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Application of the new extended (G'/G)-expansion method to find exact solutions for nonlinear partial differential equation

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Abstract In recent years, numerous approaches have been utilized for finding the exact solutions to nonlinear partial differential equations. One such method is known as the new extended (G'/G)-expansion method and was proposed by Roshid et al. In this paper, we apply this method and achieve exact solutions to nonlinear partial differential equations (NLPDEs), namely the Benjamin-Ono equation. It is establish that the method by Roshid et al. is a very well-organized method which can be used to find exact solutions of a large number of NLPDEs. equations.

Keywords. New extended (G'/G)-expansion method, the Benjamin-Ono equation, exact solutions.

2010 Mathematics Subject Classification. 35C07, 35C08, 35P99.

1. Introduction

NLPDEs play a significant role in different scientific and engineering fields, such as, fluid mechanics, propagation of shallow water waves, solid-state physics, plasma physics, plasma waves, biology, optical fibers, the heat flow and the wave propagation phenomena, quantum mechanics etc. Nonlinear wave phenomena of diffusion, reaction, dispersion, dissipation, and convection are very important in nonlinear wave equations. The exact solutions of NLEEs play an important role in the study of nonlinear physical phenomena and one of the fundamental problems for this is to obtain their traveling wave solutions. In soliton theory, there are many methods and techniques to deal with the problem of solitary wave solutions for NLPDEs such as F-expansion method [1, 25, 39], tanh - sech method [19, 20, 30], Jacobi elliptic function method [12, 13, 18, 41], the collocation method [27, 28], new generalized (G'/G)-expansion

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method [3-6], the wave of translation method [24], the inverse scattering transform [2], the Adomian decomposition method [31], the Darboux transformation method [21], the Backlund transformation method [22], the homogeneous balance method [29, 33], the Sumudu transform method [9-11], the $exp(-\varphi(\eta))$ -expansion method [14, 15], (G'/G)-expansion method [32, 34, 35, 40], new (G'/G)-expansion method [7, 16], the modified simple equation method [17, 36-38], the new extended (G'/G)-expansion method [8, 26] and so on. The objective of this work is to show that the new extended (G'/G)-expansion method and the renowned the basic (G'/G)-expansion methods are not alike. Further many new solutions are achieved via the offered the new extended (G'/G)-expansion method. This approach will play an imperative role in constructing many exact traveling wave solutions for the Benjamin-Ono equation.

2. The Method

For given nonlinear evolution equations with independent variables x, y, z and t, we consider the following form

$$F(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}....) = 0. \tag{2.1}$$

By using traveling wave transformation

$$u(x, y, z, t) = u(\xi), \xi = x + y + z - Vt, \tag{2.2}$$

where u is an unknown function depending on x, y, z, t and is a polynomial F in $u(\xi) = u(x, y, z, t)$ and its partial derivatives and V is a constant to be determined later. The existing steps of method are as follows :

Step 1: Using the Eq.(2.2) in Eq.(2.1), we can convert Eq.(2.1) to an ordinary differential equation

$$Q(u, -Vu', u', V^2u'', u'', -Vu''....) = 0 \tag{2.3}$$

Step 2: Assume the solutions of Eq.(2.3) can be expressed in the form

$$u(\xi) = \sum_{i=-n}^{n} \{ \frac{a_i(G'/G)^i}{[1 + \lambda(G'/G)]^i} + b_i(G'/G)^{i-1}\sqrt{\sigma[1 + \frac{(G'/G)^2}{\mu}]} \}, \tag{2.4}$$

with $G = G(\xi)$ satisfying the differential equation

$$G'' + \mu G = 0. \tag{2.5}$$

In which the value of σ must be ± 1, $\mu \neq 0$, $a_i, b_i, (i = -n, ..., n)$ and λ are constants to be determined later. We can evaluate n by balancing the highest-order derivative term with the nonlinear term in the reduced equation (2.3).

Step 3: Inserting Eq.(2.4) into Eq.(2.3) and making use of Eq.(2.5) and then extracting all terms of powers of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma[1+\frac{(G'/G)^2]}{\mu}}$ together with each coefficient of them to zero yield a over-determined system of algebraic equations and then solving this system of algebraic equations for $a_i, b_i (i=-n,...,n)$ and λ, V, we obtain several sets of solutions.

Step 4: For the general solutions of Eq.(2.5), we have
$\mu < 0$,

$$\frac{G'}{G} = \sqrt{-\mu}\left(\frac{A\sinh(\sqrt{-\mu}\xi)+B\cosh(\sqrt{-\mu}\xi)}{A\cosh(\sqrt{-\mu}\xi)+B\sinh(\sqrt{-\mu}\xi)}\right) = f_1(\xi)$$

and when $\mu > 0$,then

$$\frac{G'}{G} = \sqrt{\mu}\left(\frac{Acos(\sqrt{\mu}\xi)-Bsin(\sqrt{\mu}\xi)}{Asin(\sqrt{\mu}\xi)+Bcos(\sqrt{\mu}\xi)}\right) = f_2(\xi), \tag{2.6}$$

where A, B are arbitrary constants. At last, inserting the values of $a_i, b_i (i=-n,...,n)$, λ, V and (2.6) into Eq.(2.4) and obtain required traveling wave solutions of Eq.(2.1).

3. Application of our Method

As an example of our method, Let us consider the Benjamin-Ono equation,

$$u_t + Hu_{xx} + uu_x = 0, \tag{3.1}$$

where H is the Hilbert transform. The BO equation describes internal waves. It is a completely integrable equation that gives N-soliton solutions. We utilize the traveling wave variable $u(\xi) = u(x,t), \xi = x - Vt$, Eq. (3.1) is carried to an ODE

$$-Vu' + Hu'' + 1/2(u^2)' = 0. \tag{3.2}$$

Eq. (3.2) is integrable, therefore, integrating with respect to ξ once yields:

$$K - Vu + Hu' + 1/2u^2 = 0, \tag{3.3}$$

where K is an integration constant. By balancing the highest-order derivative term u' and nonlinear term u^2 in Eq. (3.3) gives $n = 1$, thus, we have the solutions of Eq.(3.1), according to Eq. (2.4) is

$$\begin{aligned} u(\xi) = a_0 + \frac{a_1(G'/G)}{1+\lambda(G'/G)} + \frac{a_{-1}[1+\lambda(G'/G)]}{(G'/G)} \\ +(b_0(G'/G)^{-1} + b_1 + b_{-1}(G'/G)^{-2}) \\ \times\sqrt{\sigma[1+(G'/G)^2/\mu]}, \end{aligned} \tag{3.4}$$

where $G = G(\xi)$ satisfies Eq.(2.5). Substituting Eq.(3.4) and Eq.(2.5) into Eq.(3.3), collecting all terms with the like powers of $(G'/G)^j$and $(G'/G)^j\sqrt{\sigma[1+(G'/G)^2/\mu]}$, and setting them to zero, we obtain a over-determined system that consists of tweenty-five algebraic equations (we omitted these for convenience). Solving this over-determined system with the assist of Maple, we have the following results.

Case-1: $K = 2a_0H\mu\lambda + a_0^2/2 + 2H^2\mu^2\lambda^2 + 2H^2\mu, V = 2H\mu\lambda + a_0, \lambda = const, a_{-1} =$

$0, a_1 = 2H\lambda^2\mu + 2H, a_0 = const, b_{-1} = b_0 = b_1 = 0.$
Now when $\mu > 0$, then using (2.6) and (3.4), we have

$$u(\xi) = a_0 + (2H\mu\lambda^2 + 2H)\frac{f_2(\xi)}{[1+\lambda f_2(\xi)]}, \tag{3.5}$$

where $\xi = x - (2H\lambda\mu + a_0)t$ and when $\mu < 0$, then using (2.6) and (3.4), we have

$$u(\xi) = a_0 + (2H\mu\lambda^2 + 2H)\frac{f_1(\xi)}{[1+\lambda f_1(\xi)]}, \tag{3.6}$$

where $\xi = x - (2H\lambda\mu + a_0)t$
Case-2: $K = 2H^2\mu^2\lambda^2 - 2a_0H^2\mu\lambda + a_0^2/2 + 2H^2\mu, V = -2H\mu\lambda + a_0, \lambda = const, a_{-1} = -2H\mu, a_1 = 0, a_0 = const, b_{-1} = b_0 = b_1 = 0.$
Now when $\mu > 0$, then using (2.6) and (3.4), we have

$$u(\xi) = a_0 - 2H\mu\frac{(1+\lambda f_2(\xi))}{f_2(\xi)}, \tag{3.7}$$

where $\xi = x - (a_0 - 2H\lambda\mu)t$ and when $\mu < 0$, then using (2.6) and (3.4), we have

$$u(\xi) = a_0 - 2H\mu\frac{(1+\lambda f_1(\xi))}{f_1(\xi)}, \tag{3.8}$$

where $\xi = x - (a_0 - 2H\lambda\mu)t$
Case-3: $K = a_0^2/2 + 8H^2\mu, V = a_0, \lambda = 0, a_{-1} = -2H\mu, a_1 = 2H, a_0 = const, b_{-1} = b_0 = b_1 = 0.$
Now when $\mu > 0$, then using (2.6) and (3.4), we have

$$u(\xi) = a_0 + 2Hf_2(\xi) - \frac{2H\mu}{f_2(\xi)}, \tag{3.9}$$

where $\xi = x - a_0t$ and when $\mu < 0$, then using (2.6) and (3.4), we have

$$u(\xi) = a_0 + 2Hf_1(\xi) - \frac{2H\mu}{f_1(\xi)}, \tag{3.10}$$

where $\xi = x - a_0t$
Case-4: $K = \frac{1}{2}H^2\mu^2\lambda^2 - a_0H\mu\lambda + a_0^2/2 + \frac{1}{2}H^2\mu, V = -H\mu\lambda + a_0, \lambda = const, a_{-1} = -H\mu, a_1 = 0, a_0 = const, b_0 = \pm H\mu\sqrt{1/\sigma}, b_{-1} = b_1 = 0.$
Now when $\mu > 0$, then using (2.6) and (3.4), we have

$$u(\xi) = a_0 - \mu\frac{(1+\lambda f_2(\xi))}{f_2(\xi)} \pm H\mu\sqrt{\frac{1}{\sigma}}\frac{1}{f_2(\xi)}\sqrt{\sigma[1 + \frac{(f_2(\xi))^2}{\mu}]}, \tag{3.11}$$

where $\xi = x - (a_0 - H\lambda\mu)t$ and when $\mu < 0$, then using (2.6) and (3.4), we have

$$u(\xi) = a_0 - \mu\frac{(1+\lambda f_1(\xi))}{f_1(\xi)} \pm H\mu\sqrt{\frac{1}{\sigma}}\frac{1}{f_1(\xi)}\sqrt{\sigma[1 + \frac{(f_1(\xi))^2}{\mu}]}, \tag{3.12}$$

where $\xi = x - (a_0 - H\lambda\mu)t$
Case-5: $K = \frac{1}{2}V^2 + \frac{1}{2}H^2\mu, V = const, \lambda = 0, a_{-1} = 0, a_0 = V, a_1 = H, b_0 = b_{-1} = 0, b_1 = \pm H\frac{\mu}{\sigma}.$
Now when $\mu > 0$, then using (2.6) and (3.4), we have

$$u(\xi) = a_0 + H\mu f_2(\xi) \pm H\sqrt{\frac{\mu}{\sigma}}\frac{1}{f_2(\xi)}\sqrt{\sigma[1 + \frac{(f_2(\xi))^2}{\mu}]}, \tag{3.13}$$

where $\xi = x - Vt$ and when $\mu < 0$, then using (2.6) and (3.4), we have

$$u(\xi) = a_0 + H\mu f_1(\xi) \pm H\sqrt{\frac{\mu}{\sigma}}\frac{1}{f_1(\xi)}\sqrt{\sigma[1 + \frac{(f_1(\xi))^2}{\mu}]}, \tag{3.14}$$

where $\xi = x - Vt$

Remark 3.1: Some of these solutions presented in this latter have been checked with Maple by putting them back into the original equations.

Remark 3.2: New extended (G'/G)-expansion method is simple but its results are very cumbersome. The results of this method contain many arbitrary constants compare to the results of the other method. The performance of new extended (G'/G)-expansion method is reliable, simple, direct, concise and gives more new exact solutions compared to the other method. This method allowed us to solve more complicated PDEs in the mathematical physics and engineering.

4. Discussions

The advantages and validity of the method over the basic (G'/G)-expansion method have been discussed in the following:

Advantages: The crucial advantage of the new extended (G'/G)-expansion method over the basic (G'/G)-expansion method is that the method provides more general and large amount of new exact traveling wave solutions with several free parameters. The exact solutions have its great importance to expose the inner mechanism of the complex physical phenomena. Apart from the physical application, the close-form solutions of nonlinear evolution equations assist the numerical solvers to compare the accuracy of their results and help them in the stability analysis.

Validity: In Ref. [23] Neyrame et al. used the linear ordinary differential equation as auxiliary equation and traveling wave solutions presented in the form $u(\xi) = \sum_{i=0}^{m} a_i(G'/G)^i$ where $a_m \neq 0$. It is notable to point out that some of our solutions are coincided with already published results, if parameters taken particular values which authenticate our the solutions. Moreover, in Ref. [23] Neyrame et al. investigated the Benjamin-Ono equation to obtain exact solutions via the basic (G'/G)-expansion method and achieved only three solutions (A. 1)-(A. 3) (see appendix). Moreover, ten solutions of the Benjamin-Ono equation are constructed by applying the new extended (G'/G)-expansion method.

Graphical representations of the solutions: The graphical illustrations of the solutions are depicted in the figures 1 to 10 with the aid of commercial software Maple.

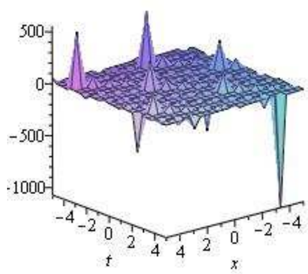


Fig. 1: Periodic solution of (3.5) with $\mu = 1, \lambda = 3, A = 1, B = 2, h = 2, a_0 = 1$ and $-5 \leq x, t \leq 5$.

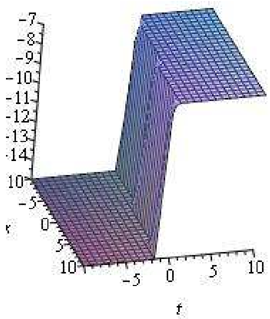


Fig. 2: Kink solution of (3.6) with $\mu = -1, \lambda = 3, A = 1, B = 2, h = 1, a_0 = 1$ and $-10 \leq x, t \leq 10$.

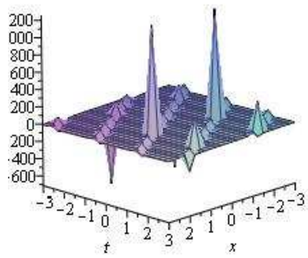


Fig. 3: Periodic solutions of (3.7) with $\mu = 1, \lambda = 3, A = 1, B = 2, h = 2, a_0 = 1$ and $-3 \leq x, t \leq 3$.

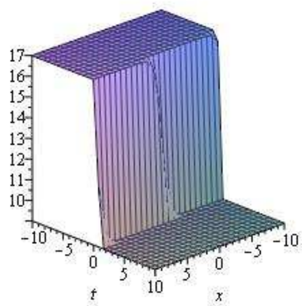


Fig. 4: Kink solution of (3.8) with $\mu = -1, \lambda = 3, A = 1, B = 2, h = 2, a_0 = 1$ and $-10 \leq x, t \leq 10$.

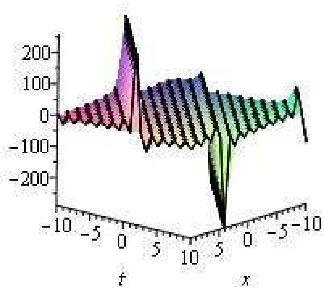


Fig. 5: Periodic solution of (3.9) with $\mu = 1, \lambda = 3, A = 1, B = 2, h = 2, a_0 = 1$ and $-10 \leq x, t \leq 10$

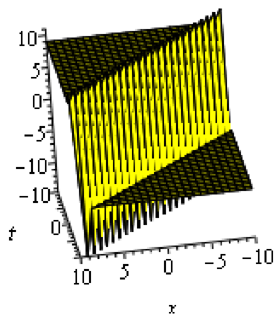


Fig. 6: Modulas plot of Singular Kink of (3.10) with $\mu = -1, \lambda = 3, A = 1, B = 2, h = 2, a_0 = 1$ and $-10 \leq x, t \leq 10$.

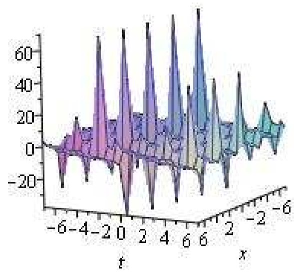


Fig. 7: Modulas plot of periodic solution of (3.11) with $\mu = 1, \lambda = 3, A = 1, B = 2, h = 2, a_0 = 1$ and $-7 \leq x, t \leq 7$.

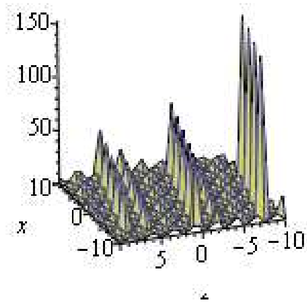


Fig. 8: Modulas plot of periodic solution of (3.12) with $\mu = -1, \lambda = 3, A = 1, B = 2, h = 2, a_0 = 1$ and $-10 \leq x, t \leq 10$.

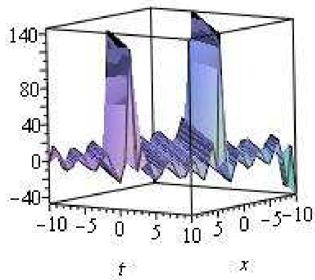
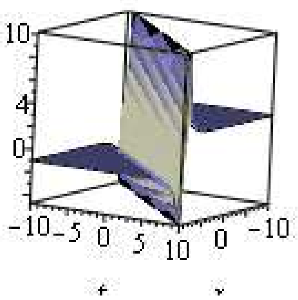


Fig. 9: Periodic solution of (3.13) with $\mu = 1, \lambda = 3, A = 1, B = 2, h = 2, a_0 = 1$ and $-10 \leq x, t \leq 10$.



5. Conclusions

The new extended (G'/G)-expansion method was applied effectively to solve one important NLPDE, including the Benjamin-Ono equation, analytically. Some exact solutions for this equation was formally obtained by using the new extended (G'/G)-expansion method. Due to the excellent acting of the new extended (G'/G)-expansion method, we feel that it is an influential scheme in handling a wide variety of NLPDEs.

Appendix: Neyrame et al. solutions [23]

Neyrame et al. [23] established exact solutions of the well-known the Benjamin-Ono equation by using the basic (G'/G)-expansion method which are as follows:

When $\lambda^2 - 4\mu > 0$, $u_1 = 2h\sqrt{\lambda^2 - 4\mu} \times (\frac{C_1 \sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi + C_2 \cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}{C_1 \cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi + C_2 \sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}) - \frac{\lambda}{2} + \alpha_0$ $(A.1)$
where $\xi = x - (h\lambda - \alpha_0)t$ and C_1, C_2 are arbitrary constants. When $\lambda^2 - 4\mu < 0$,
$u_2 = 2h\sqrt{4\mu - \lambda^2} \times (\frac{-C_1 \sin(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi + C_2 \cos(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi}{C_1 \cos(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi + C_2 \sin(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi}) - \frac{\lambda}{2} + \alpha_0$ $(A.2)$
where $\xi = x - (h\lambda - \alpha_0)t$ and C_1, C_2 are arbitrary constants. When $\lambda^2 - 4\mu = 0$,
$u_3 = \frac{2hC_2}{C_1 + C - 2\xi}$ $(A.3)$
where $\xi = x - (h\lambda - \alpha_0)t$ and C_1, C_2 are arbitrary constants.

Acknowledgment

This section should come before the References and should be unnumbered. Funding information may also be included here.

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