has given which is a form of Chebyshev polynomials. Mason et al. [12] have also used double Chebyshev polynomials expression for an infinitely differentiable function $u(x,y)$ defined on the square $S(-\infty < x, y < \infty)$, where $T_r(x)$ and $T_s(y)$ are Chebyshev polynomials of the first kind. Now we employ our new definition $E_n^U(x)$ to double form.

Definition 2.1. The double ESC functions are in the following form

$$E_{r,s}^U(x,y) = E_r^U(x)\cdot E_s^U(y), \tag{2.2}$$

and the recurrence relation takes the form

$$\begin{aligned} E_{r+1,s}^U(x,y) &= \left\{2\left(\tfrac{e^x-1}{e^x+1}\right)E_r^U(x) - E_{r-1}^U(x)\right\}\cdot E_s^U(y), \qquad r\geq 1,\\ E_{r,s+1}^U(x,y) &= E_r^U(x)\cdot\left\{2\left(\tfrac{e^y-1}{e^y+1}\right)E_s^U(y) - E_{s-1}^U(y)\right\}, \qquad s\geq 1. \end{aligned} \tag{2.3}$$

2.1. **Orthogonality of double ESC functions.** The functions $E_{r,s}^U(x,y)$ are orthogonal with respect to the weight function

$$w(x,y) = 4e^{\frac{3}{2}(x+y)}\left(e^x+1\right)^{-3}\left(e^y+1\right)^{-3},$$

with the orthognality condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{i,j}^U(x,y)E_{k,l}^U(x,y)w(x,y)\,dx\,dy = \begin{cases} \dfrac{\pi^2}{4}, & i=j \text{ and } k=l,\\ 0, & \text{otherwise.}\end{cases} \tag{2.4}$$

Also the product relation of double ESC functions used in the partial derivatives relations is given by

$$\left(\frac{e^x-1}{e^x+1}\right)\left(\frac{e^y-1}{e^y+1}\right)E^U_{m,n}(x,y)=\frac{1}{4}[E^U_{m+1,n+1}(x,y)+E^U_{m+1,n-1}(x,y)$$
$$+E^U_{m-1,n+1}(x,y)+E^U_{m-1,n-1}(x,y)].$$

2.2. **Function expansion in terms of double ESC functions.** A function $u(x,y)$ well defined over the square $S(-\infty<x,\ y<\infty)$, can be expanded as

$$u(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a_{r,s}E^U_{r,s}(x,y), \tag{2.5}$$

where

$$a_{r,s}=\frac{4}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(x,y)E^U_{r,s}(x,y)w(x,y)dxdy.$$

If $u(x,y)$ in expression (2.5) is truncated to $n,\ m<\infty$ in terms of the double ESC functions then, it will takes the form

$$U(x,y)=\sum_{r=0}^{m}\sum_{s=0}^{n}a_{r,s}E^U_{r,s}(x,y),=\mathbf{E}(x,y)\cdot\mathbf{A}, \tag{2.6}$$

where $\mathbf{E}(x,y)$ is $1\times(m+1)(n+1)$ vector with elements $E^U_{r,s}(x,y)$ and $\mathbf{A}$ is an unknown coefficient column vector, where

$$\mathbf{E}(x,y)=[E^U_{0,0}(x,y)\ \ E^U_{0,1}(x,y)\ \ ...\ \ E^U_{0,n}(x,y)\ E^U_{1,0}(x,y)\ \ E^U_{1,1}(x,y)\ \ ...$$
$$E^U_{1,n}(x,y)\ .....\ E^U_{m,0}(x,y)\ \ E^U_{m,1}(x,y)\ \ ...\ \ E^U_{m,n}(x,y)], \tag{2.7}$$

and

$$\mathbf{A}=[a_{0,0}\ \ a_{0,1}\ \ ...\ \ a_{0,n}\ \ a_{1,0}\ \ a_{1,1}\ \ ...\ \ a_{1,n}\ \ .....\ \ a_{m,0}\ \ a_{m,1}\ \ ....\ \ a_{m,n}]^T. \tag{2.8}$$

2.3. **The partial derivatives of double ESC functions.** The operational matrices of derivatives of the double ESC functions are given in the next proposition

Proposition 2.2. *The relation between the row vector* $\boldsymbol{E}(x,y)$ *and its (i,j)th-order partial derivative is given as*

$$\boldsymbol{E}^{(i,j)}(x,y)=\boldsymbol{E}(x,y)(\boldsymbol{D}_x)^i(\boldsymbol{D}_y)^j, \tag{2.9}$$

where, $\boldsymbol{D}_x$ and $\boldsymbol{D}_y$ are the $(m+1)(n+1)\times(m+1)(n+1)$ operational matrices for the partial derivatives, and the general form of them is

$$\boldsymbol{D}_x = diag\left(\left(\frac{\alpha}{4}+\frac{1}{2}\gamma_\alpha\right)\boldsymbol{I},\ \ \boldsymbol{0},\ \ \frac{-\alpha}{4}\boldsymbol{I}\right)^T, \qquad \alpha = 0,\ 1,\ ...,\ m,\ \ \gamma_\alpha = \begin{cases} 0, & \text{if } \alpha = 0,\\ 1, & \text{if } \alpha \neq 0,\end{cases} \tag{2.10}$$

and

$$\boldsymbol{D}_y = \begin{bmatrix} \boldsymbol{\eta} & 0 & \cdots & 0\\ 0 & \boldsymbol{\eta} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \boldsymbol{\eta}\end{bmatrix}^T, \quad \boldsymbol{\eta} = diag\left(\frac{\beta}{4}+\frac{1}{2}\gamma_\beta,\ 0,\ \frac{-\beta}{4}\right), \qquad \beta = 0,\ 1,\ ...,\ n,\quad \gamma_\beta = \begin{cases} 0, & \text{if } \beta = 0,\\ 1, & \text{if } \beta \neq 0,\end{cases} \tag{2.11}$$

where $\boldsymbol{I}$ and $\boldsymbol{0}$ are $(n+1)\times(n+1)$ identity and zero matrices in the block matrix $\boldsymbol{D}_x$ which is $(m+1)\times(m+1)$. Also $\boldsymbol{\eta}$ is the matrix of $\boldsymbol{(n+1)\times(n+1)}$ in the block matrix $\boldsymbol{D_y}$ which is $\boldsymbol{(m+1)\times(m+1)}$.

Proof. The partial derivatives of the ESC functions can be obtained by differentiating relation (2.3), first with respect to the variable x, and with the help of (2.6) we get

$$\frac{\partial}{\partial x}\left(E^U_{0,s}(x,y)\right) = 0,\ \text{for all } s, \tag{2.12}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left(E^U_{1,s}(x,y)\right) &= \tfrac{4e^x}{(1+e^x)^2}E^U_s(y) = \left(\tfrac{3}{4}E^U_0(x) - \tfrac{1}{4}E^U_2(x)\right)E^U_s(y)\\ &= \tfrac{3}{4}E^U_{0,s}(x,y) - \tfrac{1}{4}E^U_{2,s}(x,y),\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\left(E^U_{r+1,s}(x,y)\right) &= \frac{\partial}{\partial x}\left[2E^U_{1,s}(x,y)E^U_{r,s}(x,y) - E^U_{r-1,s}(x,y)\right]\\ &= \frac{\partial}{\partial x}\left[2\left(E^U_{1,s}(x,y)\right)^{(0,0)}\left(E^U_{r,s}(x,y)\right)^{(0,0)} - \left(E^U_{r-1,s}(x,y)\right)^{(0,0)}\right]\\ &= [2\left(E^U_{1,s}(x,y)\right)^{(1,0)}\left(E^U_{r,s}(x,y)\right)^{(0,0)} + 2\left(E^U_{1,s}(x,y)\right)^{(0,0)}\left(E^U_{r,s}(x,y)\right)^{(1,0)}\\ &\quad - \left(E^U_{r-1,s}(x,y)\right)^{(1,0)}],\end{aligned} \tag{2.14}$$

by using the relations (2.12)-(2.14) and with the help of product relation for $r = 0, 1, ..., m$, the elements of the matrix of dersvatives $\mathbf{D}_x$ can be obtained from the following equalities

$$\begin{cases} \left(E^U_{0,s}(x,y)\right)^{(1,0)} = 0, \\ \left(E^U_{1,s}(x,y)\right)^{(1,0)} = \frac{3}{4}E^U_{0,s}(x,y) - \frac{1}{4}E^U_{2,s}(x,y), \\ \left(E^U_{2,s}(x,y)\right)^{(1,0)} = E^U_{1,s}(x,y) - \frac{1}{2}E^U_{3,s}(x,y), \\ \quad\vdots \\ \left(E^U_{r,s}(x,y)\right)^{(1,0)} = \frac{r+2}{4}E_{r-1,s}(x,y) - \frac{r}{4}E_{r+1,s}(x,y), \quad r > 1, \text{ for all } s. \end{cases} \tag{2.15}$$

Similarly, we get the partial derivative with respect to the variable y as

$$\begin{cases} \left(E^U_{r,0}(x,y)\right)^{(0,1)} = 0, \\ \left(E^U_{r,1}(x,y)\right)^{(0,1)} = \frac{3}{4}E^U_{r,0}(x,y) - \frac{1}{4}E^U_{r,2}(x,y), \\ \left(E^U_{r,2}(x,y)\right)^{(0,1)} = E^U_{r,1}(x,y) - \frac{1}{2}E^U_{r,3}(x,y), \\ \quad\vdots \\ \left(E^U_{r,s}(x,y)\right)^{(0,1)} = \frac{s+2}{4}E_{r,s-1}(x,y) - \frac{s}{4}E_{r,s+1}(x,y), \quad s > 1, \text{ for all } r. \end{cases} \tag{2.16}$$

Then, the previous equalities (2.15),(2.16) form $(m+1)(n+1) \times (m+1)(n+1)$ two operational matrices $\mathbf{D}_x$ and $\mathbf{D}_y$ by our consideration that

$$\left(E^U_{r,s}(x,y)\right)^{(1,0)} = \left(E^U_{r,s}(x,y)\right)^{(0,1)} = \left(E^U_{r,s}(x,y)\right)^{(0,0)} = 0, \text{ for } r > m \text{ and } s > n.$$

Thus, to obtain the matrix $\mathbf{E}^{(i,j)}(x,y)$ in terms of $\mathbf{E}(x,y)$, we can use the relation (2.15), (2.16) as

$$\begin{aligned} \mathbf{E}^{(1,0)}(x,y) &= \mathbf{E}(x,y)\mathbf{D}_x, \\ \mathbf{E}^{(2,0)}(x,y) &= \mathbf{E}^{(1,0)}(x,y)\mathbf{D}_x = \left(\mathbf{E}(x,y)\mathbf{D}_x\right)\mathbf{D}_x = \mathbf{E}(x,y)\left(\mathbf{D}_x\right)^2, \\ \mathbf{E}^{(3,0)}(x,y) &= \mathbf{E}^{(1,0)}(x,y)\left(\mathbf{D}_x,\right)^2 = \mathbf{E}(x,y)\left(\mathbf{D}_x\right)^3. \end{aligned}$$

Therefore, by induction we can write

$$\mathbf{E}^{(i,0)}(x,y) = \mathbf{E}(x,y)\left(\mathbf{D}_x\right)^i, \tag{2.17}$$

similarly

$$\mathbf{E}^{(0,j)}(x,y) = \mathbf{E}(x,y)\left(\mathbf{D}_y\right)^j, \tag{2.18}$$

hence, by using (2.17), (2.18) we get finally

$$\mathbf{E}^{(i,j)}(x,y) = \mathbf{E}(x,y)\left(\mathbf{D}_x\right)^i\left(\mathbf{D}_y\right)^j, \tag{2.19}$$

which end the proof. □

3. Application of the introduced partial derivatives for high-order PDEs

The forms of high-order linear non-homogeneous partial differential equations with variable coefficients in unbounded domains are

$$\sum_{i=0}^{p}\sum_{j=0}^{r} q_{i,j}(x,y)\, u^{(i,j)}(x,y) = f(x,y), -\infty < x,\ y < \infty, \tag{3.1}$$

with the non-local conditions [7], [16]

$$\sum_{t=1}^{\rho}\sum_{i=0}^{p}\sum_{j=0}^{r} b_{i,j}^{t} u^{(i,j)}\left(\omega_t, \eta_t\right) = \lambda\,,$$

and / or

$$\sum_{t=1}^{\nu}\sum_{i=0}^{p}\sum_{j=0}^{r} c_{i,j}^{t}(x) u^{(i,j)}\left(x, \gamma_t\right) = g(x)\,, \tag{3.2}$$

and / or

$$\sum_{t=1}^{\theta}\sum_{i=0}^{p}\sum_{j=0}^{r} d_{i,j}^{t}(y)\, u^{(i,j)}\left(\varepsilon_t, y\right) = h(y)\,,$$

where the $u^{(0,0)}(x,y) = u(x,y)$, $u^{(i,j)}(x,y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} u(x,y)$ and $q_{i,j}(x,y)$, $f(x,y)$, $c_{i,j}^t(x)$, $g(x)$, $d_{i,j}^t(y)$ and $h(y)$ are known functions on the square $S\,(-\infty < x,\ y < \infty,)$, and $\omega_t, \eta_t,\ \gamma_t,\ \varepsilon_t$ are constants $\in\ (-\infty, \infty)$ and may be one or more of them tends to infinity. Now, we consider that the approximate solution $U(x,y)$ to the exact solution $u(x,y)$ of Eq. (3.1) defined by expression (2.6) and its (i,j)-th partial derivatives defined by Eq. (2.19) as

$$U(x,y) = \sum_{r=0}^{m}\sum_{s=0}^{n} a_{r,s} E_{r,s}(x,y) = \mathbf{E}(x,y)\cdot\mathbf{A}, \tag{3.3}$$

and

$$U^{(i,j)}(x,y) = [\mathbf{E}(x,y)(\mathbf{D}_x)^i(\mathbf{D}_y)^j]\mathbf{A}. \tag{3.4}$$

4. Fundamental matrix relations

Let us define the collocation points [10,14], so that $-\infty < x_i,\, y_i < \infty$, as

$$\begin{aligned} &x_k = Ln\left(\frac{1+\cos\left(\frac{k\pi}{m}\right)}{1-\cos\left(\frac{k\pi}{m}\right)}\right), \quad y_l = Ln\left(\frac{1+\cos\left(\frac{l\pi}{n}\right)}{1-\cos\left(\frac{l\pi}{n}\right)}\right), \\ &(k=1,\ ...,\ m-1,\ l=1,\ ...,\ n-1) \end{aligned} \tag{4.1}$$

and at the boundaries

$$(k=0, k=m)\ x_0 \to \infty,\ x_m \to -\infty, (l=0, l=n)\ y_0 \to \infty,\ y_n \to -\infty.$$

Since the double ESC functions are convergent at both boundaries $\pm\infty$, then the appearance of infinity in the collocation points does not cause a loss or divergence in the method. Now, we substitute the collocation points (4.1) into Eq. (3.1) to obtain

$$\sum_{i=0}^{p}\sum_{j=0}^{r} q_{i,j}(x_k, y_l)\, u^{(i,j)}(x_k, y_l) = f(x_k, y_l), \tag{4.2}$$

the system (4.2) can be written in the matrix form

$$\sum_{i=0}^{p}\sum_{j=0}^{r} \mathbf{Q}_{i,j}\ \mathbf{U}^{(i,j)} = \mathbf{F}, \qquad p \le m, \quad r \le n, \tag{4.3}$$

where $\mathbf{Q}_{i,j}$ denotes the diagonal matrix with inner elements are $q_{i,j}(x_k, y_l)$ where, $(k = 0,\ 1,\ 2,\ ...,\ m\ ;\ l = 0,\ 1,\ 2,\ ...,\ n)$ and $\mathbf{F}$ denotes the column matrix with the elements $f(x_k, y_l)$ where, $(k = 0,\ 1,\ 2,\ ...,\ m\ ;\ l = 0,\ 1,\ 2,\ ...,\ n)$, by substituting the collocation points (4.1) into derivatives of the unknown function as in Eq. (3.4) yields

$$\mathbf{U}^{(i,j)} = \begin{bmatrix} U^{(i,j)}(x_0,\ y_0) \\ \vdots \\ U^{(i,j)}(x_0,\ y_n) \\ U^{(i,j)}(x_1,\ y_0) \\ \vdots \\ U^{(i,j)}(x_1,\ y_n) \\ \vdots \\ U^{(i,j)}(x_n,\ y_m) \end{bmatrix} = [\mathbf{E}(\mathbf{D}_x)^i(\mathbf{D}_y)^j]\mathbf{A}\ , \tag{4.4}$$

where

$$\begin{aligned} \mathbf{E} = [&\mathbf{E}(x_0, y_0)\quad \mathbf{E}(x_0, y_1)\ ...\ \mathbf{E}(x_0, y_n)\ \ \mathbf{E}(x_1, y_0)\quad \mathbf{E}(x_1, y_1)\ ...\ \mathbf{E}(x_1, y_n) \\ &...\ \ \mathbf{E}(x_m, y_0)\quad \mathbf{E}(x_m, y_1)\quad ...\quad \mathbf{E}(x_m, y_n)]^T, \end{aligned}$$

therefore, from Eq. (4.3), we get a system of matrix equation "fundamental matrix" for the PDE in the following form

$$\left(\sum_{i=0}^{p}\sum_{j=0}^{r} \mathbf{Q}_{i,j}\ \left\{\mathbf{E}(x,y)(\mathbf{D}_x)^i(\mathbf{D}_y)^j\right\}\right)\mathbf{A}\ = \mathbf{F}, \tag{4.5}$$

which corresponds to a system of $(m+1)(n+1)$ linear algebraic equations with $(m+1)(n+1)$ double ESC coefficients $a_{r,s}$ unknowns. By substituting the collocation points (4.1) in the condition (3.2) by same procedure before we get the fundamental matrices for conditions as

$$\begin{array}{l} \sum_{t=1}^{\rho}\sum_{i=0}^{p}\sum_{j=0}^{r} b_{i,j}^{t}\left\{\mathbf{E}\left(\omega_t,\eta_t\right)(\mathbf{D}_x)^i(\mathbf{D}_y)^j\right\}\mathbf{A}=\lambda,\\ \sum_{t=1}^{\nu}\sum_{i=0}^{p}\sum_{j=0}^{r} c_{i,j}^{t}(x_k)\left\{\mathbf{E}\left(x_k,\gamma_t\right)(\mathbf{D}_x)^i(\mathbf{D}_y)^j\right\}\mathbf{A}=g(x_k),\\ \sum_{t=1}^{\theta}\sum_{i=0}^{p}\sum_{j=0}^{r} d_{i,j}^{t}(y_l)\ \left\{\mathbf{E}\left(\varepsilon_t,y_l\right)(\mathbf{D}_x)^i(\mathbf{D}_y)^j\right\}\mathbf{A}=h(y_l)\,. \end{array} \tag{4.6}$$

It is also noted that the structure of matrices $\mathbf{Q}_{i,j}$ and $\mathbf{F}$ vary according to the number of collocation points and the structure of the problem. However, $\mathbf{E}$, $\mathbf{D}_x$ and $\mathbf{D}_y$ do not change their nature for fixed values of m and n which are truncation limits of the ESC series. In the other words, the changes in $\mathbf{E}$, $\mathbf{D}_x$ and $\mathbf{D}_y$ are only dependent on the number of collocation points.

5. Method of solution

The fundamental matrix (4.5) for Eq. (3.1) corresponding to a system of $(m+1)(n+1)$ algebraic equations for the $(m+1)(n+1)$ unknown ESC coefficients $a_{0,0}, a_{0,1}, \cdots, a_{0,n}$, $a_{1,0}, a_{1,1}, \cdots, a_{1,n}, \cdots, a_{m,0}, a_{m,1}, \cdots, a_{m,n}$.

We can write the matrix (4.5) as

$$\mathbf{WA{=}F,}\quad \text{or}\quad [\mathbf{W};\ \mathbf{F}], \tag{5.1}$$

and we can obtain the matrix form for the conditions by means of (4.6) in a compact form as

$$\mathbf{VA}=\mathbf{R},\qquad \text{or}\qquad [\mathbf{V};\mathbf{R}]\,, \tag{5.2}$$

where $\mathbf{V}$ is a $h\times(m+1)(n+1)$ matrix and $\mathbf{R}$ is a $h\times 1$ matrix, so that h is the rank of the all row matrices as in (4.6) belong to the given conditions. Then (5.1) together with (5.2) can be written in the following compact form:

$$\mathbf{W}^{*}\mathbf{A}=\mathbf{F}^{*},\quad \text{or}\quad [\mathbf{W}^{*};\mathbf{F}^{*}]\,. \tag{5.3}$$

Furthermore, the system (5.3) can be formed by appending the rows (5.2) on conditions to the system (5.1). Then the size of the system of algebraic equations increases and therefore $\mathbf{W}^*$ becomes a rectangular matrix. To solve this new system, the generalized inverse of $\mathbf{W}^*$ can be used [7], and so the double ESC coefficients can be found as

$$\mathbf{A} = \text{geninv}(\mathbf{W}^*) \cdot \mathbf{F}^*.$$

The method procedure can be summarized by the following algorithm:

1. Calculating the matrix $\mathbf{W}$;
2. Forming the matrix $\mathbf{W}^*$ by adding $\mathbf{V}$;
3. Solving the system of algebraic equations and finding ESC coefficients.

6. Test examples

We consider here some test examples that will be numerically treated by the above proposed method. The numerical computations are carried out by the Mathematica. 7.0.

Example: 6.1

Consider the following partial differential equation

$$u^{(2,1)} + \frac{1}{1+e^x} u^{(1,0)} = f(x,y), \quad x, y \in (-\infty,\ \infty)\,, \tag{6.1}$$

to be the first test problem, with exact solution

$$u(x,y) = (-5 + 3\,Coshy)\,Sech^2\left(\frac{y}{2}\right)\left(Tanh\frac{x}{2}\right),$$

where, the function $f(x,y)$ takes the form

$$f(x,y) = \frac{Sech^2\left(\frac{x}{2}\right)\left(-5 + 3\,Coshy - 8(-1+e^x)Tanh\left(\frac{x}{2}\right)\right)}{(-1+e^x)(1+Coshy)},$$

and the subjected conditions are

$$\begin{aligned}
&u(x,y) = 6\,Tanh\left(\tfrac{x}{2}\right), \ \text{ at } \ y \to \infty,\\
&u(x,y) = 6 \quad \text{at} \ \ x \to \infty \quad \text{and at} \ \ y \to -\infty,\\
&u(0,0) = 0,\\
&u(x,0) \ = 2 \quad \text{at} \ \ x \to -\infty\ .
\end{aligned}$$

The fundamental matrix takes the form

$$\left\{\mathbf{Q}_{1,0}\left[\mathbf{E}\left(\mathbf{D}_x\right)^1\right] + \mathbf{Q}_{2,1}\left[\mathbf{E}\left(\mathbf{D}_x\right)^2(\mathbf{D}_y)^1\right]\right\}\mathbf{A} = \mathbf{F},$$

we take $m = n = 8$, where, the approximate solution given by

$$U(x,y) = a_{0,0}E^U_{0,0}(x,y) + a_{0,1}E^U_{0,1}(x,y) + \cdots + a_{8,8}E^U_{8,8}(x,y).$$

Then, after the augmented matrix of the system and conditions are computed, we obtain the coefficients solution as:

$$\begin{aligned}
&a_{0,0} = \ a_{0,1} = \ ... \ = a_{0,8} = 0,\\
&a_{1,0} = 0,\ a_{1,1} = 0,\ a_{1,2} = 1,\ ...,\ a_{1,8} = 0,\\
&a_{2,0} = \ a_{2,1} = ... \ = a_{2,8} = 0,\\
&\qquad \vdots\\
&a_{8,0} = \ a_{8,1} = \ ... = \ a_{8,8} = 0,
\end{aligned}$$

and the solution is given as: $U(x,y) = E^U_{1,2}(x,y)$,

$$\begin{aligned}
U(x,y) &= 2\left(\tfrac{e^x-1}{e^x+1}\right)\left(-1+4\left(\tfrac{e^y-1}{e^y+1}\right)^2\right)\\
&= (-5+3Coshy)\, Sech^2\left(\frac{y}{2}\right) Tanh\left(\tfrac{x}{2}\right),
\end{aligned}$$

which represent the exact solution of the problem.

Example: 6.2

Consider the following differential equation [10]

$$u_{xy} - \frac{2}{1+e^x}u_y = \frac{4e^y}{(1+e^x)^2(1+e^y)^2}, \qquad x,y \in (-\infty,\ \infty)\,, \tag{6.2}$$

with conditions

$$u_y(0,y) = 0, \qquad u(x,0) = 0.$$

The fundamental matrix takes the form

$$\left\{\mathbf{Q}_{0,1}\left[\mathbf{E}\left(\mathbf{D}_y\right)^1\right] + \mathbf{Q}_{1,1}\left[\mathbf{E}\left(\mathbf{D}_x\right)^1\left(\mathbf{D}_y\right)^1\right]\right\}\mathbf{A} = \mathbf{F},$$

we take $m = n = 4$, where the approximate solution given by

$$U(x,y) = a_{0,0}E^U_{0,0}(x,y) + a_{0,1}E^U_{0,1}(x,y) + \cdots + a_{4,4}E^U_{4,4}(x,y),$$

then, after the augmented matrix of the system and conditions are computed, we obtain the coefficients:

$$\begin{aligned}
&a_{0,0} = \ a_{0,1} = \ ... \ = a_{0,4} = 0,\\
&a_{1,0} = 0,\ a_{1,1} = \tfrac{1}{4},\ a_{1,2} = a_{1,3} = a_{1,4} = 0,\\
&a_{2,0} = \ a_{2,1} = \ ... \ = a_{4,4} = 0,
\end{aligned}$$

and the solution is given by: $U(x,y) = \frac{1}{4}E^U_{1,1}(x,y)$, or

$$U(x,y) = \frac{1}{4}\left[2\left(\frac{e^x-1}{e^x+1}\right)2\left(\frac{e^y-1}{e^y+1}\right)\right] = \left(\frac{e^{x+y}-e^x-e^y+1}{(e^x+1)\,(e^y+1)}\right),$$

which represent the exact solution of Eq (6.2). On the other hand, the approximate solution given in [10] at $n = m = 15$ the approximate solution doesn't give the exact solution.

Example: 6.3

The Cauchy problem [16], for the one-dimensional homogeneous wave equation is given by

$$\begin{aligned} &u_{yy} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \quad y \in [0,\ \infty), \\ &u\,(x, 0) = f(x), \quad u_y(x, 0) = g(x), \quad -\infty < x < \infty. \end{aligned} \tag{6.3}$$

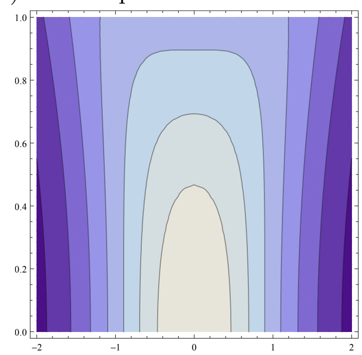
The solution of this problem can be interpreted as the amplitude of a sound wave propagating in very long and narrow pipe, which in practice can be considered as one-dimensional infinite medium. The initial conditions f, g are given functions that represent the amplitude $u(x, y)$ and the velocity u_y of the string at time $y = 0$. The exact solution of (6.3) is given by DAlemberts formula

$$u(x, y) = \frac{1}{2}\left[f(x + cy) + f(x - cy)\right] + \frac{1}{2c}\int_{x-cy}^{x+cy} g(s)ds.$$

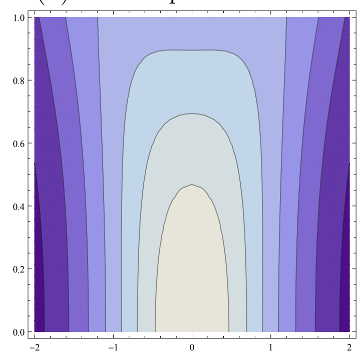
Thus, if we take $f(x) = Sech(x)$ and $g(x) = 0$, and applying our present method to solve (6.3), at $n = m = 8$, and 10 by using double ESC collocation points, we obtain the approximate solution $U(x, y)$. In Table 1, the exact and approximate solutions are listed according to different values of x, y. The calculation of L_2 norm $\left(L_2 = \sqrt{h\sum_{i=0}^{I}\left(u^i - U^i\right)^2}\right)$ presented in Table 2, shows that the grater n, m give us good accuracy at step size $h = 0.1, x \in [-2,\ 2], y \in [0,\ 1]$. In Figure 1 we seek contour plots of the exact and approximate solutions ($n = m = 8, 10$), such that $x \in [-2,\ 2], y \in [0,\ 1]$. Also the error functions for $n = m = 8$ and 10 are plotted in Figure 2.

Figure 1. The contour plots of the exact and approximate solutions

(A) contour plot of the exact solution



(B) contour plot $n = m = 8$



(C) contour plot $n = m = 10$

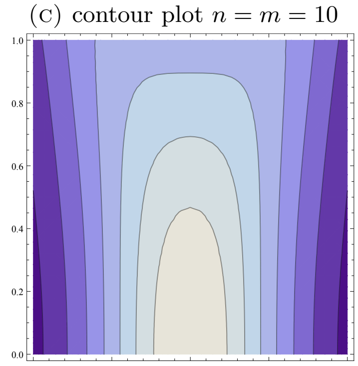
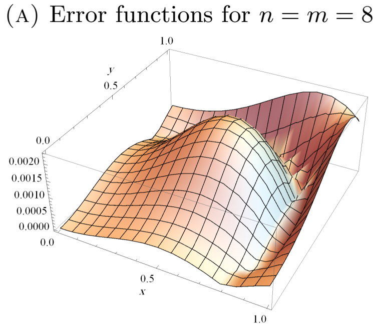


Figure 2. The error function of exact and approximate solutions

(A) Error functions for $n = m = 8$



(B) Error functions for $n = m = 10$

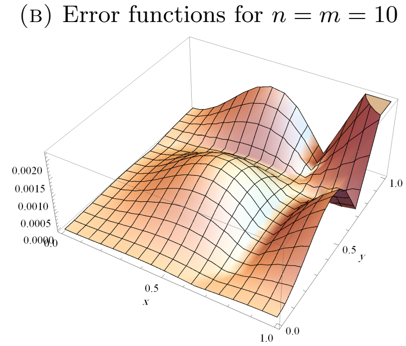


Table. 1 comparing the approximate and exact solution

x	y	Exact solution	Our method $n=m=8$	Abs error	Our method $n=m=10$	Abs error
0	1	0.648054	0.64752	5.34×10^{-4}	0.647913	1.41×10^{-4}
0.1	0.9	0.697877	0.697663	2.13×10^{-4}	0.697525	4.94×10^{-4}
0.2	0.5	0.876667	0.877628	9.60×10^{-4}	0.874907	8.92×10^{-4}
0.3	0.7	0.786531	0.787771	1.23×10^{-3}	0.786684	1.52×10^{-4}
0.4	0.6	0.814191	0.816115	1.92×10^{-3}	0.815118	9.27×10^{-4}
0.5	0.5	0.824027	0.826211	2.18×10^{-3}	0.825253	1.22×10^{-3}
0.6	0.3	0.827211	0.828661	1.44×10^{-3}	0.82791	6.98×10^{-4}
0.7	0.4	0.777981	0.77964	1.65×10^{-3}	0.778372	3.90×10^{-4}
0.8	0.7	0.710058	0.710421	3.62×10^{-4}	0.710033	2.50×10^{-5}
0.9	0.2	0.69802	0.697939	8.03×10^{-5}	0.697525	4.94×10^{-4}
1	1	0.632901	0.631296	1.60×10^{-3}	0.63639	3.48×10^{-3}

Table 2. Comparing the L_2-norm

	our method
$n=m=8$	1.04597×10^{-3}
$n=m=10$	4.16219×10^{-4}

Example: 6.4

Let us consider the Poisson equation [2, 16]

$$\bigtriangledown^2 u = f\left(x,y\right), \quad 0 \le x,\ y \le 1. \tag{6.4}$$

Poisson equation arises in steady state heat problems with time independent heat sources, where the Dirichlet boundary conditions in general form is

$$u(0,y) = f_1(y), \quad u(x,0) = g_1(x),$$
$$u(1,y) = f_2(y), \quad u(x,1) = g_2(x).$$

If we chose the exact solution to be as

$$u(x,y) = \left(1+e^x\right)^{-1}\left(1+e^y\right)^{-1},$$

then, we find

$f_1(y) = \dfrac{1}{2}\left(1+e^y\right)^{-1}, \quad g_1(x) = \dfrac{1}{2}\left(1+e^x\right)^{-1},$

$f_2(y) = \left(1+e^y\right)^{-1}\left(1+e\right)^{-1}, \quad g_2(x) = \left(1+e\right)^{-1}\left(1+e^x\right)^{-1}.$

Appling our present method to solve (6.4), at $n = m = 8$ by using double ESC collocation points, we obtain the approximate solution

$$U(x,y) = 0.25E^U_{0,0}(x,y) - 0.125E^U_{0,1}(x,y) - 0.125E^U_{1,0}(x,y) + 0.0625E^U_{1,1}(x,y).$$

By simplifying the previous relation we reach to

$$U(x,y) = (1+e^x)^{-1}(1+e^y)^{-1},$$

which represent the exact solution of Poisson equation (6.4) with the connected conditions.

7. Conclusion

In this paper, a collocation method for solving high-order linear partial differential equations with variable coefficients under more general form of conditions is investigated. The method is based on the approximation by truncated double exponential second kind Chebyshev (ESC) series, and the definition of the partial derivatives is presented. All principles and properties of this type are derived and introduced by us as new definitions. The definition of the partial derivatives of ESC functions is presented and derived as new operational matrices of derivatives. The PDEs and conditions are transformed into block matrix equations, which correspond to system of linear algebraic equations with unknown ESC coefficients, by using ESC collocation points. The generalized invers is used to solve this linear system and to find the ESC coefficients. Illustrative examples are used to demonstrate the applicability and the effectiveness of the proposed technique. In addition, an interesting feature of this method is to find the analytical exact solution if the equation has an exact solution of rational exponential form. The method can also be extended to high-order nonlinear partial differential equation with variable coefficients, but some modifications are required.

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References

[1] W. M. Abd-Elhameed, Y. H. Youssri and E. H. Doha, *A novel operational matrix method based on shifted Legendre polynomials for solving second-order boundary value problems involving singular, singularly perturbed and Bratu-type equations*, Math. Sci., *9* (2015), 93–102.

[2] N. H. Asmar, *Partial differential equations with fourier series and boundary value problems*, PERSON, Prentic Hall, 2004.

[3] A. Basiri Parsa, M. M. Rashidi, O. Anwar Bg and S.M. Sadri, *Semi-computational simulation of magneto-hemodynamic flow in a semi-porous channel using optimal Homotopy and differential transform methods*, Computers in Biology and Medicine, *43: 9* (2013), 1142–1153.

[4] N. K. Basu, *On double Chebyshev series approximation*,SIAM J. Numer. Anal., *10: 3* (1973), 496–505.

[5] A. H. Bhrawy, *An efficient Jacobi pseudo spectral approximation for nonlinear complex generalized Zakharov system*, Applied Math. and Comput., *247* (2014), 30–46.

[6] R. L. Burden and J. D. Faires, *Numerical analysis*, Cengage Learning, 2011.

[7] A. Dascioglu, *Chebyshev polynomial approximation for high-order partial differential equations with complicated conditions*, Numer. methods for partial Differ. Equ., *25* (2009), 610–621.

[8] E. Erfani, M. M. Rashidi, and A. Basiri Parsa, *The modified differential transform method for solving off-centered stagnation flow toward a rotating disc*, Int. J. of Comput. methods, *7: 4* (2010) 655–670.

[9] B. Y. Guo, and J. P. Yan, *LegendreGauss collocation method for initial value problems of second order ordinary differential equations*. Applied Numer. Math., *59* (2009), 1386–1408.

[10] A. B. Koc and A. Kurnaz, *A new kind of double Chebyshev polynomial approximation on unbounded domains*, Boundary value problems, *2013: 1* (2013), 1687–2770.

[11] S. J. Liao, *Beyond perturbation: introduction to the Homotopy analysis method*, CRC Press, Boca Raton: Chapman & Hall, 2003.

[12] J.C. Mason and D.C. Handscomb, *Chebyshev polynomials*, CRC Press, Boca Raton, 2003.

[13] A. A. Ragab, K. M. Hemida, M. S. Mohamed, and M. A. Abd El Salam, *Solution of time-fractional Navier-Stokes equation by using Homotopy analysis method*, Gen. Math. notes, *13: 2* (2012) 13–21.

[14] M. A. Ramadan, K. R. Raslan T. S. El Danaf and M. A. Abd El salam, *On the exponential Chebyshev approximation in unbounded domains: A comparison study for solving high-order ordinary differential equations*, Int. J. of pure and applied Math., *105: 3* (2015), 399–413.

[15] M. M. Rashidi, E. Momoniat, and B. Rostami, *Analytic approximate solutions for MHD boundary-layer viscoelastic fluid flow over continuously moving stretching surface by homotopy analysis method with two auxiliary parameters*, J. of Applied Math., *2012* (2012), 1–19.

[16] M. Sezer, *A Chebyshev series approximation for linear second-order partial differential equations with complicated conditions*, Gazi Uni. J. of Sci., *26: 4* (2013), 515–525.

[17] G. D. Smith, *Numerical Solution of Partial Differential Equations*, Clarendon Press, Oxford, 2005.

[18] AM. Wazwaz, *Partial differential equations and solitary waves theory*, Springer, 2009.